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Some geometric properties of lacunary Zweier Sequence Spaces of order α

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Abstract

In this paper we introduce a new sequence space using Zweier matrix operator and lacunary sequence of order α. Also we study some geometrical properties such as order continuous, the Fatou property and the Banach-Saks property of the new space.

Keywords and phrases : *Lacunary sequence; Zweier operator; order continuous; Fatou property; Banach-Saks property*

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1. Introduction

Throughout the article w, c, c_0 and ℓ_∞ denotes the spaces of all, convergent, null and bounded sequences, respectively. Also, by ℓ_1 and ℓ_p, we denote the spaces of all absolutely summable and p-absolutely summable series, respectively. Recall that a sequence $(x(i))_{i=1}^{\infty}$ in a Banach space X is called *Schauder* (or *basis*) of X if for each $x \in X$ there exists a unique sequence $(a(i))_{i=1}^{\infty}$ of scalars such that $x = \sum_{i=1}^{\infty} a(i)x(i)$, i.e. $\lim_{n\to\infty} \sum_{i=1}^{n} a(i)x(i) = x$. A sequence space X with a linear topology is called a *K-space* if each of the projection maps $P_i : X \to \mathbf{C}$ defined by $P_i(x) = x(i)$ for $x = (x(i))_{i=1}^{\infty} \in X$ is continuous for each natural i. A *Fréchet space* is a complete metric linear space and the metric is generated by a *F-norm* and a Fréchet space which is a *K-space* is called an *FK-space* i.e. a *K-space* X is called an *FK-space* if X is a complete linear metric space. In other words, X is an *FK-space* if X is a Fréchet space with continuous coordinatewise projections. All the sequence spaces mentioned above are *FK*-space except the space c_{00} which is the space of real sequences which have only a finite number of non-zero coordinates. An *FK*-space X which contains the space c_{00} is said to have the *property AK* if for every sequence $(x(i))_{i=1}^{\infty} \in X, x = \sum_{i=1}^{\infty} x(i)e(i)$ where $e(i) = (0, 0, ...1^{i^{th} place}, 0, 0, ...)$.

A Banach space X is said to be a *Köthe sequence space* if X is a subspace of w such that

(a) if $x \in w, y \in X$ and $|x(i)| \leq |y(i)|$ for all $i \in \mathbf{N}$, then $x \in X$ and $||x|| \leq ||y||$

(b) there exists an element $x \in X$ such that $x(i) > 0$ for all $i \in \mathbf{N}$.

We say that $x \in X$ is *order continuous* if for any sequence $(x_n) \in X$ such that $x_n(i) \leq |x(i)|$ for all $i \in \mathbf{N}$ and $x_n(i) \to 0$ as $n \to \infty$ we have $||x_n|| \to 0$ holds.

A Köthe sequence space X is said to be *order continuous* if all sequences in X are order continuous. It is easy to see that $x \in X$ order continuous if and only if $||(0, 0, ..., 0, x(n+1), x(n+2), ...)|| \to 0$ as $n \to \infty$.

A Köthe sequence space X is said to be the *Fatou property* if for any real sequence x and (x_n) in X such that $x_n \uparrow x$ coordinatewisely and $\sup_n ||x_n|| < \infty$, we have that $x \in X$ and $||x_n|| \to ||x||$.

A Banach space X is said to have the *Banach-Saks property* if every bounded sequence (x_n) in X admits a subsequence (z_n) such that the sequence $(t_k(z))$ is convergent in X with respect to the norm, where

$$t_k(z) = \frac{z_1 + z_2 + ... + z_k}{k} \text{ for all } k \in \mathbf{N}.$$

Some of works on geometric properties of sequence space can be found in [3, 4, 8, 9, 13, 16, 17, 18, 19, 20, 22, 23].

Şengönül [24] defined the sequence $y = (y_k)$ which is frequently used as the Z^i-transformation of the sequence $x = (x_k)$ i.e.

$$y_k = ix_k + (1-i)x_{k-1}$$

where $x_{-1} = 0, k \neq 0, 1 < k < \infty$ and Z^i denotes the matrix $Z^i = (z_{nk})$ defined by

$$z_{nk} = \begin{cases} i, & \text{if } n = k; \\ 1-i, & \text{if } n-1 = k; \\ 0, & \text{otherwise.} \end{cases}$$

Şengönül [24] introduced the Zweier sequence spaces $\mathcal{Z}$ and $\mathcal{Z}_0$ as follows

$$\mathcal{Z} = \{x = (x_k) \in w : Z^i x \in c\}$$

and

$$\mathcal{Z}_0 = \{x = (x_k) \in w : Z^i x \in c_0\}.$$

For details on Zweier sequence spaces we refer to [5, 10, 11, 12, 14, 15].

2. Lacunary Zweier sequence spaces of order α

by lacunary sequence we mean an increasing sequence $\theta = (k_r)$ of positive integers satisfyling $k_0 = 0$ and $h_r := k_r - k_{r-1} \to \infty$ as $r \to \infty$. We denote the intervals, by $I_r = (k_{r-1}, k_r]$, which determines θ. Let $\alpha \in (0, 1]$ be any real number and let p be a positive real number such that $1 \leq p < \infty$. Now we define the following sequence space.

$$[\mathcal{Z}_\theta^\alpha]_\infty(p) = \left\{ x \in w : \sup_r \frac{1}{h_r^\alpha} \sum_{k \in I_r} | \left(Z^i x \right)_k |^p < \infty \right\}.$$

Special cases:

(a) For $p = 1$ we have $[\mathcal{Z}_\theta^\alpha]_\infty(p) = [\mathcal{Z}_\theta^\alpha]_\infty$.

(b) For $\alpha = 1$ and $p = 1$ we have $[\mathcal{Z}_\theta^\alpha]_\infty(p) = [\mathcal{Z}_\theta]_\infty$.

For details on sequence spaces of order α we refer to [1, 2, 6, 7].

Theorem 2.1. *Let $\alpha \in (0,1]$ and p be a positive real number such that $1 \le p < \infty$. Then the sequence space $[\mathcal{Z}_\theta^\alpha]_\infty(p)$ is a BK-space normed by*

$$||x||_\alpha = \sup_r \frac{1}{h_r^\alpha} \left(\sum_{k \in I_r} |\left(Z^i x\right)_k|^p \right)^{\frac{1}{p}}. \tag{2.1}$$

Proof. Since the matrix Z^i is a triangle, we have the result by norm (2.1) and the Theorem 4.3.12 of Wilansky [[25], p. 63]. □ $[\mathcal{Z}_\theta^\alpha]_\infty \subset [\mathcal{Z}_\theta^\alpha]_\infty(p)$.

Theorem 2.2. *Let α and β be fixed real numbers such that $0 < \alpha \le \beta \le 1$ and p be a positive real number such that $1 \le p < \infty$. Then $[\mathcal{Z}_\theta^\alpha]_\infty(p) \subset [\mathcal{Z}_\theta^\beta]_\infty(p)$.*

Proof. The proof of theorem follows from the following inequality. For all $r \in \mathbf{N}$ we have is straightforward, so omitted.

$$\frac{1}{h_r^\alpha} \sum_{k \in I_r} |\left(Z^i x\right)_k|^p \le \frac{1}{h_r^\beta} \sum_{k \in I_r} |\left(Z^i x\right)_k|^p.$$

□

Theorem 2.3. *Let α and β be fixed real numbers such that $0 < \alpha \le \beta \le 1$ and p be a positive real number such that $1 \le p < \infty$. For two lacunary sequences $\theta = (h_r)$ and $\phi = (l_r)$ for all r, then $[\mathcal{Z}_\theta^\alpha]_\infty(p) \subset [\mathcal{Z}_\phi^\beta]_\infty(p)$ if and only if $\sup_r \left(\frac{h_r^\alpha}{l_r^\beta}\right) < \infty$.*

Proof. Let $x = (x_k) \in [\mathcal{Z}_\theta^\alpha]_\infty(p)$ and $\sup_r \left(\frac{h_r^\alpha}{l_r^\beta}\right) < \infty$. Then

$$\sup_r \frac{1}{h_r^\alpha} \sum_{k \in I_r} |\left(Z^i x\right)_k|^p < \infty$$

and there exists a positive number K such that $h_r^\alpha \leq K l_r^\beta$ and so that $\frac{1}{l_r^\beta} \leq \frac{K}{h_r^\alpha}$ for all r. Therefore, we have

$$\frac{1}{l_r^\beta} \sum_{k \in I_r} |\left(Z^i x\right)_k|^p \leq \frac{K}{h_r^\alpha} \sum_{k \in I_r} |\left(Z^i x\right)_k|^p.$$

Now taking supremum over r, we get

$$\sup_r \frac{1}{l_r^\beta} \sum_{k \in I_r} |\left(Z^i x\right)_k|^p \leq \sup_r \frac{K}{h_r^\alpha} \sum_{k \in I_r} |\left(Z^i x\right)_k|^p$$

and hence $x \in [\mathcal{Z}_\phi^\beta]_\infty(p)$.

Next suppose that $[\mathcal{Z}_\theta^\alpha]_\infty(p) \subset [\mathcal{Z}_\phi^\beta]_\infty(p)$ and $\sup_r \left(\frac{h_r^\alpha}{l_r^\beta}\right) = \infty$. Then there exists an incresing sequence (r_i) of natural numbers such that $\lim_i \left(\frac{h_{r_i}^\alpha}{l_{r_i}^\beta}\right) = \infty$. Let L be a positive real number, then there exists $i_0 \in \mathbf{N}$ such that $\frac{h_{r_i}^\alpha}{l_{r_i}^\beta} > L$ for all $r_i \geq i_0$. Then $h_{r_i}^\alpha > L l_{r_i}^\beta$ and so $\frac{1}{l_{r_i}^\beta} > \frac{L}{h_{r_i}^\alpha}$. Therefore we can write

$$\frac{1}{l_{r_i}^\beta} \sum_{k \in I_r} |\left(Z^i x\right)_k|^p > \frac{L}{h_{r_i}^\alpha} \sum_{k \in I_r} |\left(Z^i x\right)_k|^p \text{ for all } r_i \geq i_0.$$

Now taking supremum over $r_i \geq i_0$ then we get

$$\sup_{r_i \geq i_0} \frac{1}{l_{r_i}^\beta} \sum_{k \in I_{r_i}} |\left(Z^i x\right)_k|^p > \sup_{r_i \geq i_0} \frac{L}{h_{r_i}^\alpha} \sum_{k \in I_{r_i}} |\left(Z^i x\right)_k|^p. \tag{2.2}$$

Since the relation (2.2) holds for all $L \in \mathbf{R}^+$ (we may take the number L sufficienlty large), we have

$$\sup_{r_i \geq i_0} \frac{1}{l_{r_i}^\beta} \sum_{k \in I_{r_i}} |\left(Z^i x\right)_k|^p = \infty$$

but $x = (x_k) \in [\mathcal{Z}_\theta^\alpha]_\infty(p)$ with

$$\sup_r \left(\frac{h_r^\alpha}{l_r^\beta}\right) < \infty.$$

Therefore $x \notin [\mathcal{Z}_\phi^\beta]_\infty(p)$ which contradicts that $[\mathcal{Z}_\theta^\alpha]_\infty(p) \subset [\mathcal{Z}_\phi^\beta]_\infty(p)$. Hence $\sup_{r \geq 1} \left(\frac{h_r^\alpha}{l_r^\beta}\right) < \infty$. □

Corollary 2.4. *Let α and β be fixed real numbers such that $0 < \alpha \leq \beta \leq 1$ and p be a positive real number such that $1 \leq p < \infty$. For any two lacunary sequences $\theta = (h_r)$ and $\phi = (l_r)$ for all $r \geq 1$, then*

(a) $[\mathcal{Z}_\theta^\alpha]_\infty(p) = [\mathcal{Z}_\phi^\beta]_\infty(p)$ *if and only if* $0 < \inf_r \left(\frac{h_r^\alpha}{l_r^\beta}\right) < \sup_r \left(\frac{h_r^\alpha}{l_r^\beta}\right) < \infty$.

(b) $[\mathcal{Z}_\theta^\alpha]_\infty(p) = [\mathcal{Z}_\phi^\alpha]_\infty(p)$ *if and only if* $0 < \inf_r \left(\frac{h_r^\alpha}{l_r^\alpha}\right) < \sup_r \left(\frac{h_r^\alpha}{l_r^\alpha}\right) < \infty$.

(c) $[\mathcal{Z}_\theta^\alpha]_\infty(p) = [\mathcal{Z}_\theta^\beta]_\infty(p)$ *if and only if* $0 < \inf_r \left(\frac{h_r^\alpha}{h_r^\beta}\right) < \sup_r \left(\frac{h_r^\alpha}{h_r^\beta}\right) < \infty$.

Theorem 2.5. $\ell_p \subset [\mathcal{Z}_\theta^\alpha]_\infty(p) \subset \ell_\infty$.

Proof. The proof of the result is straightforward, so omitted. □

Theorem 2.6. *If $0 < p < q$, then $[\mathcal{Z}_\theta^\alpha]_\infty(p) \subset [\mathcal{Z}_\theta^\alpha]_\infty(q)$.*

Proof. The proof of the result is straightforward, so omitted. □

3. Some geometric properties

In this section we study some of the geometric properties like order continuous, the Fatou property and the Banach-Saks property in this new sequence space.

Theorem 3.1. *The space $[\mathcal{Z}_\theta^\alpha]_\infty(p)$ is order continuous.*

Proof. We have to show that the space $[\mathcal{Z}_\theta^\alpha]_\infty(p)$ is an AK-space. It is easy to see that $[\mathcal{Z}_\theta^\alpha]_\infty(p)$ contains c_{00} which is the space of real sequences which have only a finite number of non-zero coordinates. By using the definition of AK-properties, we have that $x = (x(i)) \in [\mathcal{Z}_\theta^\alpha]_\infty(p)$ has a unique representation $x = \sum_{i=1}^\infty x(i)e(i)$ i.e. $||x - x^{[j]}||_\alpha = ||(0, 0, .., x(j), x(j+1), ...)||_\alpha \to 0$ as $j \to \infty$, which means that $[\mathcal{Z}_\theta^\alpha]_\infty(p)$ has AK. Therefore BK-space $[\mathcal{Z}_\theta^\alpha]_\infty(p)$ containing c_{00} has AK-property, hence the space $[\mathcal{Z}_\theta^\alpha]_\infty(p)$ is order continuous. □

Theorem 3.2. *The space $[\mathcal{Z}_\theta^\alpha]_\infty(p)$ has the Fatou property.*

Proof. Let x be a real sequence and (x_j) be any nondecreasing sequence of non-negative elements form $[\mathcal{Z}_\theta^\alpha]_\infty(p)$ such that $x_j(i) \to x(i)$ as $j \to \infty$ coordinatewisely and $\sup_j ||x_j||_\alpha < \infty$.

Let us denote $T = \sup_j ||x_j||_\alpha$. Since the supremum is homogeneous, then we have

$$\frac{1}{T}\sup_r \frac{1}{h_r^\alpha}\left(\sum_{k\in I_r} |\left(Z^i x_j(i)\right)_k|^p\right)^{\frac{1}{p}}$$

$$\leq \sup_r \frac{1}{h_r^\alpha}\left(\sum_{k\in I_r}\left|\frac{\left(Z^i x_j(i)\right)_k}{||x_n||_\alpha}\right|^p\right)^{\frac{1}{p}}$$

$$= \frac{1}{||x_n||_\alpha}||x_n||_\alpha = 1.$$

Also by the assumptions that (x_j) is non-dreceasing and convergent to x coordinatewisely and by the Beppo-Levi theorem, we have

$$\frac{1}{T}\lim_{j\to\infty}\sup_r \frac{1}{h_r^\alpha}\left(\sum_{k\in I_r} |\left(Z^i x_j(i)\right)_k|^p\right)^{\frac{1}{p}}$$

$$= \sup_r \frac{1}{h_r^\alpha}\left(\sum_{k\in I_r}\left|\frac{\left(Z^i x(i)\right)_k}{T}\right|^p\right)^{\frac{1}{p}} \leq 1,$$

whence

$$||x||_\alpha \leq T = \sup_j ||x_j||_\alpha = \lim_{j\to\infty} ||x_j||_\alpha < \infty.$$

Therefore $x \in [\mathcal{Z}_\theta^\alpha]_\infty(p)$. On the other hand, since $0 \leq x$ for any natural number j and the sequence (x_j) is non-decreasing, we obtain that the sequence $(||x_j||_\alpha)$ is bounded form above by $||x||_\alpha$. Therefore $\lim_{j\to\infty} ||x_j||_\alpha \leq ||x||_\alpha$ which contadicts the above inequality proved already, yields that $||x||_\alpha = \lim_{j\to\infty} ||x_j||_\alpha$. $\square$

Theorem 3.3. *The space $[\mathcal{Z}_\theta^\alpha]_\infty(p)$ has the Banach-Saks property.*

Proof. The proof of the result follows from the standard technique. $\square$

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