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Asymptotically Double Lacunary Statistically Equivalent Sequences of Interval Numbers

Ayhan Esi
Adiyaman University, Turkey
Shyamal Debnath
Tripura University, India
and
Subrata Saha
Tripura University, India
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Abstract

In this paper we have introduced the concept of asymptotically double lacunary statistically equivalent of interval numbers and strong asymptotically double lacunary statistically equivalent of interval numbers. We have investigated the relations related to these spaces.

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1. Introduction

Interval arithmetic was first suggested by Dwyer [22] in 1951. Development of interval arithmetic as a formal system and evidence of its value as a computational device was provided by Moore [25] in 1959 and Moore and Yang [26] in 1962. Furthermore, Moore and others [23, 24] have developed applications to differential equations.

Chiao in [18] introduced sequence of interval numbers and defined usual convergence of sequences of interval number. Şengönül and Eryilmaz in [21] introduced and studied bounded and convergent sequence spaces of interval numbers and showed that these spaces are complete metric space.

In the recent days, Esi in [1] and [2] introduced and studied strongly almost $\lambda-$ convergence and statistically almost $\lambda-$ convergence of interval numbers and lacunary sequence spaces of interval numbers, respectively. For more information about interval numbers one may refer to Debnath and Saha [31], Debnath et al. [29, 30].

The idea of statistical convergence for ordinary sequences was introduced by Fast [12] in 1951. Schoenberg [15] studied statistical convergence as a summability method and listed some of elementary properties of statistical convergence. Both of these authors noted that if bounded sequence is statistically convergent, then it is Cesaro summable. Existing work on statistical convergence appears to have been restricted to real or complex sequence, but several authors extended the idea to apply to sequences of fuzzy numbers and also introduced and discussed the concept of statistically convergent sequences of fuzzy numbers. For some very interesting investigations concerning statistical convergence, one may consult the papers of Cakalli [11], Miller [13], Maddox [14] and many others, where more references on this important summability method can be found.

In 1993, Marouf [19] presented definitions for asymptotically equivalent sequences and asymptotic regular matrices. In 2003, Patterson [27] extended these concepts by presenting as asymptotically statistical equivalent analog of these definitions and natural regularity conditions for nonnegative summability matrices.

2. Preliminaries

We denote the set of all real valued closed intervals by $\mathbf{R}$. Any elements of $\mathbf{R}$ is called interval number and denoted by $\overline{A} = [x_l, x_r]$. Let x_l and x_r be first and last points of $\overline{x}$ interval number, respectively. For $\overline{A}_1, \overline{A}_2 \in \mathbf{R}$, we have

$\overline{A}_1 = \overline{A}_2 \Leftrightarrow x_{1_l}{=}x_{2_l}, x_{1_r}{=}x_{2_r}$. $\overline{A}_1{+}\overline{A}_2 = \{x \in \mathbf{R} : x_{1_l} + x_{2_l} \leq x \leq x_{1_r} + x_{2_r}\}$, and if $\alpha \geq 0$, then
$\alpha\overline{A} = \{x \in \mathbf{R} : \ \alpha x_{1_l} \leq x \leq \alpha x_{1_r}\}$ and if $\alpha < 0$, then
$\alpha\overline{A} = \{x \in \mathbf{R} : \ \alpha x_{1_r} \leq x \leq \alpha x_{1_l}\}$,

$$\overline{A}_1.\overline{A}_2$$

$= \{x \in \mathbf{R} : \min\{x_{1_l}.x_{2_l}, x_{1_l}.x_{2_r}, x_{1_r}.x_{2_l}, x_{1_r}.x_{2_r}\} \leq x$
$\leq \max\{x_{1_l}.x_{2_l}, x_{1_l}.x_{2_r}, x_{1_r}.x_{2_l}, x_{1_r}.x_{2_r}\}$.

The set of all interval numbers $\mathbf{R}$ is a complete metric space defined by

$$d\left(\overline{A}_1, \overline{A}_2\right) = \max\{|x_{1_l} - x_{2_l}|, |x_{1_r} - x_{2_r}|\}.$$

In the special case $\overline{A}_1 = [a,a]$ and $\overline{A}_2 = [b,b]$, we obtain usual metric of $\mathbf{R}$. Let us define transformation $f : \mathbf{N} \rightarrow I\mathbf{R}$ by $k \rightarrow f(k) = \overline{A^k}$, $\overline{A} = \left(\overline{A}_k\right)$. Then $\overline{A} = \left(\overline{A}_k\right)$ is called sequence of interval numbers. The $\overline{A}_k$ is called k^{th} term of sequence $\overline{A} = \left(\overline{A}_k\right)$. w^i denotes the set of all interval numbers with real terms.

A double sequence of real numbers is a function x: N $\times$ N $\rightarrow$ R. We shall use the notation $x = (x_{k,l})$.

A double sequence $x = (x_{k,l})$ has a Pringsheim limit L (denoted by P - lim x = L) provided that given an $\varepsilon > 0$ there exists an $N_0 \in N$ such that $|(x_{k,l}) - L| < \varepsilon$ whenever $k, l > L$. We shall describe such an $x = (x_{k,l})$ more briefly as "P-convergent". The double sequence $x = (x_{k,l})$ is bounded if there exists a positive number M such that $|(x_{k,l}) - L| < M$ for all k and l, and

$$\|x\| = sup_{k,l}\,|x_{k,l}| < \infty.$$

Let $p = (p_{k,l})$ be a double sequence of positive real numbers. If $0 < h = \inf{}_{k,l} p_{k,l} \leq p_{k,l} \leq H = \sup_{k,l} p_{k,l} < \infty$ and $D = max(1, 2^{H-1})$, then for all $a_{k,l}, b_{k,l} \in C$ for all $k, l \in N$, we have

$$|a_{k,l} + b_{k,l}|^{p_{k,l}} \leq \text{D}\ (|a_{k,l}|^{p_{k,l}} + |b_{k,l}|^{p_{k,l}}).$$

We should note that in contrast to the case for single sequences, a convergent double sequence need not be bounded. A sequence $x = (x_k)$ is said to be statistically convergent to the number L if for every $\varepsilon > 0$

$$lim_n \frac{1}{n}\ |\{k \leq n : |x_k - L| \geq \varepsilon\}| = 0$$

where the vertical bars indicate the number of elements in the enclosed set. Later, Mursaleen and Edely [20] defined the statistical analogue for double sequence $x = (x_{k,l})$ as follows:

A real double sequence $x = (x_{k,l})$ is said to be P - statistical convergence to L provided that for each $\varepsilon > 0$

$$P - lim_{m,n}\frac{1}{mn} \left|\{(k,l) : k < m,\ l < n;\ |x_{k,l} - L| \geq \varepsilon\}\right| = 0$$

In this case, we write $St_2 - lim_{k,l}x_{k,l} = L$ and we denote the set of all P- statistical convergent double sequences by St_2.

By a lacunary sequence $\theta = (k_r)$; r = 0, 1, 2, ... where $k_0 = 0$, we shall mean an increasing sequence of non-negative integers with $h_r = k_r - k_{r-1} \to \infty$ as $r \to \infty$. The intervals determined by θ will be denoted by $I_r = (k_{r-1}, k_r]$. The ratio $\frac{k_r}{k_{r-1}}$ will be denoted by q_r, where $r > 1$. The space of lacunary strongly convergent sequence space N_θ was defined by Freedman et al. [10] as follows:

$$N_\theta = \{x = (x_k) : lim_r \frac{1}{h_r}\ \Sigma_{k\in I_r}|x_k - L| = 0,\ for\ some\ L\}.$$

The double sequence $\theta_{r,s} = \{(k_r, l_s)\}$ is called double lacunary sequence if there exist two increasing of integers such that $k_o = 0$, $h_r = k_r - k_{r-1} \to \infty$ as $r \to \infty$and $l_o = 0$, $\overline{h_s} = l_s - l_{s-1} \to \infty$ as $s \to \infty$.

Notations: $k_{r,s} = k_r l_s$, $h_{r,s} = h_r\overline{h_s}$ and $\theta_{r,s}$ is determined by $I_{r,s} = \{(k,l) :\ k_{r-1} < k \leq k_r \text{ and } l_{s-1} < l \leq l_s\}$, $q_r = \frac{k_r}{k_{r-1}}, \overline{q_s} = \frac{l_s}{l_{s-1}}\ and\ q_{r,s} = q_r\overline{q_s}$.

The set of all double lacunary sequences denoted by $N_{\theta_{r,s}} = \{x = (x_k) : P - lim_{r,s}\frac{1}{h_{r,s}}\Sigma_{(k,l)\in I_{r,s}}\ |x_{k,l} - L| = 0,\ for\ some\ L\}$

Definition 1.1. [1] A sequence $\overline{A} = \left(\overline{A}_k\right)$ of interval numbers is said to be convergent to the interval number $\overline{A}_o$ if for each $\varepsilon > 0$ there exists a positive integer k_o such that $d\left(\overline{A}_k, \overline{A}_o\right) < \varepsilon$ for all $k \geq k_o$ and we denote it by $\lim_k \overline{A}_k = \overline{A}_o$.

Thus, $\lim_k \overline{A}_k = \overline{A}_o \Leftrightarrow \lim_k x_{k_l} = x_{o_l}$ and $\lim_k x_{k_r} = x_{o_r}$.

For more information about interval numbers one can refer to Esi [1-9] and Debnath and Saha [29-31]:

Definition 1.2. [19] Two non-negative sequences $x = (x_k)$ and $y = (y_k)$ are said to be asymptotically equivalent if $lim_k\frac{x_k}{y_k} = 1$ (denoted by $x \sim y$).

Definition 1.3. [16] The sequence $x = (x_k)$ has statistical limit L, denoted by st - lim x= L provided that for every $\varepsilon > 0$,

$$lim_n \frac{1}{n}\{the\ number\ of\ k \leq n :\ |x_k - L| \geq \varepsilon\} = 0.$$

Definition 1.4.[28] Two non-negative sequences $x = (x_k)$ and $y = (y_k)$ are said to be asymptotically statistical equivalent of multiple L provided that for every $\varepsilon > 0$,

$$\lim_n \frac{1}{n}\{the\ number\ of\ k \leq n :\ |\frac{x_k}{y_k} - L| \geq \varepsilon\} = 0.$$

(denoted by xS_Ly), and simply asymptotically statistical equivalent if $L = 1$.

Definition 1.5.[28] Let θ be a lacunary sequence; the two non-negative sequences $x = (x_k)$ and $y = (y_k)$ are said to be asymptotically lacunary statistical equivalent of multiple L provided that for every $\varepsilon > 0$,

$$\lim_r \frac{1}{h_r}\left|\left\{k \in I_r : |\frac{x_k}{y_k} - L|) \geq \varepsilon\right\}\right| = 0$$

$\Big($denoted by $xS_\theta^L y\Big)$ and simply asymptotically lacunary statistical equivalent if $L = 1$.

Definition 1.6. Let θ be a lacunary sequence; two number sequences $x = (x_k)$ and $y = (y_k)$ are strong asymptotically lacunary equivalent of multiple L provided that

$$\lim_r \frac{1}{h_r}\Sigma_{k\in I_r}|\frac{x_k}{y_k} - L| = 0$$

$\Big($denoted by $xN_\theta^L y\Big)$ and simply strong asymptotically lacunary equivalent if $L = 1$.

3. Main Results

Definition 3.1 Let $\theta_{r,s} = \{(k_r, l_s)\}$ be a double lacunary sequence. Two non negative double interval sequences $\overline{x} = (\overline{x}_{kl})$ and $\overline{y} = (\overline{y}_{kl}) \neq \overline{0} = [0,0]$ are said to be asymptotically double lacunary statistical equivalent of interval number $\overline{x}_0$ provided that for every $\epsilon > 0$

$$P-\lim_{r,s}\frac{1}{h_{r,s}}\left|\left\{(k,l)\in I_{r,s}:\ d\left(\frac{\overline{x}_{k,l}}{\overline{y}_{k,l}},\overline{x}_0\right)\geq\varepsilon\right\}\right|=0$$

$\Big($denoted by $\overline{x}S_{\theta_{r,s}}\overline{y}\Big)$ and simply $\theta_{r,s}-$asymptotically double lacunary statistical equivalent if

$\overline{x}_0=\overline{1}$. In the special case $\theta_{r,s}=\{(2^r,2^s)\}$, we shall write $S_{r,s}$ instead of $S_{\theta_{r,s}}$.

Definition 3.2 Let $\theta_{r,s}=\{(k_r,l_s)\}$ be a double lacunary sequence and p $=(p_{k,l})$ be any double sequence of strictly positive real numbers. Two non negative double sequences of interval numbers $\overline{x}=(\overline{x}_{kl})$ and $\overline{y}=(\overline{y}_{kl})$ are said to be strong asymptotically double lacunary statistical equivalent of interval number $\overline{x}_0$ provided that for every $\epsilon>0$

$$P-\lim_{r,s}\frac{1}{h_{r,s}}\Sigma_{k\in I_{r,s}}[\ d\left(\frac{\overline{x}_{k,l}}{\overline{y}_{k,l}},\overline{x}_0\right)]^{p_{k,l}}=0$$

$\Big($denoted by $\overline{x}2^{N^p_{\theta_{r,s}}}\overline{y}\Big)$ and simply $\theta_{r,s}-$ strong asymptotically double lacunary statistical equivalent if $\overline{x}_0=\overline{1}$. In the special case $\theta_{r,s}=\{(2^r,2^s)\}$, we shall write 2^{N^p} instead of $2^{N^p_{\theta_{r,s}}}$.

Theorem 3.1 Let $\theta_{r,s}=\{(k_r,l_s)\}$ be a double lacunary sequence. Then

(i)If $\overline{x}2^{N^p_{\theta_{r,s}}}\overline{y}$ then $\overline{x}S_{\theta_{r,s}}\overline{y}$

(ii) If $\overline{x}=(\overline{x}_{kl})\in\overline{l}_\infty$ and $\overline{x}S_{\theta_{r,s}}\overline{y}$ then $\overline{x}2^{N^p_{\theta_{r,s}}}\overline{y}$.

(iii) If $\overline{x}=(\overline{x}_{kl})\in\overline{l}_\infty$ then $S_{\theta_{r,s}}\bigcap l_\infty=2^{N^p_{\theta_{r,s}}}\bigcap\overline{l}_\infty$, where $\overline{l}_\infty$ denote the set of bounded sequences.

Proof : (i) Let $\epsilon>0$ and $\overline{x}2^{N^p_{\theta_{r,s}}}\overline{y}$, then

$\left|\left\{(k,l)\in I_{r,s}:\ d\left(\frac{\overline{x}_{k,l}}{\overline{y}_{k,l}},\overline{x}_0\right)\geq\varepsilon\right\}\right|\geq\Sigma_{(k,l)\in I_{r,s} and\ d\left(\frac{\overline{x}_{k,l}}{\overline{y}_{k,l}},\overline{x}_0\right)\geq\epsilon}\ d\left(\frac{\overline{x}_{k,l}}{\overline{y}_{k,l}},\overline{x}_0\right)$

and P-$\lim_{r,s}\frac{1}{h_{r,s}}\Sigma_{k\in I_{r,s}}[\ d\left(\frac{\overline{x}_{k,l}}{\overline{y}_{k,l}},\overline{x}_0\right)]^{p_{k,l}}=0$

This implies that P-$\lim_{r,s}\frac{1}{h_{r,s}}\left|\left\{(k,l)\in I_{r,s}:\ d\left(\frac{\overline{x}_{k,l}}{\overline{y}_{k,l}},\overline{x}_0\right)\geq\varepsilon\right\}\right|=0$ Therefore, $\overline{x}S_{\theta_{r,s}}\overline{y}$

(ii) Suppose that $\overline{x}=(\overline{x}_{kl})$ and $\overline{y}=(\overline{y}_{kl})$ in $\overline{l}_\infty$ and $\overline{x}S_{\theta_{r,s}}\overline{y}$. Then there is a constant M >0 such that $d\left(\frac{\overline{x}_{k,l}}{\overline{y}_{k,l}},\overline{x}_0\right)\leq M$

Given $\epsilon > 0$. So we have
$\frac{1}{h_{r,s}} \Sigma_{(k,l)\in I_{r,s}} [\ d\left(\frac{\overline{x}_{k,l}}{\overline{y}_{k,l}}, \overline{x}_0\right)]^{p_{k,l}}$
$= \frac{1}{h_{r,s}} \Sigma_{(k,l)\in I_{r,s} and\ d\left(\frac{\overline{x}_{k,l}}{\overline{y}_{k,l}}, \overline{x}_0\right)\geq \epsilon} [\ d\left(\frac{\overline{x}_{k,l}}{\overline{y}_{k,l}}, \overline{x}_0\right)]^{p_{k,l}}$
$+ \frac{1}{h_{r,s}} \Sigma_{(k,l)\in I_{r,s}}$ and $d\left(\frac{\overline{x}_{k,l}}{\overline{y}_{k,l}}, \overline{x}_0\right) < \epsilon [\ d\left(\frac{\overline{x}_{k,l}}{\overline{y}_{k,l}}, \overline{x}_0\right)]^{p_{k,l}}$
$\leq \frac{1}{h_{r,s}} \Sigma_{(k,l)\in I_{r,s} and\ d\left(\frac{\overline{x}_{k,l}}{\overline{y}_{k,l}}, \overline{x}_0\right)\geq \epsilon} max(M^h, M^H)$
$+ \frac{1}{h_{r,s}} \Sigma_{(k,l)\in I_{r,s} and\ d\left(\frac{\overline{x}_{k,l}}{\overline{y}_{k,l}}, \overline{x}_0\right)< \epsilon} \epsilon^{p_{k,l}}$
$\leq max(M^h, M^H)\ \frac{1}{h_{r,s}} \left|\left\{(k,l) \in I_{r,s} :\ d\left(\frac{\overline{x}_{k,l}}{\overline{y}_{k,l}}, \overline{x}_0\right) \geq \varepsilon\right\}\right| + max(\epsilon^h, \epsilon^H)$

Therefore, $\overline{x}2^{N^p_{\theta_{r,s}}}\overline{y}$.

(iii) It follows from (i) and (ii).

Theorem 3.2 Let $\theta_{r,s} = \{(k_r, l_s)\}$ be a double lacunary sequence with lim $\inf_r q_r > 1$ and lim inf $_s\overline{q}_s > 1$ then $\overline{x}S_{r,s}\overline{y}$ implies $\overline{x}S_{\theta_{r,s}}\overline{y}$.

Proof : Suppose that lim $\inf_r q_r > 1$ and lim $\inf_s q_s > 1$ then there exists a $\delta > 0$ such that $q_r > 1 + \delta$, $\overline{q}_s > 1 + \delta$ for sufficiently large r and s, which implies $\frac{h_r}{k_r} \geq \frac{\delta}{1+\delta}$ and $\frac{\overline{h}_s}{l_s} \geq \frac{\delta}{1+\delta}$.

Since $h_{r,s} = k_r l_s - k_r l_{s-1} - k_{r-1} l_s - k_{r-1} l_{s-1}$

We granted the following

$$\frac{k_r l_s}{h_{r,s}} \leq \frac{(1+\delta)^2}{\delta^2} \text{ and}$$
$$\frac{k_{r-1} l_{s-1}}{h_{r,s}} \leq \frac{1}{\delta}$$

If $\overline{x}S_{r,s}\overline{y}$ then for every $\epsilon > 0$ and for sufficiently large r and s, we have

$$\frac{1}{k_r l_s} \left|\left\{(k,l) \in I_{r,s}; k \leq k_r\ and\ l \leq l_s :\ d\left(\frac{\overline{x}_{k,l}}{\overline{y}_{k,l}}, \overline{x}_0\right) \geq \varepsilon\right\}\right|$$
$$\geq \frac{1}{k_r l_s} \left|\left\{(k,l) \in I_{r,s} :\ d\left(\frac{\overline{x}_{k,l}}{\overline{y}_{k,l}}, \overline{x}_0\right) \geq \varepsilon\right\}\right|$$
$$\geq \frac{(1+\delta)^2}{\delta^2}\ \frac{1}{h_{r,s}} \left|\left\{(k,l) \in I_{r,s} :\ d\left(\frac{\overline{x}_{k,l}}{\overline{y}_{k,l}}, \overline{x}_0\right) \geq \varepsilon\right\}\right|$$

Hence, $\overline{x}S_{\theta_{r,s}}\overline{y}$.

Theorem 3.3 Let $\theta_{r,s} = \{(k_r, l_s)\}$ be a double lacunary sequence with lim $sup_r q_r < \infty$ and lim $sup_s \overline{q}_s < \infty$ then $\overline{x}S_{\theta_{r,s}}\overline{y}$ implies $\overline{x}S_{r,s}\overline{y}$.

Proof: If lim $sup_r q_r < \infty$ and lim $sup_s \overline{q}_s < \infty$ then there exists D > 0 such that $q_r < D$ and $\overline{q}_s < D$ for all r, s ≥ 1. Let $\overline{x}S_{\theta_{r,s}}\overline{y}$, and $\epsilon > 0$. Then there exist $r_0 < 0$ and $s_0 > 0$ such that for every i $\geq r_0$ and j $\geq s_0$

$$C_{i,j} = \frac{1}{h_{i,j}} \left|\left\{(k,l) \in I_{i,j} :\ d\left(\frac{\overline{x}_{k,l}}{\overline{y}_{k,l}}, \overline{x}_0\right) \geq \varepsilon\right\}\right| < \varepsilon$$

Let M = max $\{C_{i,j} : 1 \leq i \leq r_0 \ and \ 1 \leq j \leq s_0\}$ and m and n be such that $k_{r-1} < m \leq k_r$ and $l_{s-1} < n \leq l_s$. Thus we obtain the following

$$\frac{1}{mn} \left|\left\{(k,l) \in I_{i,j}; k \leq m \ and \ l \leq n :\ d\left(\frac{\overline{x}_{k,l}}{\overline{y}_{k,l}}, \overline{x}_0\right) \geq \varepsilon\right\}\right|$$

$$\leq \frac{1}{k_{r-1}l_{s-1}} \left|\left\{(k,l) \in I_{i,j}; k \leq k_r \ and \ l \leq l_s :\ d\left(\frac{\overline{x}_{k,l}}{\overline{y}_{k,l}}, \overline{x}_0\right) \geq \varepsilon\right\}\right|$$

$$\leq \frac{1}{k_{r-1}l_{s-1}} \Sigma_{(1\leq t\leq r_0)\bigcup(1\leq u\leq s_0)} h_{t,u}C_{t,u} + \frac{1}{k_{r-1}l_{s-1}} \Sigma_{(r_0<t\leq r)\bigcup(s_0<u\leq s)} h_{t,u}C_{t,u}$$

$$\leq \frac{M}{k_{r-1}l_{s-1}} \Sigma_{(1\leq t\leq r_0)\bigcup(1\leq u\leq s_0)} h_{t,u} + \frac{1}{k_{r-1}l_{s-1}} \Sigma_{(r_0<t\leq r)\bigcup(s_0<u\leq s)} h_{t,u}C_{t,u}$$

$$\leq \frac{Mk_{r_0}l_{s_0}r_0s_0}{k_{r-1}l_{s-1}} + \frac{1}{k_{r-1}l_{s-1}} \Sigma_{(r_0<t\leq r)\bigcup(s_0<u\leq s)} h_{t,u}C_{t,u}$$

$$\leq \frac{Mk_{r_0}l_{s_0}r_0s_0}{k_{r-1}l_{s-1}} + (sup_{\ t\geq r_0\bigcup u\geq s_0} C_{t,u}) \ \frac{1}{k_{r-1}l_{s-1}} \Sigma_{(r_0<t\leq r)\bigcup(s_0<u\leq s)} h_{t,u}$$

$$\leq \frac{Mk_{r_0}l_{s_0}r_0s_0}{k_{r-1}l_{s-1}} + \frac{\varepsilon}{k_{r-1}l_{s-1}} \Sigma_{(r_0<t\leq r)\bigcup(s_0<u\leq s)} h_{t,u}$$

$$\leq \frac{Mk_{r_0}l_{s_0}r_0s_0}{k_{r-1}l_{s-1}} + \varepsilon D^2$$

Since k_r and l_s both approach infinity as both m and n approach infinity it follows that

$$\frac{1}{mn} \left|\left\{(k,l) \in I_{i,j}; k \leq m \ and \ l \leq n :\ d\left(\frac{\overline{x}_{k,l}}{\overline{y}_{k,l}}, \overline{x}_0\right) \geq \varepsilon\right\}\right| \to 0.$$

This completes the proof.

Theorem 3.4 Let $\theta_{r,s} = \{(k_r, l_s)\}$ be a double lacunary sequence with $1 < lim \ inf_{r,s} q_{r,s} \leq lim \ sup_{r,s} \overline{q}_{r,s} \ < \infty$ then

$$\overline{x}S_{r,s}\overline{y} = \overline{x}S_{\theta_{r,s}}\overline{y}$$

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Proof : The result clearly follows from theorem 3.2 and theorem 3.3.

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Ayhan Esi
Department of Mathematics
Adiyaman university,
Turkey
e-mail : aesi23@hotmail.com

Shyamal Debnath
Department of Mathematics

Tripura University,
India
e-mail : shyamalnitamath@gmail.com

and

Subrata Saha
Department of Mathematics
Tripura University,
India
e-mail : subratasaha2015@gmail.com