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On a sequence of functions $V_n^{(\alpha,\beta,\delta)}(x; a, k, s)$

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Abstract

In this paper, authors established various properties of a sequence of functions $\{V_n^{(\alpha,\beta,\delta)}(x; a, k, s)/n = 0, 1, 2, ...\}$ *such as generating relations, bilateral generating relations, finite summation formulae, generating functions involving Stirling number, explicit representation and integral transforms.*

Keyword : *Sequence of functions, operational techniques, generating functions, finite summation formulae, Srivastava's theorem, Singhal-Srivastava generating function and Srivastava-Lavoie theorem.*

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1. Introduction

Differential operators plays important role in study of various kind of generalized Rodrigues formulae, generating relations, Finite summation formulae. In this paper, operators considered as follows

$$\theta \equiv x^a(s+xD) \text{ and } \theta_1 \equiv x^a(1+xD), \tag{1.1}$$

where a and s are arbitrary and $D = \dfrac{d}{dx}$.

Shukla and Prajapati [6] studied some properties of a class of polynomials suggested by Mittal. A sequence of functions $\{V_n^{(\alpha,\beta,\delta)}(x;a,k,s);\ n = 0,1,2,...\}$ introduced by Prajapati and Ajudia [4] as,

$$V_n^{(\alpha,\beta,\delta)}(x;a,k,s) = \frac{1}{n!}x^{-\beta}W(\alpha,\delta;p_k(x))\theta^n\left\{x^\beta W(\alpha,\delta;-p_k(x))\right\} \tag{1.2}$$

where $W(\alpha,\delta;z)$ is Wright function defined as

$\mathrm{W}(\alpha,\delta;z) = \sum\limits_{n=0}^{\infty}\frac{z^n}{n!\Gamma(\alpha n+\delta)}$

Some noteworthy operational techniques required as follows

$$\theta^n\{x^{\alpha+k}\} = a^n\left(\frac{s+\alpha+k}{a}\right)_n x^{\alpha+k+na}, \tag{1.3}$$

where k is an integer, n a non-negative integer and α is arbitrary.

$$\theta^n\{xuv\} = x\sum_{k=0}^{n}\binom{n}{k}\theta^k u\ \theta_1^{n-k}v \tag{1.4}$$

where n is non negative integer.

$$\theta^n\{x^\alpha f(x)\} = x^\alpha(\theta+\alpha x^a)^n\{f(x)\} \tag{1.5}$$

2. Generating Relations

Following generating relations (2.1) to (2.3) of equation (1.2) obtained by Prajapati and Ajudia [4] as

$$\sum_{n=0}^{\infty}x^{-an}V_n^{(\alpha,\beta,\delta)}(x;a,k,s)t^n = \tag{2.1}$$

$$(1-at)^{-\left(\frac{\beta+s}{a}\right)} W\left(\alpha,\delta;p_k(x)\right) W\left(\alpha,\delta;-p_k\{x(1-at)^{-\frac{1}{a}}\}\right)$$

with $|t|<|a|^{-1}; a\neq 0$

$$\sum_{n=0}^{\infty} x^{-a\,n}\, V_n^{(\alpha,\beta-an,\delta)}(x;a,k,s)\, t^n = \tag{2.2}$$

$$(1+at)^{\frac{\beta+s}{a}-1} W\left(\alpha,\delta;p_k(x)\right) W\left(\alpha,\delta;-p_k\{x(1+at)^{\frac{1}{a}}\}\right)$$

with $|t|<|a|^{-1}; a\neq 0$

For $n\in\mathbf{N}$,

$$\sum_{m=0}^{\infty} \binom{m+n}{m} V_{n+m}^{(\alpha,\beta,\delta)}(x;a,k,s) x^{-am} t^m \tag{2.3}$$

$$= (1-at)^{-\left(\frac{\beta+s}{a}\right)} \frac{W(\alpha,\delta;p_k(x))}{W\left(\alpha,\delta;p_k\{x(1-at)^{-\frac{1}{a}}\}\right)} V_n^{(\alpha,\beta,\delta)}\{x(1-at)^{-\frac{1}{a}};a,k,s\}$$

with $|t|<|a|^{-1}; a\neq 0$.

In similar pattern, one can prove the following generating relation:

For $n\in\mathbf{N}$,

$$\sum_{m=0}^{\infty} \binom{n+m}{m} x^{-am} V_{n+m}^{(\alpha,\beta-an,\delta)}(x;a,k,s) t^m \tag{2.4}$$

$$= (1+at)^{\frac{\beta+s}{a}-1} \frac{W\left(\alpha,\delta;p_k(x)\right)}{W\left(\alpha,\delta;p_k\{x(1+at)^{\frac{1}{a}}\}\right)} V_n^{(\alpha,\beta,\delta)}(x(1+at)^{\frac{1}{a}};a,k,s)$$

with $|t|<|a|^{-1}; a\neq 0$.

The following theorem plays an important role to get several generating relations.

Theorem 2.1 (Srivastava's Theorem [8]). *If the functions* $A(z), B_i(z)$ *and* $z^{-1}C_j(z)$ *are analytic about the origin such that*

$$A(0), B_i(0), C_j(0)\neq 0, i=1,2,\ldots,r; j=1,2,\ldots,l \tag{2.5}$$

and if the sequence of function $\{g_n^{(p_1,p_2,\ldots,p_r)}(x_1,x_2,\ldots,x_l)\}_{n=0}^{\infty}$ *defined by*

$$A(z)\prod_{i=1}^{r}\{[B_i(z)]^{p_i}\}\exp\left(\sum_{j=1}^{l}x_jC_j(z)\right)=\sum_{n=0}^{\infty}g_n^{(p_1,p_2,\ldots,p_r)}(x_1,x_2,\ldots,x_l)\frac{z^n}{n!} \tag{2.6}$$

then, for arbitrary p_i,λ_i,x_j *and* $y_j(i=1,2,\ldots,r;j=1,2,\ldots,l)$ *independent of* z,

$$\sum_{n=0}^{\infty}g_n^{(p_1+\lambda_1 n,\ldots,p_r+\lambda_r n)}(x_1+ny_1,\ldots,x_l+ny_l)\frac{t^n}{n!}=\frac{A(w)\prod_{i=1}^{r}\{[B_i(w)]^{p_i}\}\exp\left\{\sum_{j=1}^{l}x_jC_j(w)\right\}}{1-w\left[\sum_{i=1}^{r}\lambda_i\left(\frac{B_i'(w)}{B_i(w)}\right)+\sum_{j=1}^{l}y_jC_j'(w)\right]} \tag{2.7}$$

where

$$w=t\prod_{i=1}^{r}\{[B_i(w)]^{\lambda_i}\}\exp\left\{\sum_{j=1}^{l}y_jC_j(w)\right\} \tag{2.8}$$

By considering $\alpha=0,\delta=1$ and $p_k(x)=px^k$ in generating relation (2.1) reduces into the form

$$\sum_{n=0}^{\infty}x^{-an}V_n^{(0,\beta,1)}(x;a,k,s,p)t^n=(1-at)^{-\left(\frac{\beta+s}{a}\right)}\exp\{px^k(1-(1-at)^{-\frac{k}{a}})\} \tag{2.9}$$

The above generating relation (2.9) is equivalently of the type (2.6), if we consider $r=2,l=1,A(z)=1$, $B_1(z)=(1-at)^{-\frac{1}{a}}=B_2(z)$, $x_1=p$, $C_1(z)=x^k(1-(1-at)^{-\frac{k}{a}})$ and $g_n^{(\beta,s)}(p)=n!V_n^{(0,\beta,1)}(x;a,k,s,p)$.

By employing theorem (2.1) on (2.9) gives

$$\sum_{n=0}^{\infty}V_n^{(0,\beta+\lambda_1 n,1)}(x;a,k,s+\lambda_2 n,p+ny)t^n= \tag{2.10}$$

$$\frac{(1-aw)^{-\left(\frac{\beta+s}{a}\right)}\exp\left\{px^k[1-x(1-aw)^{-\frac{1}{a}}]\right\}}{1-w(1-aw)^{-1}\left\{\lambda_1+\lambda_2-ykx^k(1-aw)^{-\frac{k}{a}}\right\}}$$

where,

$$w=t(1-aw)^{-\left(\frac{\lambda_1+\lambda_2}{a}\right)}\exp\left\{yx^k[1-(1-aw)^{-\frac{k}{a}}]\right\};a\neq 0$$

replacing aw by w in the generating relation (2.10), we get

$$\sum_{n=0}^{\infty}V_n^{(0,\beta+\lambda_1 n,1)}(x;a,k,s+\lambda_2 n,p+ny)t^n \tag{2.11}$$

$$=\frac{(1-w)^{-\left(\frac{\beta+s}{a}\right)}\exp\left\{px^k[1-x(1-w)^{-\frac{1}{a}}]\right\}}{1-w\{a(1-w)\}^{-1}\left\{\lambda_1+\lambda_2-ykx^k(1-w)^{-\frac{k}{a}}\right\}}$$

where,

$$w=at(1-w)^{-\left(\frac{\lambda_1+\lambda_2}{a}\right)}\exp\left\{yx^k[1-(1-w)^{-\frac{k}{a}}]\right\};a\neq 0$$

On setting w by $w(1+w)^{-1}$ in the generating relation (2.11), one can get

$$\sum_{n=0}^{\infty}V_n^{(0,\beta+\lambda_1 n,1)}(x;a,k,s+\lambda_2 n,p+ny)t^n= \tag{2.12}$$

$$\frac{(1+w)^{\left(\frac{\beta+s}{a}\right)}\exp\left\{px^k[1-x(1+w)^{\frac{1}{a}}]\right\}}{1-\frac{w}{a}\left\{\lambda_1+\lambda_2-ykx^k(1+w)^{\frac{k}{a}}\right\}}$$

where,

$$w=at(1+w)^{\left(\frac{\lambda_1+\lambda_2}{a}+1\right)}\exp\left\{yx^k[1-(1+w)^{\frac{k}{a}}]\right\};a\neq 0$$

In particular, $\lambda_1=-a-\lambda_2$ and $y=0$, then generating relation (2.12) reduces as

$$\sum_{n=0}^{\infty}V_n^{(0,\beta-(a+\lambda_2)n,1)}(x;a,k,s+\lambda_2 n,p+ny)t^n= \tag{2.13}$$

$$(1+at)^{\left(\frac{\beta+s}{a}-1\right)}\exp\left\{px^k[1-x(1+at)^{\frac{1}{a}}]\right\}$$

with $|t|<|a|^{-1};a\neq 0$ and if $\lambda_2=0$ gives

$$\sum_{n=0}^{\infty} V_n^{(0,\beta-an,1)}(x;a,k,s,p+ny)t^n = (1+at)^{\left(\frac{\beta+s}{a}-1\right)} \exp\left\{px^k[1-x(1+at)^{\frac{1}{a}}]\right\}$$

(2.14)

with $|t| < |a|^{-1}; a \neq 0$.

3. Bilateral Generating Functions

A class of function $\{S_n(x), n = 0, 1, 2, ...\}$ is generated by

$$\sum_{n=0}^{\infty} A_{m,n} S_{m+n} t^n = f(x,t)\{g(x,t)\}^{-m} S_m(h(x,t)) \tag{3.1}$$

where m is non negative integer, the coefficient $A_{m,n}$ are arbitrary constant and f, g, h are suitable functions of x and t. Equation (3.1) is well-known in the literature as the Singhal-Srivastava generating function[7].

Theorem 3.1 (Singhal-Srivastava Theorem [12, 7]). *For the sequence $\{S_n(x)\}$ generated by (3.1), let*

$$F(x,t) = \sum_{n=0}^{\infty} a_n S_n(x) t^n \tag{3.2}$$

where $a_n \neq 0$ are arbitrary constants, then

$$f(x,t)F[h(x,t), yt\{g(x,t)\}^{-1}] = \sum_{n=0}^{\infty} S_n(x)\sigma_n(y)t^n \tag{3.3}$$

where $\sigma_n(y)$ is a polynomial (of degree n in y) defined by

$$\sigma_n(y) = \sum_{k=0}^{n} a_k A_{n-k} y^k \tag{3.4}$$

Theorem 3.1 play an important role to obtain a bilateral generating relation from (2.3).

Considering $S_n(x) = V_n^{(\alpha,\beta,\delta)}(x;a,k,s), A_{m,n} = \binom{m+n}{n}, g(x,t) = (1-at), h(x,t) = x(1-at)^{-\frac{1}{a}}$ *and* $f(x,t) = (1-at)^{-\frac{\beta+s}{a}} \frac{W(\alpha,\delta;p_k(x))}{W\left(\alpha,\delta;p_k\{x(1-at)^{-\frac{1}{a}}\}\right)}$ *in above theorem, produces*

$$\sum_{n=0}^{\infty} V_n^{(\alpha,\beta,\delta)}(x;a,k,s)\,\sigma_n(y)t^n = \tag{3.5}$$

$$(1-at)^{-\left(\frac{\beta+s}{a}\right)} \frac{W(\alpha,\delta;p_k(x))}{W\left(\alpha,\delta;p_k\{x(1-at)^{-\frac{1}{a}}\}\right)} F\left[\frac{x}{(1-at)^{\frac{1}{a}}}, \frac{yt}{(1-at)}\right]$$

where $F(x,t) = \sum\limits_{n=0}^{\infty} a_n S_n(x) t^n$

In 1962, Buchholz [1] given identity as

$$\sum_{n=0}^{\infty} (-1)^n \zeta_N^{(\mu+n)}(x) \frac{t^n}{n!} = e^{-t} \zeta_N^{(\mu)}(x+t). \tag{3.6}$$

Theorem 3.2 (Hubble and Srivastava [2]). *For the sequence* $\{S_n(x)\}$ *generated by (3.1), let*

$$\Phi_N(x,y,t) = \sum_{n=0}^{\infty} a_n S_n(x) \zeta_N^{(\mu+n)}(y) t^n, (a_n \neq 0) \tag{3.7}$$

where μ *is an arbitrary(real or complex) parameter.*
Suppose that

$$\phi_{m,n}(u,v,x) = \sum_{r=0}^{min\{m,n\}} \frac{(-1)^r}{r!} a_{m-r} A_{m-r,n-r} u^{m-r} v^{n-r} S_{m+n-2r}(x) \tag{3.8}$$

Then

$$\sum_{m,n=0}^{\infty} \phi_{m,n}(u,v,x) \zeta_N^{(\mu+n)}(y) t^m = e^{-t} f(x,v) \Phi_N\left[h(x,v), y+t, \frac{ut}{g(x,v)}\right] \tag{3.9}$$

provided that each member exists.

Following corollaries followed by theorem (3.2)

Corollary 3.3. *If*

$$\Phi_N(x,y,t)=\sum_{n=0}^{\infty}a_nV_n^{(\alpha,\beta,\delta)}\left(x;a,k,s\right)\zeta_N^{(\mu+n)}(y)t^n,(a_n\neq 0) \tag{3.10}$$

where μ *is an arbitrary(real or complex) parameter*
and

$$\phi_{m,n}(u,v,x)=$$

$$\sum_{r=0}^{min\{m,n\}}\frac{(-1)^r}{r!}\binom{m+n-2r}{n-r}a_{m-r}u^{m-r}v^{n-r}V_{m+n-2r}^{(\alpha,\beta,\delta)}\left(x;a,k,s\right) \tag{3.11}$$

then

$$\sum_{m,n=0}^{\infty}\phi_{m,n}(u,v,x)\zeta_N^{(\mu+n)}(y)t^m \tag{3.12}$$

$$=e^{-t}\left(1-av\right)^{-\left(\frac{\beta+s}{a}\right)}\frac{W\left(\alpha,\delta;p_k(x)\right)}{W\left(\alpha,\delta;p_k\{x(1-av)^{-\frac{1}{a}}\}\right)}$$

$$\Phi_N\left[x(1-av)^{-\frac{1}{a}},y+t,\frac{ut}{(1-av)}\right]$$

Corollary 3.4. *If*

$$\Psi_N(x,y,t)=\sum_{n=0}^{\infty}a_nV_n^{(\alpha,\beta-an,\delta)}\left(x;a,k,s\right)\zeta_N^{(\mu+n)}(y)t^n,(a_n\neq 0)$$

(3.13)

where μ *is an arbitrary(real or complex) parameter*

and

$$\psi_{m,n}(u,v,x)=$$

$$\sum_{r=0}^{min\{m,n\}}\frac{(-1)^r}{r!}\binom{m+n-2r}{n-r}a_{m-r}u^{m-r}v^{n-r}V_{m+n-2r}^{(\alpha,\beta-a(m+n-2r),\delta)}\left(x;a,k,s\right)$$

(3.14)

Then

$$\sum_{m,n=0}^{\infty} \psi_{m,n}(u,v,x)\zeta_N^{(\mu+n)}(y)t^m = e^{-t}\left(1+av\right)^{-\left(\frac{\beta+s}{a}\right)} \frac{W\left(\alpha,\delta;p_k(x)\right)}{W\left(\alpha,\delta;p_k\{x(1+av)^{-\frac{1}{a}}\}\right)}$$

(3.15)

$$\Psi_N\left[x(1+av)^{-\frac{1}{a}}, y+t, \frac{ut}{(1+av)}\right]$$

Srivastava and Lavoie have generalized McBride's theorem [3] in 1975 as follows

Theorem 3.5 (Srivastava and Lavoie [11, 12]). *If sequence* $\{\zeta_\mu(x)$: μ*is a complex number*$\}$ *is generated by*

$$\sum_{n=0}^{\infty} A_{\mu,n}\zeta_{\mu+n}(x)t^n = f(x,t)\{g(x,t)\}^{-\mu}\zeta_\mu(h(x,t)) \tag{3.16}$$

where $A_{\mu,n}$ *arbitrary constants, and* f, g, h *are arbitrary functions of* x *and* t.

Let

$$\Phi_{q,\nu}(x,t) = \sum_{n=0}^{\infty} a_{\nu,n}\zeta_{\mu+qn}(x)t^n, (a_{\nu,n} \neq 0), \tag{3.17}$$

where q *is positive integer and* ν *is an arbitrary complex parameter.*

Then

$$\sum_{n=0}^{\infty} \zeta_{\mu+n}(x)P_{n,\nu}^q(y)t^n = f(x,t)\{g(x,t)\}^{-\mu}\Phi_{q,\nu}\left[h(x,t), yt^q\{g(x,t)\}^{-q}\right],$$

(3.18)

where $P_{n,\nu}^q(y)$ *is a polynomial of degree* $\left[\frac{n}{q}\right]$ *in* y*, which is defined as,*

$$P_{n,\nu}^q(y) = \sum_{r=0}^{\left[\frac{n}{q}\right]} A_{\nu+qr,n-qr}\ a_{\nu,n}\ y^r$$

In the generating relation (2.3), consider
$\mu = m, A_{\mu,n} = \binom{m+n}{n}, \zeta_n(x) = V_n^{(\alpha,\beta,\delta)}(x;a,k,s),$
$f(x,t) = (1-at)^{-\left(\frac{\beta+s}{a}\right)} \frac{W(\alpha,\delta;p_k(x))}{W\left(\alpha,\delta;p_k\{x(1-at)^{-\frac{1}{a}}\}\right)}, g(x,t) = 1, h(x,t)$
$= x(1-at)^{-\frac{1}{a}}$. *Then*

$$\sum_{n=0}^{\infty} V_{m+n}^{(\alpha,\beta,\delta)}(x;a,k,s) P_{n,\nu}^{q}(y) t^n \tag{3.19}$$
$$= (1-at)^{-\left(\frac{\beta+s}{a}\right)} \frac{W(\alpha,\delta;p_k(x))}{W\left(\alpha,\delta;p_k\{x(1-at)^{-\frac{1}{a}}\}\right)} \Phi_{q,\nu}\left[x(1-at)^{-\frac{1}{a}}, yt^q\right]$$

where,

$$\Phi_{q,\nu}(x,t) = \sum_{n=0}^{\infty} a_{\nu,n} V_{m+qn}^{(\alpha,\beta,\delta)}(x;a,k,s) t^n, \text{ with} a_{\nu,n} \neq 0, \tag{3.20}$$

and $P_{n,\nu}^{q}(y) = \sum_{r=0}^{\left[\frac{n}{q}\right]} \binom{\nu+n}{n-qr} a_{\nu,n}\, y^r$

is a polynomial of degree $\left[\frac{n}{q}\right]$ *in* y, q *is positive integer and* ν *is an arbitrary complex number.*

Theorem 3.6. *(Srivastava [9]) For the function* $\zeta_\mu(x)$ *defined by (3.16), let*

$$\Theta_{q,\mu}^{p,\nu}[x;y_1,...,y_l;t] = \sum_{n=0}^{\infty} C_n^{\mu,\nu} \zeta_{\mu+qn}(x) \Omega_{\nu+pn}(y_1,...,y_l) t^n, C_n^{\mu,\nu} \neq 0,$$

(3.21)

where μ *and* ν *are arbitrary complex numbers,* p *and* q *are positive integers, and* $\Omega_\nu(y_1,...,y_l)$ *is non-vanishing function of* l *variables* $y_1,...,y_l$, $l \geq 1$.

Then

$$\sum_{n=0}^{\infty} \zeta_{\mu+n}(x) Q_{n,q,\mu}^{p,\nu}(y_1,...,y_l;z) t^n \tag{3.22}$$
$$= f(x,t)\{g(x,t)\}^{-\mu} \Theta_{q,\mu}^{p,\nu}\left[h(x,t);y_1,...,y_l;z\left\{\frac{t}{g(x,t)}\right\}^q\right],$$

where $Q_{n,q,\mu}^{p,\nu}(y_1,...,y_l;z)$ *is a polynomial of degree* $\left[\frac{n}{q}\right]$ *in* z *(with coefficient's dependent on* $y_1,...,y_l$*) defined by*

$$Q_{n,q,\mu}^{p,\nu}(y_1,...,y_l;z) = \sum_{r=0}^{\left[\frac{n}{q}\right]} A_{\mu+qr,n-qr}\, C_r^{\mu,\nu}\Omega_{\nu+pr}(y_1,...,y_l)\, z^r \tag{3.23}$$

On considering

$$\Theta_{q,m}^{p,\nu}[x;y_1,...,y_l;t] = \sum_{n=0}^{\infty} C_n^{m,\nu} V_{m+qn}^{(\alpha,\beta,\delta)}(x;a,k,s)\Omega_{\nu+pn}(y_1,...,y_l)t^n, C_n^{m,\nu} \neq 0,$$

(3.24)

where ν *is arbitrary complex number,* p *and* q *are positive integers and* $\Omega_\nu(y_1,...,y_l)$ *is non-vanishing function of* l *variables* $y_1,...,y_l$, $l \geq 1$.

Taking $A_{m,n} = \binom{m+n}{n}, \zeta_n(x) = V_n^{(\alpha,\beta,\delta)}(x;a,k,s), f(x,t)$ $= (1-at)^{-\left(\frac{\beta+s}{a}\right)} \frac{W(\alpha,\delta;p_k(x))}{W\left(\alpha,\delta;p_k\{x(1-at)^{-\frac{1}{a}}\}\right)}, g(x,t) = 1, h(x,t) = x(1-at)^{-\frac{1}{a}}$ *in above theorem, and generating relation (2.3) yields bilateral generating relation as*

$$\sum_{n=0}^{\infty} V_{m+n}^{(\alpha,\beta,\delta)}(x;a,k,s) Q_{n,q,\mu}^{p,\nu}(y_1,...,y_l;z)t^n \tag{3.25}$$

$$= (1-at)^{-\left(\frac{\beta+s}{a}\right)} \frac{W\left(\alpha,\delta;p_k(x)\right)}{W\left(\alpha,\delta;p_k\{x(1-at)^{-\frac{1}{a}}\}\right)} \Theta_{q,\mu}^{p,\nu}\left[x(1-at)^{-\frac{1}{a}};y_1,...,y_l;zt^q\right]$$

where $Q_{n,q,m}^{p,\nu}(y_1,...,y_l;z)$ *is a polynomial of degree* $\left[\frac{n}{q}\right]$ *in* z *(with coefficient's dependent on* $y_1,...,y_l$*) defined by*

$$Q_{n,q,m}^{p,\nu}(y_1,...,y_l;z) = \sum_{r=0}^{\left[\frac{n}{q}\right]} \binom{m+n}{n-qr} C_r^{m,\nu}\Omega_{\nu+pr}(y_1,...,y_l)\, z^r \tag{3.26}$$

4. Generating Functions involving Stirling Number

Riordan [5], defined Stirling number of second kind as

$$S(n,m)=\frac{1}{m!}\sum_{j=1}^{m}(-1)^{m-j}\binom{m}{j}j^n \tag{4.1}$$

so that

$$S(n,1)=S(n,n)=1, S(n,n-1)=\binom{n}{2} and S(n,0)=\begin{cases}1, & n=0\\ 0 & n\in\mathbf{N}\end{cases}$$

The following theorem is useful to get some generating relation for $V_n^{(\alpha,\beta,\delta)}(x;a,k,s)$;

Theorem 4.1 (Srivastava [10]). *Let the sequence* $\{\zeta_n(x)\}_{n=0}^{\infty}$ *be generated by*

$$\sum_{m=0}^{\infty}\binom{n+m}{m}\zeta_{m+n}(x)t^m=f(x,t)\{g(x,t)\}^{-n}\zeta_n(h(x,t)) \tag{4.2}$$

where f,g *and* h *are suitable function of* x *and* t.

Then the following family of generating function

$$\sum_{m=0}^{\infty}m^n\zeta_m(h(x,-z))z^m\{g(x,-z)\}^{-m}=\{f(x,-z)\}^{-1}\sum_{m=0}^{n}m!S(n,m)\zeta_m(x)z^m \tag{4.3}$$

holds true provided that each member of (4.3) exists.

The generating functions (2.3) and (2.4) related to the family given by (4.2).

By comparing (2.3) and (4.2); $\zeta_m(x)=V_m^{(\alpha,\beta,\delta)}(x;a,k,s), f(x,t)=(1-ax^at)^{-\left(\frac{\beta+s}{a}\right)}\frac{W(\alpha,\delta;p_k(x))}{W(\alpha,\delta;p_k\{x(1-ax^at)^{-\frac{1}{a}}\})}, h(x,t)=x(1-ax^at)^{-\frac{1}{a}},$ *and* $g(x,t)=(1-ax^at)$, *we get*

$$\sum_{m=0}^{\infty}m^nV_m^{(\alpha,\beta,\delta)}(x(1+ax^az)^{-\frac{1}{a}};a,k,s)(1+ax^az)^{-m}z^m \tag{4.4}$$

$$= (1+ax^a z)^{\left(\frac{\beta+s}{a}\right)} \frac{W\left(\alpha,\delta;p_k\{x(1+ax^a z)^{-\frac{1}{a}}\}\right)}{W(\alpha,\delta;p_k(x))}$$

$$\sum_{m=0}^{n} m!S(n,m)V_m^{(\alpha,\beta,\delta)}(x;a,k,s)z^m$$

Replacing $x^a z$ *by* $\frac{z}{1-az}$ *and* x *by* $\frac{x}{(1-az)^{\frac{1}{a}}}$ *above equation gives*

$$\sum_{m=0}^{\infty} m^n V_m^{(\alpha,\beta,\delta)}(x;a,k,s)z^m = (1-az)^{\left(-\frac{\beta+s}{a}\right)} \frac{W(\alpha,\delta;p_k(x))}{W\left(\alpha,\delta;p_k\{x(1-az)^{-\frac{1}{a}}\}\right)}$$

(4.5)

$$\sum_{m=0}^{n} m!S(n,m)V_m^{(\alpha,\beta,\delta)}\left(x(1-az)^{-\frac{1}{a}};a,k,s\right)\left(\frac{z}{1-az}\right)^m$$

with $|t| < |a|^{-1}; a \neq 0.$

*Similarly, theorem 4.1, applied to generating relation (**??**) gives generating relation as*

$$\sum_{m=0}^{\infty} m^n V_m^{(\alpha,\beta-am,\delta)}(x(1-ax^a z)^{\frac{1}{a}};a,k,s)(1-ax^a z)^{-m}z^m \tag{4.6}$$

$$= (1-ax^a z)^{1-\left(\frac{\beta+s}{a}\right)} \frac{W\left(\alpha,\delta;p_k\{x(1-ax^a z)^{\frac{1}{a}}\}\right)}{W(\alpha,\delta;p_k(x))}$$

$$\sum_{m=0}^{n} m!S(n,m)V_m^{(\alpha,\beta-am,\delta)}(x;a,k,s)z^m$$

Replacing $x^a z$ *by* $\frac{z}{1+az}$ *and* x *by* $\frac{x}{(1+az)^{\frac{1}{a}}}$ *above equation gives*

$$\sum_{m=0}^{\infty} m^n V_m^{(\alpha,\beta-am,\delta)}(x;a,k,s)z^m = (1+az)^{\left(\frac{\beta+s}{a}\right)-1} \frac{W(\alpha,\delta;p_k(x))}{W\left(\alpha,\delta;p_k\{x(1+az)^{\frac{1}{a}}\}\right)}$$

(4.7)

$$\sum_{m=0}^{n} m!S(n,m)V_m^{(\alpha,\beta-am,\delta)}\left(x(1+az)^{\frac{1}{a}};a,k,s\right)\left(\frac{z}{1+az}\right)^m$$

with $|t| < |a|^{-1}; a \neq 0.$

5. Finite Summation Formulae

Finite summation formulae for (1.2) obtained as follows,

$$(5.1)\, V_n^{(\alpha,\sigma,\delta)}(x;a,k,s) = \sum_{m=0}^{n} \frac{(ax^a)^m}{m!}\left(\frac{\sigma}{a}\right)_m (ax^a)^m V_{n-m}^{(\alpha,\beta,\delta)}(x;a,k,s-\beta)$$

$$(5.2)\; V_n^{(\alpha,\beta+\mu+1,\delta)}(x;a,k,s) = \sum_{m=0}^{n} \frac{(ax^a)^m}{m!}\left(\frac{\mu+1}{a}\right)_m V_{n-m}^{(\alpha,\gamma,\delta)}(x;a,k,s)$$

Proof. *From (1.2) and (1.5) it follows that,*

$$V_n^{(\alpha,\beta,\delta)}(x;a,k,s) = \frac{1}{n!}W(\alpha,\delta;p_k(x))(\theta+\beta x^a)^n\left\{W(\alpha,\delta;-p_k(x))\right\}$$

replacing s *by* $s-\beta$*, above equation can be written as*

$$(5.3) \quad \theta^n\left\{W(\alpha,\delta;-p_k(x))\right\} = \frac{n!}{W(\alpha,\delta;p_k(x))}V_n^{(\alpha,\beta,\delta)}(x;a,k,s-\beta)$$

employing (1.4) in (1.2) gives

$$(5.4) \qquad V_n^{(\alpha,\sigma,\delta)}(x;a,k,s) =$$
$$\frac{x^{-\sigma+1}}{n!}W(\alpha,\delta;p_k(x))\sum_{m=0}^{n}\binom{n}{m}\theta_1^m\{x^{\sigma-1}\}\theta^{n-m}\left\{W(\alpha,\delta;-p_k(x))\right\}$$

*Using (5.3), simplification of (5.4) reduces to (**??**).* □

Proof. *Using (1.2), equation (1.4) can be written as*

$$(5.5) \qquad V_n^{(\alpha,\beta+\mu+1,\delta)}(x;a,k,s)$$
$$= \frac{x^{-(\beta+\mu)}}{n!}W(\alpha,\delta;p_k(x))\sum_{m=0}^{n}\binom{n}{m}\theta_1^m\{x^{\mu}\}\theta^{n-m}\left\{x^{\beta}W(\alpha,\delta;-p_k(x))\right\}$$

this leads to

$$V_n^{(\alpha,\beta+\mu+1,\delta)}(x;a,k,s) = \sum_{m=0}^{n}\left(\frac{\mu+1}{a}\right)_m V_{(n-m)}^{(\alpha,\beta,\delta)}(x;a,k,s)\frac{(ax^a)^m}{m!} \tag{5.6}$$

□

6. Explicit representation of $V_n^{(\alpha,\beta,\delta)}(x;a,k,s)$

Now consider $p_k(x) = x^k$ in (1.1), reduces as

$$V_n^{(\alpha,\beta,\delta)}(x;a,k,s) = \tfrac{1}{n!}x^{-\beta}W(\alpha,\delta;x^k)\theta^n\left[x^\beta W(\alpha,\delta;-x^k)\right]$$

this can be written as

$$V_n^{(\alpha,\beta,\delta)}(x;a,k,s) = \tfrac{a^n}{n!}W(\alpha,\delta;x^k)\sum_{r=0}^{\infty}\tfrac{(-1)^r}{r!\Gamma(\alpha r+\delta)}\left(\tfrac{s+\beta+kr}{a}\right)_n x^{kr+an} \tag{6.1}$$

Further simplification of above expression leads to

$$V_n^{(\alpha,\beta,\delta)}(x;a,k,s) = \frac{a^n}{n!}\sum_{m=0}^{\infty}\sum_{r=0}^{\infty}\frac{(-1)^r\left(\frac{s+\beta+kr}{a}\right)_n}{m!r!\Gamma(\alpha m+\delta)\Gamma(\alpha r+\delta)}x^{kr+km+an}$$

Using, Series Manipulation technique, we get

$$V_n^{(\alpha,\beta,\delta)}(x;a,k,s) = \frac{a^n x^{an}}{n!}\sum_{m=0}^{\infty}\sum_{r=0}^{m}\frac{(-1)^{m-r}\left(\frac{s+\beta+kr}{a}\right)_n}{(m-r)!r!\Gamma(\alpha m-\alpha r+\delta)\Gamma(\alpha r+\delta)}x^{km}$$

This can be expressed as

$$V_n^{(\alpha,\beta,\delta)}(x;a,k,s) = \frac{a^n x^{an}}{n!}\sum_{m=0}^{\infty}C_{m,n}^{(\alpha,\beta,\delta,k,a,s)}\frac{x^{km}}{m!} \tag{6.2}$$

where

$$C_{m,n}^{(\alpha,\beta,\delta,k,a,s)} = \sum_{r=0}^{m}(-1)^{m-r}\binom{m}{r}\left(\frac{s+\beta+kr}{a}\right)_n\frac{1}{\Gamma(\alpha m-\alpha r+\delta)\Gamma(\alpha r+\delta)} \tag{6.3}$$

Equation (6.1) and (6.2) are explicit representation of $V_n^{(\alpha,\beta,\delta)}(x;a,k,s)$.

6.1. Particular cases:

Setting particular values $\alpha = 0, \delta = 1$, Wright function reduces to exponential function
and

$$C_{m,n}^{(0,\beta,1,k,a,s)} = \sum_{r=0}^{m} (-1)^{m-r} \binom{m}{r} \left(\frac{s+\beta+kr}{a} \right)_n$$

which is m^{th} difference of a polynomial of degree n, which vanishes whenever m exceeds n.

Consider Δ_k is forward difference with common difference k in argument define as

(6.4) $$\Delta_k f(z) = f(z+k) - f(z)$$

hence

(6.5) $$\Delta_k^m f(z) = \sum_{r=0}^{m} (-1)^{m-r} \binom{m}{r} f(z+kr)$$

and equation (6.2) can be written as

(6.6) $$V_n^{(0,\beta,1)}(x;a,k,s) = \frac{a^n x^{an}}{n!} \sum_{m=0}^{n} \frac{x^{km}}{m!} \Delta_k^m \left(\frac{s+\beta}{a} \right)_n$$

7. Integral Transforms on the sequence of functions

Equation (6.2) follows that

(7.1) $$\int_0^t x^{p-1}(t-x)^{q-1} V_n^{(\alpha,\beta,\delta)}(x;a,k,s)dx$$

$$= \frac{a^n}{n!} \sum_{m=0}^{\infty} \frac{C_{m,n}^{(\alpha,\beta,\delta,k,a,s)}}{m!} \int_0^t x^{km+an+p-1}(t-x)^{q-1}dx$$

Therefore,

(7.2) $$\int_0^t x^{p-1}(t-x)^{q-1} V_n^{(\alpha,\beta,\delta)}(x;a,k,s)dx$$

$$= \frac{a^n}{n!}\sum_{m=0}^{\infty}\frac{C_{m,n}^{(\alpha,\beta,\delta,k,a,s)}}{m!}\beta(km+an+p,q)t^{km+an+p+q-1}$$

Beta (Euler) Transform

Setting $t=1$ *in (7.2) gives Beta(Euler) transform of* $V_n^{(\alpha,\beta,\delta)}(x;a,k,s)$

$$B\left(V_n^{(\alpha,\beta,\delta)}(x;a,k,s);p,q\right) = \beta(p,q)\frac{a^n(p)_{an}}{n!(p+q)_{an}}\sum_{m=0}^{\infty}\frac{C_{m,n}^{(\alpha,\beta,\delta,k,a,s)}(an+p)_{km}}{m!(an+p+q)_{km}} \tag{7.3}$$

The following integral transforms follows from equation (6.2).

Finite Laplace Transform

$$L_T\left\{V_n^{(\alpha,\beta,\delta)}(x;a,k,s)\right\} = \frac{a^n}{n!}\sum_{m=0}^{\infty}\frac{C_{m,n}^{(\alpha,\beta,\delta,k,a,s)}}{m!}\frac{\gamma(an+km+1,pT)}{p^{km+an+1}} \tag{7.4}$$

Laplace Transform

$$L\left\{V_n^{(\alpha,\beta,\delta)}(x;a,k,s)\right\} = \frac{a^n}{n!}\sum_{m=0}^{\infty}\frac{C_{m,n}^{(\alpha,\beta,\delta,k,a,s)}}{m!}\frac{(an+1)_{km}(a)_n\Gamma a}{p^{km+an+1}} \tag{7.5}$$

Laguerre Transform

$$(7.6) \qquad Ł\left\{V_n^{(\alpha,\beta,\delta)}(x;a,k,s)\right\}$$

$$= \frac{a^n}{n!}\sum_{m=0}^{\infty}\frac{C_{m,n}^{(\alpha,\beta,\delta,k,a,s)}}{m!}\frac{\Gamma(km+an+\mu+1)\Gamma(r-km-an)}{r!\Gamma(-km-an)}$$

provided each gamma function exist.

Generalized Stieltjes Transform

$$(7.7) \qquad S_g\left\{V_n^{(\alpha,\beta,\delta)}(x;a,k,s)\right\}$$

$$= \frac{a^n x^{an+1-\rho}}{n!\Gamma(\rho)}\sum_{m=0}^{\infty}\frac{C_{m,n}^{(\alpha,\beta,\delta,k,a,s)}}{m!}\Gamma(km+an+1)\Gamma(\rho-km-an-1)x^{km}$$

where $Re(km+an+1)>0$, $|arg(x)|<\pi$.

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