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Uniform Convergence in β-Duals

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Abstract

Let E be a vector valued sequence space with β-dual $E^{\beta Y}$. We consider sufficient conditions on E for the series in a pointwise bounded subset of $E^{\beta Y}$ to be uniformly convergent over certain subsets of E. The conditions involve gliding hump assumptions on the multiplier space E. Applications to matrix mappings between vector valued sequence spaces are given.

In the papers [Sw6],[Sw7] gliding hump assumptions were used to establish uniform convergence results for pointwise bounded subsets of the β-dual of vector valued sequence spaces. Similar gliding hump properties were used in [Sw3] to establish Orlicz-Pettis Theorems for multiplier convergent series with respect to strong topologies on locally convex spaces and spaces of continuous linear operators. In [CL] Li and Chen used other assumptions on the space of multipliers to establish similar Orlicz-Pettis results for strong topologies. In this paper we show that the assumptions on the multiplier space employed in [CL] can be used to establish uniform convergence results for pointwise bounded subsets of β-duals similar to those given in [Sw6],[Sw7].

We begin by fixing the notation and terminology. Throughout X, Y will be Hausdorff locally convex spaces with $L(X, Y)$ the space of all continuous linear operators from X into Y. E will be a vector space of X-valued sequences which contains $c_{00}(X)$, the space of all X-valued sequences which are eventually 0. If x is any sequence (scalar or vector), the j^{th} coordinate of x is x_j so $x = \{x_j\}$; if $z \in X$ and $j \in \mathbf{N}$, the sequence with z in the j^{th} coordinate and 0 in the other coordinates will be denoted by $e^j \otimes z$. The β-dual of E with respect to Y, $E^{\beta Y}$, is defined to be

$$E^{\beta Y} = \{\{T_j\} \subset L(X, Y) : \sum_{j=1}^{\infty} T_j x_j \; converges \; for \; every \; x = \{x_j\} \in E\}.$$

If $T = \{T_j\} \in E^{\beta Y}$ and $x = \{x_j\} \in E$, we write

$$T \cdot x = \sum_{j=1}^{\infty} T_j x_j.$$

The weakest topology on E such that all of the linear maps $x \to T \cdot x$ from E into Y are continuous for all $T \in E^{\beta Y}$ will be denoted by $w(E, E^{\beta Y})$.

We next introduce the conditions which will be employed to obtain our main results. If $x = \{x_j\}$ is a sequence (scalar or vector), and $t = \{t_j\}$ is a scalar sequence, the coordinate product of x and t will be denoted by $tx = \{t_j x_j\}$. The space E is c_0-factorable if whenever $u \in E$, there exist $t \in c_0$ and $v \in E$ such that $u = tv$. For example, l^p $(0 < p < \infty)$, c_0 and cs are c_0-factorable; see [Sw4] for this and other examples. The space E is monotone if whenever $x \in E$ and $I \subset \mathbf{N}$, then $\chi_I x \in E$, where χ_I denotes the characteristic function of I. For example, l^p $(0 < p \leq \infty)$, c_0 are monotone.

We will also impose a gliding hump assumption in our main results. Let $\mathcal{F}$ be a family of subsets of E which contains the finite subsets. An interval in $\mathbf{N}$ is a set of the form $I = \{j \in \mathbf{N} : m \leq j \leq n\}$ for $m \leq n$, $m, n \in \mathbf{N}$. A sequence of intervals $\{I_j\}$ is increasing if $\max I_j < \min I_{j+1}$.

Definition 1. *The space E has the signed $\mathcal{F}$-gliding hump property (signed $\mathcal{F}$-GHP) if for every $F \in \mathcal{F}$ whenever $\{x^j\} \subset F$ and $\{I_j\}$ is an increasing sequence of intervals, there exist a sequence of signs $\{s_j\}$ and a subsequence $\{n_j\}$ such that the coordinate sum of the series $\sum_{j=1}^{\infty} s_j \chi_{I_{n_j}} x^{n_j}$ belongs to E. If all of the signs $\{s_j\}$ can be chosen equal to 1, then E has the $\mathcal{F}$-gliding hump property ($\mathcal{F}$-GHP).*

When $\mathcal{F}$ is the family of all finite subsets, the signed $\mathcal{F}$-GHP ($\mathcal{F}$-GHP) is the signed weak gliding hump property [signed-WGHP] (weak gliding hump property [WGHP]) which has been utilized in [St], [No]. When E has a vector topology and $\mathcal{F}$ is the family of all bounded subsets of E, the signed $\mathcal{F}$-GHP ($\mathcal{F}$-GHP) is the signed strong gliding hump property [signed-SGHP](strong gliding hump property [SGHP]) which has been utilized in [LS2]. Further examples can be found in [Sw5].

We record the preliminary results which will be used in the proofs of the main results. First, a matrix theorem of Antosik and Mikusinski.

Theorem 2. *(Antosik-Mikusinski) Let $\{x_{ij} : i, j \in \mathbf{N}\} \subset X$. If (1) $\lim_i x_{ij}$ exists for every $j \in \mathbf{N}$ and (2) for every increasing sequence of positive integers $\{m_j\}$ there is a subsequence $\{n_j\}$ of $\{m_j\}$ such that the series $\sum_{j=1}^{\infty} x_{in_j}$ converges and $\lim_i \sum_{j=1}^{\infty} x_{in_j}$ exists, then $\lim_i x_{ii} = 0$ (diagonal$[x_{ij}] = \{x_{ii}\} \to 0$).*

Stronger versions of the Antosik-Mikusinski Matrix Theorem can be found in [Sw2],[Sw4]. A matrix $[x_{ij}]$ satisfying the conditions (1) and (2) is called a $\mathcal{K}$-matrix.

Next, we state an interesting result of Li and Wang ([LW]) which is central to our proofs.

Lemma 3. *(Li/Wang) Let Z be a vector space and $K \subset Z$ a convex set which contains 0. If $x_1, ..., x_n \in K$ and $M > 0$ is such that*

$$M \sum_{j \in \Delta} x_j \in K \ for\ every\ \Delta \subset \{1, ..., n\},$$

then

$$\sum_{j=1}^{n} s_j x_j \in K \ for\ every\ 0 \leq s_j \leq M,\ j = 1, ..., n.$$

Finally, we need a lemma concerning uniform convergence.

Lemma 4. *Let $A \subset E^{\beta Y}$ and $B \subset E$. If the series $\sum_{j=1}^{\infty} T_j x_j$ do not converge uniformly for $T \in A$ and $x \in B$, then there exist a balanced neighborhood of 0,V, in Y, $\{T^k\} \subset A$, $\{x^k\} \subset B$ and an increasing sequence of intervals $\{I_k\}$ such that $\sum_{l \in I_k} T_l^k x_l^k \notin V$ for all k.*

For the proof, see [Sw6] Lemma3, [Sw4] Lemma 2.15.

Now we can establish our first main result.

Theorem 5. *Assume E is c_0-factorable, monotone and has signed $\mathcal{F}$-GHP. If $A \subset E^{\beta Y}$ is pointwise bounded and $B \in \mathcal{F}$, then the series $\sum_{j=1}^{\infty} T_j x_j$ converge uniformly for $T \in A, x \in B$.*

Proof. If the conclusion fails to hold, there exist a symmetric, convex neighborhood of 0 in Y, an increasing sequence of intervals $\{I_k\}$, $T^k \in A$ and $x^k \in B$ such that $\sum_{l \in I_k} T_l^k x_l^k \notin V$ for every $k \in \mathbf{N}$ (Lemma 4). Since E has signed $\mathcal{F}$-GHP, by passing to a subsequence if necessary, we may assume $x = \sum_{j=1}^{\infty} s_j \chi_{I_j} x^j \in E$ with signs s_j. Then $\sum_{l \in I_j} T_l^j x_l = s_j \sum_{l \in I_j} T_l^j x_l^j \notin V$. Since E is c_0-factorable, $x = tu$ with $t \in c_0$ and $u \in E$ and since E is monotone, we may assume $t \geq 0$. Set $r_j = \max\{t_l : l \in I_j\}$ so $r_j > 0$ and $r_j \to 0$. Then $\sum_{l \in I_j} t_l T_l^j u_l \notin V$. Lemma 3 implies there exists $\Delta_j \subset I_j$ such that

$$(\&) \quad r_j \sum_{l \in \Delta_j} T_l^j u_l \notin V.$$

Consider the matrix

$$M = [m_{ij}] = [r_i \sum_{l \in \Delta_j} T_l^i u_l] = [r_i T^i \cdot \chi_{\Delta_j} u].$$

We show M satisfies the conditions of the Antosik-Mikusinski Theorem 2. First, the columns of M converge to 0 since the $\{T^i\}$ are pointwise bounded and $r_i \to 0$. For condition (2) set $v = \sum_{j=1}^{\infty} \chi_{\Delta_j} u$ and note $v \in E$ since E is monotone. Then

$$\sum_{j=1}^{\infty} m_{ij} = \sum_{j=1}^{\infty} r_i T^i \cdot \chi_{\Delta_j} u = r_i T^i \cdot v \to 0$$

since $\{T^i\}$ is pointwise bounded and $r_i \to 0$. Since the same argument applies to any subsequence, by Theorem 2 the diagonal of M converges to 0. But, this contradicts (&).

A similar result was established in Theorem 4 of [Sw6] under the assumptions that the space E satisfies the infinite gliding hump property (∞-GHP) and has $\mathcal{F}$-GHP (see definition 2 of [Sw7] or [Sw4] for the definition of the infinite gliding hump property (∞-GHP) and examples of spaces which have this property). Li and Chen ([CL]) have given examples which show that the conditions, infinite gliding hump property and c_0-factorable/monotone are independent of one another. Example 5 of [Sw6] shows that the conclusion of Theorem 5 fails to hold if E does not have the infinite gliding hump property or is c_0-factorable/monotone.

Corollaries 8 and 10 of [Sw6] give characterizations of pointwise bounded subsets of $E^{\beta Y}$ and subsets of $E^{\beta Y}$ which are uniformly bounded on members of $\mathcal{F}$ under the assumption that E has the infinite gliding hump property and signed $\mathcal{F}$-GHP, i.e., a uniform boundedness principle for β-duals. From Theorem 5 these results likewise hold if E is c_0-factorable and monotone and give Uniform Boundedness results for such spaces.

We next establish the analogue of Theorem 5 of [Sw7] under the assumption that E is c_0-factorable and monotone.

A vector valued sequence space E is a K-space if E has a vector topology under which the coordinate maps $x = \{x_j\} \to x_j$ are continuous from E into X for every j.

Definition 6. *The K-space E has the zero gliding hump property (0-GHP) if for every null sequence $\{x^k\} \subset E$ and for every increasing sequence of intervals $\{I_k\}$, there is a subsequence $\{n_k\}$ such that the coordinate sum of the series $\sum_{j=1}^{\infty} \chi_{I_{n_j}} x^{n_j} \in E$.*

The notion of a space having 0-GHP was introduced by Lee Peng Yee.([LPY]). Examples of spaces with 0-GHP are given in Appendices B and C of [Sw4]. For example, if X is a normed space, then $c_0(X)$ and $l^p(X)$ $(0 < p \leq \infty)$ have 0-GHP.

We require a lemma analogous to Lemma 4.

Lemma 7. *Let E be a K-space with $A \subset E^{\beta Y}$ and $x^k \to 0$ in E. Assume*

$$(*)\ \textit{for every } x \in E, \textit{ the series } \sum_{j=1}^{\infty} T_j x_j \textit{ converge uniformly for } T \in A.$$

If the series $\sum_{j=1}^{\infty} T_j x_j^k$ do not converge uniformly for $T \in A$ and $k \in \mathbf{N}$, then there exist a symmetric neighborhood of 0,V, in Y, $T_k \in A$ and a subsequence $\{n_k\}$ and an increasing sequence of intervals $\{I_k\}$ such that $\sum_{l \in I_k} T_l^k x_l^{n_k} \notin V$.

See Lemma 3 of [Sw7] for a proof.

Concerning condition (*), we have

Proposition 8. *Assume that E is c_0-factorable and monotone. If $A \subset E^{\beta Y}$ is pointwise bounded on E, then condition $(*)$ holds.*

Proof. If the conclusion fails to hold, there exist a symmetric neighborhood , V, of 0 in Y, $T^k \in A$ and an increasing sequence of intervals $\{I_k\}$ such that

$$(\#)\ \sum_{l \in I_k} T_l^k x_l \notin V.$$

Since E is c_0-factorable, $x = tu$ with $t \in c_0, u \in E$ and since E is monotone we may assume $t \geq 0$. Then $\sum_{l \in I_k} t_l T_l^k u_l \notin V$. Put $r_k = \max\{t_l : l \in I_k\}$ and note $r_k > 0$ and $r_k \to 0$. By the Li/Wang Lemma 3 there exists $\Delta_k \subset I_k$ such that

$$(\&)\ r_k \sum_{l \in \Delta_k} T_l^k u_l \notin V.$$

Define the matrix M by

$$M = [m_{ij}] = [r_i \sum_{l \in \Delta_j} T_l^i u_l] = [r_i T^i \cdot \chi_{\Delta_j} u].$$

We claim that E is a $\mathcal{K}$-matrix so we can apply the Antosik-Mikusinski Matrix Theorem. First, the columns of M converge to 0 since $\{T^i \cdot x\}$ and $r_i \to 0$. Next, $v = \sum_{j=1}^{\infty} \chi_{\Delta_j} u \in E$ since E is monotone. Then

$$\sum_{j=1}^{\infty} m_{ij} = r_i \sum_{j=1}^{\infty} T^i \cdot \chi_{\Delta_j} u = r_i T^i \cdot v \to 0$$

since $\{T^i \cdot v\}$ is bounded and $r_i \to 0$. Since the same argument can be applied to any subsequence, M is a $\mathcal{K}$-matrix and by the Antosik-Mikusinski Matrix Theorem 2 the diagonal of M converges to 0. But, this contradicts (&).

It is shown in [Sw4],2.32 that condition (*) holds when E has ∞-GHP.

Theorem 9. *Assume E is c_0-factorable, monotone and has 0-GHP. If $A \subset E^{\beta Y}$ is pointwise bounded and $x^k \to 0$ in E, then the series $\sum_{j=1}^{\infty} T_j x_j$ converge uniformly for $T \in A, k \in \mathbf{N}$.*

Proof. Condition (*) in Lemma 7 is satisfied by Proposition 8. Let the notation be as in Lemma 7. The 0-GHP implies there exists a subsequence $\{m_k\}$ of $\{n_k\}$ and an increasing sequence of intervals $\{I_k\}$ such that $x = \sum_{k=1}^{\infty} \chi_{I_k} x^{m_k} \in E$. To avoid double subscripts assume $m_k = n_k$ so

$$\sum_{l \in I_k} T_l^k x_l^{n_k} = \sum_{l \in I_k} T_l^k x_l \notin V.$$

Since E is c_0-factorable, $x = tu$ with $t \in c_0$ and $u \in E$ and since E is monotone, we may assume $t \geq 0$. Hence, $\sum_{l \in I_k} t_l T_l^k u_l \notin V$. Put $r_k = \max\{t_l : l \in I_k\}$ so $r_k > 0$ and $r_k \to 0$. By Lemma 3 there exists $\Delta_k \subset I_k$ such that

$$(\#) \quad r_k \sum_{l \in \Delta_k} T_l^k u_l \notin V.$$

Set

$$M = [m_{ij}] = [r_i \sum_{l \in \Delta_j} T_l^i u_l].$$

We show M satisfies the conditions of the Antosik-Mikusinski Theorem 2. First the columns of M converge to 0 since $\{T^i\}$ is pointwise bounded and $r_i \to 0$. For condition (2) set $v = \sum_{j=1}^{\infty} \chi_{\Delta_j} u$ and note $v \in E$ since E is monotone. Then

$$\sum_{j=1}^{\infty} m_{ij} = \sum_{j=1}^{\infty} r_i T^i \cdot \chi_{\Delta_j} u = r_i T^i \cdot v \to 0$$

since $\{T^i\}$ is pointwise bounded and $r_i \to 0$. Since the same argument applies to any subsequence, by Theorem 2 the diagonal of M converges to 0. But, this contradicts (#).

A similar result was established in Theorem 5 of [Sw7] under the assumptions that the space E has the infinite gliding hump property and 0-GHP.

From Theorem 9 we can obtain a sequential equicontinuity result for pointwise bounded subsets of $E^{\beta Y}$. The pair (X, Y) has the sequential uniform boundedness property (SUB) if every pointwise bounded subset $B \subset L(X, Y)$ is sequentially equicontinuous. For example, if X is a complete metric linear space or a metrizable barrelled space, (X, Y) has SUB ([Ko]39.5(1), [Ro]2.2.1,[Wi]9.3.4).

Theorem 10. *Assume that E is c_0-factorable and monotone, has 0-GHP and (X, Y) has SUB. If $A \subset E^{\beta Y}$ is pointwise bounded on E and $x^k \to 0$ in E, then $T \cdot x^k \to 0$ uniformly for $T \in A$.*

Proof. This follows from Theorem 9 and Theorem 6 of [Sw5].

A sequential continuity result for the bilinear map

$$\Theta : E^{\beta Y} \times E \to Y,\ \Theta(T, x) = T \cdot x,$$

follows directly from Theorem 10 and the equicontinuity Theorem 7 of [Sw7]. Let $w(E^{\beta Y}, E)$ be the weakest topology on $E^{\beta Y}$ such that the linear maps $T \to T \cdot x$ from $E^{\beta Y}$ into Y are continuous for all $x \in E$ and let τ_E be the topology of E.

Proposition 11. *Assume that E is c_0-factorable and monotone, has 0-GHP [or has ∞-GHP and 0-GHP] and (X, Y) has SUB. The bilinear map Θ is $w(E^{\beta Y}, E) \times \tau_E$ sequentially continuous.*

We next indicate some applications of the uniform convergence results to matrix mappings from E into $l^\infty(Y)$, the space of all bounded Y valued sequences. We begin with some observations about matrix mappings between sequence spaces. Let $A = [A_{ij}]$ be an infinite matrix of linear operators with $A_{ij} \in L(X, Y)$. Let F be a vector space of Y valued sequences which contains $c_{00}(Y)$ and is a K-space. The matrix A maps E into F if the series $\sum_{j=1}^\infty A_{ij} x_j$ converge for every $x = \{x_j\} \in E$ and

$$Ax = \{\sum_{j=1}^\infty A_{ij} x_j\}_i \in F;$$

if A maps E into F, we write

$$A : E \to F.$$

Let A^i be the i^{th} row of A so $Ax = \{A^i \cdot x\}$ and $A^i \in E^{\beta Y}$ for each i. We consider maps associated with A. Define $\Theta_k : E \to F$ by

$$\Theta_k(x) = \{A^1 \cdot x, ..., A^k \cdot x, 0, 0, ...\}$$

and $F_n : E \to F$ by

$$F_n(x) = \{\sum_{j=1}^n A_{ij} x_j\}_i.$$

Henceforth, we assume E is a K-space and $A : E \to F$.

Lemma 12. *Assume E has the signed-SGHP. If $B \subset E$ is bounded, then for each i $\{\sum_{j=1}^\infty A_{ij} x_j : x \in B\}$ is bounded in Y.*

Proof. By the signed-SGHP, the series $\sum_{j=1}^{\infty} A_{ij}x_j$ converge uniformly for $x \in B$ ([Sw4]11.9). Let U be a balanced neighborhood of 0 in Y and let V be a balanced neighborhood such that $V + V \subset U$. There exists N such that $\sum_{j=N}^{\infty} A_{ij}x_j \in V$ for all $x \in B$. Since E is a K-space, $\{x_j : x \in B\}$ is bounded in X for every j so there exists $t > 1$ such that $\sum_{j=1}^{N-1} A_{ij}x_j \in tV$ for $x \in B$. Then

$$\sum_{j=1}^{\infty} A_{ij}x_j = \sum_{j=1}^{N-1} A_{ij}x_j + \sum_{j=N}^{\infty} A_{ij}x_j \in tV + V \subset tU$$

for $x \in B$

We say that F has the injection property I if the coordinate injections

$$I_j : Y \to F,\ y \to e^j \otimes y,$$

are bounded. Most familiar K-spaces have the injection property.

Corollary 13. *Assume E has signed SGHP and F has property I. Then $\Theta_k : E \to F$ is bounded.*

Proof. This follows directly from Lemma 12 and the definition of property I.

The space F is an AK-space if for each $y = \{y_j\} \in F$,

$$y = \sum_{j=1}^{\infty} e^j \otimes y_j,$$

where the series converges in F. For example, $c_{00}(Y)$, $c_0(Y)$, and $l^p(Y)$ $(0 < p < \infty)$ are AK-spaces.

Lemma 14. *If F is an AK-space, $\Theta_k \to A$ pointwise on E.*

We want to establish a result which assures the boundedness of the matrix mapping A. For this we will apply Lemma 12 and a Banach-Steinhaus Theorem for sequentially continuous linear operators. The most natural such result holds for $\mathcal{A}$ spaces; $\mathcal{A}$ spaces are the most natural domains for which the uniform boundedness conclusion in the Uniform Boundedness Principle holds (see [LS1] or 4.3.1 of [Sw2]). This result states that a pointwise bounded family of sequentially continuous linear operators from an $\mathcal{A}$ space into a topological vector space is uniformly bounded on bounded subsets of the $\mathcal{A}$ space. In particular, the limit of a pointwise convergent sequence of sequentially continuous linear operators from an $\mathcal{A}$ space into a topological vector space is bounded and is uniformly bounded on bounded subsets of the domain space; a Banach-Steinhaus result. We do not give the definition of an $\mathcal{A}$ space as this would be too great a diversion but, for example, any sequentially complete locally convex space is an $\mathcal{A}$ space (see [LS1] or [Sw2]3.3.8 and 3.4 for other examples). From Lemma 12 and this Banach-Steinhaus result we can obtain

Proposition 15. *Assume E is an $\mathcal{A}$ space with signed-SGHP and 0-GHP and F is an AK-space with property I. Then A is bounded and $A \cup \{\Theta_k\}$ is uniformly bounded on bounded subsets of E.*

Proof. By the 0-GHP each A^i is sequentially continuous ([Sw2]12.5.2) so the result follows from Lemma 14 and the description of the Banach-Steinhaus Theorem above.

A similar result can be obtained from Lemma 14 and the standard Banach-Steinhaus Theorem for continuous linear operators with domain a barrelled space.

Proposition 16. *Assume E has signed SGHP and is barrelled and bornological. Assume F is an AK-space which has property I. Then A is continuous and $A \cup \{\Theta_k\}$ is uniformly bounded on bounded subsets of E.*

Proof. Since E is bornological, each A^i is continuous by Lemma 12. Since E is barrelled, the result follows from the Banach-Steinhaus Theorem for barrelled spaces ([Wi]9.3.4,[Sw1]24.11) and Lemma 14.

Under the assumptions of Propositions 15 and 16 it follows that $L_s(E,F)$ is an AK-space, where $L_s(E,F)$ is $L(E,F)$ with the topology of pointwise convergence on E.

We next consider matrix maps $A : E \to l^\infty(Y)$, where $l^\infty(Y)$ is the vector space of all bounded Y valued sequences.

Lemma 17. *Assume $A : E \to l^\infty(Y)$. Then*

(*a*) *for every j, $\{A_{ij} : i \in \mathbf{N}\} \subset L(X,Y)$ is pointwise bounded on$X$.*

Proof. For every $z \in X$, $\{A_{ij}z : i \in \mathbf{N}\} = \{A^i \cdot (e^j \otimes z) : i \in \mathbf{N}\} \in l^\infty(Y)$.

Corollary 18. *If X is an $\mathcal{A}$ space, then F_n is bounded for each n.*

Proof. By the Uniform Boundedness Principle for $\mathcal{A}$ spaces (described above, [Sw2]4.3.1), $\{A_{ij} : i \in \mathbf{N}\}$ is uniformly bounded on bounded subsets of X.

We give necessary and sufficient conditions for a matrix A to map E into $l^\infty(Y)$.

Proposition 19. *Assume E is c_0-factorable/monotone or has ∞-GHP and $A : E \to l^\infty(Y)$. Then*

$$(b)\ for\ each\ x \in E\ the\ series\ \sum_{j=1}^{\infty} A_{ij}x_j\ converge\ uniformly\ for\ i \in \mathbf{N}.$$

Proof. $\{A^i : i \in \mathbf{N}\}$ is pointwise bounded on E so the result follows from Proposition 8 and 2.32 of [Sw4] (this theorem is stated for scalar multipliers but the proof is the same for vector valued multipliers).

Conditions (a) and (b) give necessary conditions for $A : E \to l^\infty(Y)$. We show they are sufficient.

Proposition 20. *If (a) and (b) hold, then $A : E \to l^\infty(Y)$.*

Proof. Let $x \in E$. Let U be a balanced neighborhood of 0 in Y and pick V to be a balanced neighborhood such that $V + V \subset U$. By (b) there exists N such that $\sum_{j=N}^{\infty} A_{ij}x_j \in V$. Condition (a) implies there exists $t > 1$ such that $\sum_{j=1}^{N-1} A_{ij}x_j \in tV$. Therefore,

$$\sum_{j=1}^{\infty} A_{ij}x_j = \sum_{j=N}^{\infty} A_{ij}x_j + \sum_{j=1}^{N-1} A_{ij}x_j \in V + tV \subset tU$$

and $\{A^i \cdot x : i \in \mathbf{N}\}$ is bounded.

Note that if for each $x \in E$ the series $\sum_{j=1}^{\infty} A_{ij}x_j$ converge uniformly for $i \in \mathbf{N}$, then $F_n \to A$ pointwise on E. Thus, from Proposition 8 and Theorem 2.32 of [Sw4], if $A : E \to l^\infty(Y)$, then $\{A^i\}$ is pointwise bounded on E and we have

Proposition 21. *If E is c_0-factorable/monotone or has ∞-GHP and $A : E \to l^\infty(Y)$, then $F_n \to A$ pointwise on E.*

We next consider conditions for A to map E to $c(Y)$, the space of Y valued convergent sequences.

Proof.

Proposition 22. *If $A : E \to c(Y)$, then*

$$(c)\ for\ each\ j \in \mathbf{N}, z \in X, \quad \lim_i A_{ij} z\ exists.$$

Proof. We have $A^i \cdot (e^j \otimes z) = A_{ij} z$.

Associated with condition (c) we have

$$(d)\ for\ each\ j \in \mathbf{N}, z \in X, \ \ \{A_{ij} z : i \in \mathbf{N}\}\ is\ Cauchy.$$

Lemma 23. *Conditions (b) and (d) imply that for each $x \in E$ the sequence $\{A^i \cdot x : i \in \mathbf{N}\}$ is Cauchy.*

Proof. Let U be a balanced neighborhood of 0 in Y and pick V to be a balanced neighborhood such that $V + V + V \subset U$. Condition (b) implies there exists N such that $\sum_{j=N}^{\infty} A_{ij} x_j \in V$ for all i. By condition (d) there exists k such that $p, q \geq k$ implies $\sum_{j=1}^{N-1} (A_{pj} - A_{qj}) x_j \in V$. If $p, q \geq k$, then

$$A^p \cdot x - A^q \cdot x = \sum_{j=1}^{N-1} (A_{pj} - A_{qj}) x_j + \sum_{j=N}^{\infty} A_{pj} x_j - \sum_{j=N}^{\infty} A_{qj} x_j \in V + V + V \subset U.$$

Lemma 23 gives

Corollary 24. *If Y is sequentially complete, then conditions (b) and (c) imply $A : E \to c(Y)$.*

Next, we consider boundedness and sequential continuity of matrix maps $A : E \to l^{\infty}(Y)$. The locally convex topology on $l^{\infty}(Y)$ is given by the semi-norms

$$p'(\{y_j\}) = \sup\{p(y_j) : j \in \mathbf{N}\},$$

where p runs through the family of all continuous semi-norms defining the topology of Y. We first establish necessary conditions.

Proposition 25. *If $A : E \to l^{\infty}(Y)$ is bounded and E has property I, then*

$$(i)\ for\ each\ j \in \mathbf{N}\ and\ B \subset X\ bounded,\ the\ set \{A_{ij} z : i \in \mathbf{N}, z \in B\}$$

is bounded.

Proof. $\{A_{ij}z : i \in \mathbf{N}, z \in B\} = \{A^i \cdot (e^j \otimes z) : i \in \mathbf{N}, z \in B\}$ is bounded.

Proposition 26. *Assume E is c_0-factorable/monotone or has ∞-GHP and has signed-SGHP and $A : E \to l^\infty(Y)$. Then*

(ii) *if $B \subset E$ is bounded, $\sum_{j=1}^{\infty} A_{ij}x_j$ converge uniformly for $i \in \mathbf{N}, x \in B$.*

Proof. This follows from Theorem 5 or Theorem 4 of [Sw6].

We now show conditions (i) and (ii) are sufficient for a matrix A to be a bounded map from E into $l^\infty(Y)$.

Proposition 27. *Conditions (i) and (ii) imply that $A : E \to l^\infty(Y)$ is bounded.*

Proof. Let $B \subset E$ be bounded and U a balanced neighborhood of 0 in Y. Pick a balanced neighborhood V such that $V + V \subset U$. Condition (ii) implies that there exists N such that $\sum_{j=N}^{\infty} A_{ij}x_j \in V$ for $i \in \mathbf{N}, x \in B$. Since E is a K-space, for each j, $\{x_j : x \in B\}$ is bounded so by condition (i) there exists $t > 1$ such that $\sum_{j=1}^{N-1} A_{ij}x_j \in tV$ for $i \in \mathbf{N}, x \in B$. Therefore, if $i \in \mathbf{N}$, then

$$A^i \cdot x = \sum_{j=1}^{N-1} A_{ij}x_j + \sum_{j=N}^{\infty} A_{ij}x_j \in tV + V \subset tU$$

so $\{A^i \cdot x : i \in \mathbf{N}, x \in B\}$ is bounded.

Corollary 28. *Assume E has signed-SGHP and is c_0-factorable/monotone or has ∞-GHP and $A : E \to l^\infty(Y)$. If X is an $\mathcal{A}$ space, then A is bounded.*

Proof. Lemma 17 implies (a) holds and then (i) holds by the $\mathcal{A}$ space assumption. Condition (ii) holds by Proposition 26. The result follows from Proposition 27.

Corollary 28 can be considered to be an automatic continuity/boundedness result in the sense that algebraic conditions imply a continuity/boundedness conclusion. Theorem 12.5.7 of [Sw2] also gives sufficient conditions for a matrix map $A : E \to l^\infty(Y)$ to be bounded; this result assumes E has 0-GHP and the pair (X, Y) has the uniform boundedness property. The results above connect uniform convergence of series with boundedness of the matrix maps.

We can also obtain an automatic sequential continuity result for matrix maps $A : E \to l^\infty(Y)$.

Proposition 29. *Assume E has 0-GHP and is c_0-factorable/monotone or has ∞-GHP, (X, Y) has SUB and $A : E \to l^\infty(Y)$. Then A is sequentially continuous.*

Proof. Let $x^k \to 0$ in E. Since $\{A^i\}$ is pointwise bounded on E, the series $\sum_{j=1}^\infty A_{ij}x_j$ converge uniformly for $i.k \in \mathbf{N}$ (Theorem 9 or Theorem 5 of [Sw7]). The result follows from Theorem 6 of [Sw7].

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