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On the instability of a kind of vector functional differential equations of the eighth order with multiple deviating arguments

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Abstract

In this paper, we investigate the instability of solutions to a certain class of nonlinear vector functional differential equations of the eighth order with n-deviating arguments. We employ the Lyapunov-Krasovskii functional approach and base on the Krasovskii criteria to prove two new theorems on the topic. Our results improve certain results in the literature from scalar functional differential equations to their vectorial forms.

Subjclass[2010] : *34K20.*

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1. Introduction

The instability analysis of differential equations of the eighth order has received considerable attention in the last two decades. In the literature, the Lyapunov technique has been utilized to study the instability of the solutions of differential equations of the eighth order (Bereketoğlu [2], Iyase [4], Tunç [6, 7, 8, 9, 10] and C. Tunç and E. Tunç [11]). Some respective contributions on the topic can be summarized as the following:

First, in 1991, using the Lyapunov technique, Bereketoğlu [2] established certain conditions to the instability of the zero solution of the eighth order scalar differential equation without delay

$$\begin{aligned}&x^{(8)}+a_2x^{(6)}+a_3x^{(5)}+a_4x^{(4)}+f_5(x,x',...,x^{(7)})x'''\\&+f_6(x')x''+f_7(x,x',...,x^{(7)})x'+f_8(x)=0.\end{aligned}$$

Later, in 1996, using the same method, Iyase [4] proved a theorem on the nonexistence of nontrivial periodic solutions to the nonlinear scalar differential equation of the eighth order without delay:

$$x^{(8)}+a_1x^{(7)}+a_2x^{(6)}+a_3x^{(5)}+a_4x^{(4)}+a_5x'''+f_6(x')x''+f_7(x)x'+f_8(x)=0.$$

Recently, Tunç [8, 9, 10] discussed the instability of the zero solution of the eighth order scalar nonlinear differential equations with delay

$$\begin{aligned}&x^{(8)}+a_2x^{(6)}+a_3x^{(5)}+a_4x^{(4)}+f_5(x,x(t-r),x',...,x^{(7)}(t-r))x'''\\&+f_6(x')x''+f_7(x,x(t-r),x',...,,x^{(7)}(t-r))x'+f_8(x(t-r))=0,\\&x^{(8)}+a_1x^{(7)}+a_2x^{(6)}+a_3x^{(5)}+a_4x^{(4)}+a_5x'''+f_6(x')x''\\&+f_7(x)x'+f_8(x,x(t-r),x',...,x^{(7)}(t-r))=0\end{aligned}$$

and

$$\begin{aligned}&x^{(8)}+a_2x^{(6)}+a_3x^{(5)}+a_4x^{(4)}+f_5(x,...,x(t-\tau_n),...,x^{(7)},...,x^{(7)}(t-\tau_n))x'''\\&+f_6(x')x''+f_7(x,...,x(t-\tau_n),...,x^{(7)},...,x^{(7)}(t-\tau_n))x'\\&+\sum_{i=1}^{n}h_i(x(t-\tau_i))=0,\end{aligned}$$

(1.1)

$$\begin{aligned} x^{(8)} \quad + \quad & a_1x^{(7)} + a_2x^{(6)} + a_3x^{(5)} + a_4x^{(4)} + a_5x''' + g_6(x')x'' \\ & + g_7(x, ..., x(t-\tau_n), x', ..., x'(t-\tau_n)) \\ & + g_8(x, ..., x(t-\tau_n), ..., x^{(7)}, ..., x^{(7)}(t-\tau_n)) = 0, \end{aligned}$$

(1.2)

respectively.

In this paper, we consider the eighth order nonlinear functional vector differential equations with n-deviating arguments, $\tau_i,\ (i = 1,\ 2, ..., n)$:

$$\begin{aligned} X^{(8)} \quad + \quad & A_2X^{(6)} + A_3X^{(5)} + A_4X^{(4)} \\ & + F_5(X, X(t-\tau_1), ..., X(t-\tau_n), ..., X^{(7)}, ..., X^{(7)}(t-\tau_n))X''' \\ & + F_6(X')X'' \\ & + F_7(X, X(t-\tau_1), ..., X(t-\tau_n), ..., X^{(7)}, ..., X^{(7)}(t-\tau_n))X' \\ & + \sum_{i=1}^{n} H_i(X(t-\tau_i)) = 0 \end{aligned}$$

(1.3)

and

$$\begin{aligned} X^{(8)} \quad + \quad & A_1X^{(7)} + A_2X^{(6)} + A_3X^{(5)} + A_4X^{(4)} + A_5X''' + G_6(X')X'' \\ + \quad & G_7(X, X(t-\tau_1), ..., X(t-\tau_n), ..., X', ..., X'(t-\tau_n)) \\ + \quad & G_8(X, X(t-\tau_1), ..., X(t-\tau_n), ..., X^{(7)}, ..., X^{(7)}(t-\tau_n)) = 0, \end{aligned}$$

(1.4)

respectively.

Let $X = X_1,\ X' = X_2,\ ...,\ X^{(7)} = X_8$. We can write Eq. (1.3) and Eq. (1.4) in the system form

$$
\begin{array}{l}
X_i' = X_{i+1}, (i = 1,\ 2, ..., 7),\\
X_8' = -A_2X_7 - A_3X_6 - A_4X_5\\
-F_5(X_1, ..., X_1(t-\tau_n), ..., X_8, ..., X_8(t-\tau_n))X_4\\
-F_6(X_2)X_3 - F_7(X_1, ..., X_1(t-\tau_n), ..., X_8, ..., X_8(t-\tau_n))X_2\\
-\sum\limits_{i=1}^{n} H_i(X_1) + \sum\limits_{i=1}^{n} \int\limits_{t-\tau_i}^{t} J_{H_i}(X_1(s))X_2(s)ds
\end{array}
$$

(1.5)

and

$$
\begin{array}{rcl}
X_i' & = & X_{i+1}, (i = 1,\ 2, ..., 7),\\
X_8' & = & -A_1X_8 - A_2X_7 - A_3X_6 - A_4X_5 - A_5X_4 - G_6(X_2)X_3\\
 & & -G_7(X_1, ..., X_1(t-\tau_n), X_2, ..., X_2(t-\tau_n))\\
 & & -G_8(X_1, ..., X_1(t-\tau_n), ..., X_8, ..., X_8(t-\tau_n)),
\end{array}
$$

(1.6)

respectively, where τ_i are certain positive constants, the fixed delays, $t - \tau_i \geq 0$, $A_1, ..., A_5$ are constant $n \times n-$ symmetric matrices, the primes in Eq. (1.3) and Eq. (1.4) denote differentiation with respect to t, $t \in \Re_+$, $\Re_+ = [0, \infty)$; F_5, F_6, F_7 and G_6 are continuous $n \times n-$ symmetric matrix functions for the arguments displayed explicitly, $H_i : \Re^n \to \Re^n$, $G_7 : \Re^{2n(n+1)} \to \Re^n$, $G_8 : \Re^{8n(n+1)} \to \Re^n$, $G_7(X_1, ..., X_1(t-\tau_n), 0, X_2(t-\tau_1), ..., X_2(t-\tau_n)) = 0$, $G_8(0, X_1(t-\tau_1), ..., X_8(t-\tau_n)) = 0$ and $H_i(0) = 0$, H_i, G_7 and G_8 are continuous for all of their respective arguments. The Jacobian matrices of $H_i(X)$ are given by

$$
J_{H_1}(X) = \left(\frac{\partial h_{1i}}{\partial x_j}\right), ...,\ J_{H_n}(X) = \left(\frac{\partial h_{ni}}{\partial x_j}\right)\ (i,\ j = 1,\ 2, ..., n),
$$

where $(x_1, ..., x_n)$ and $(h_{1i}), ..., (h_{ni})$ are the components of X and H_i, respectively. It is also assumed that the Jacobian matrices $J_{H_i}(X)$ exist and are continuous. The existence and uniqueness of the solutions of Eq. (1.3) and Eq. (1.4) are assumed (see È l'sgol'ts ([3], pp.14, 15). Throughout what follows $X_1(t), ..., X_8(t)$ are abbreviated as $X_1, ..., X_8$, respectively.

Consider the linear constant coefficient differential equation of the eighth order

$$x^{(8)} + a_1x^{(7)} + a_2x^{(6)} + a_3x^{(5)} + a_4x^{(4)} + a_5x''' + a_6x'' + a_7x' + a_8x = 0$$

(1.7)

and its auxiliary equation

$$\psi(\lambda) \equiv \lambda^8 + a_1\lambda^7 + ... + a_7\lambda + a_8 = 0. \tag{1.8}$$

If β is an arbitrary real number, then the real part of $\psi(i\beta)$ is given by $\phi(\beta) = \beta^8 - a_2\beta^6 + a_4\beta^4 - a_6\beta^2 + a_8$.

It is also known that if $a_2 \leq 0,\ a_4 \geq 0,\ a_6 \leq 0,\ a_8 > 0$
in which case $\phi(\beta) > 0$, then the auxiliary equation cannot have any purely imaginary root whatever. It therefore follows from general theory that Eq. (1.7) does not has a periodic solution except $x = 0$. An analogous consideration of the imaginary part of $\psi(i\beta)$ also leads to conditions on a_1, a_3, a_5 and a_7 for the nonexistence of any periodic solution of Eq. (1.7) other than $x = 0$.

Besides, if $a_1 = 0$ $a_8 \neq 0$,
then the sum of the roots of (1.8) equals zero and each of them is different from zero, respectively. Furthermore, a necessary and sufficient condition for (1.8), with $a_1 = 0$, to have a purely imaginary root $\lambda = i\beta$ (β real) is that the following two equations

$$a_3\beta^4 - a_5\beta^2 + a_7 = 0 \tag{1.9}$$

and

$$\beta^8 - a_2\beta^6 + a_4\beta^4 - a_6\beta^2 + a_8 = 0$$

are simultaneously satisfied. If $a_3 \neq 0$, the left hand side of Eq. (1.9) can be rewritten in the form
$a_3\left(\beta^2 - \frac{a_5}{2a_3}\right)^2 + a_7 - \frac{a_5^2}{4a_3}$.

Therefore, if

$$a_3 \neq 0,\ \left(a_7 - \frac{a_5^2}{4a_3}\right) sgn a_3 > 0, \tag{1.10}$$

then the estimate (1.9) cannot be satisfied, and Eq. (1.8) would not have any purely imaginary root whatever. Hence, Eq. (1.8) has at the least one root $\lambda_0 = \alpha_0 + i\beta_0$, ($\alpha_0$, β_0 real) with $\alpha_0 > 0$ provided that $a_1 = 0$, $a_8 \neq 0$ and (1.10) hold.

The aim of this paper is to give the extensions of these results to Eq. (1.3) and Eq. (1.4). We extend the results obtained in the scalar cases for Eq. (1.1) and Eq. (1.2) to vector delay differential equations, Eq. (1.3) and Eq. (1.4).

Let $r \geq 0$ be given, and let $C = C([-r, 0], \ \Re^n)$ with

$$\|\phi\| = \max_{-r \leq s \leq 0} |\phi(s)|\,, \quad \phi \in C.$$

For $H > 0$ define $C_H \subset C$ by

$$C_H = \{\phi \in C : \|\phi\| < H\}.$$

If $x : [-r, \ A) \to \Re^n$ is continuous, $0 < A \leq \infty$, then, for each t in $[0, A)$, x_t in C is defined by

$$x_t(s) = x(t + s), \ -r \leq s \leq 0, \ t \geq 0.$$

Let G be an open subset of C and consider the general autonomous differential system with finite delay

$$\dot{x} = F(x_t), \ F(0) = 0, \ x_t = x(t + \theta), \ -r \leq \theta \leq 0, t \geq 0,$$

where $F : G \to \Re^n$ is continuous and maps closed and bounded sets into bounded sets. It follows from these conditions on F that each initial value problem

$$\dot{x} = F(x_t), \ x_0 = \phi \in G$$

has a unique solution defined on some interval $[0, A)$, $0 < A \leq \infty$. This solution will be denoted by $x(\phi)(.)$ so that $x_0(\phi) = \phi$.

Definition 1.1. *The zero solution, $x = 0$, of $\dot{x} = F(x_t)$ is stable if for each $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon) > 0$ such that $\|\phi\| < \delta$ implies that $|x(\phi)(t)| < \varepsilon$ for all $t \geq 0$. The zero solution is said to be unstable if it is not stable.*

2. Main results

We need the following lemma.

Lemma 2.1. *(Bellman [1]) Let A be a real symmetric $n \times n-$ matrix and*

$$a' \geq \lambda_i(A) \geq a > 0,\ (i = 1,\ 2, ..., n),$$

where a' and a are constants, and $\lambda_i(A)$ are the eigenvalues of the matrix A. Then

$$a' \langle X, X\rangle \geq \langle AX, X\rangle \geq a \langle X, X\rangle$$

and

$$a'^2 \langle X, X\rangle \geq \langle AX, AX\rangle \geq a^2 \langle X, X\rangle .$$

The first main result is the following theorem.

Let $\tau = \max \tau_i,\ \ (i = 1, 2, ..., n)$.

Theorem 2.1. *Let all the assumptions imposed to $A_2,\ A_3,\ A_4, F_5, F_6, F_7$ and H_i and*

$$H_i(0) = 0, H_i(X_1) \neq 0,\ (X_1 \neq 0), |\lambda_i(J_{H_i}(X_1))| \leq \delta_i, \delta_i > 0, \lambda_i(A_3(.))) \geq a_3$$

and

$$\lambda_i(F_7(X_1, ..., X_8(t-\tau_n)))sgn a_3 - \tfrac{1}{4|a_3|}\lambda_i(F_5(X_1, ..., X_8(t-\tau_n)))^2 \geq \delta > 0$$

hold for arbitrary $X_1, ..., X_8(t-\tau_n)$, where $a_3(\neq 0)$, δ_i and δ are constants.

If $\tau < \frac{\delta}{\sqrt{n}(\delta_1+...+\delta_n)}$, then the zero solution of Eq. (1.3) is unstable.

Proof. Let the function $V(.) = V(X_1, ..., X_8)$ be defined by

$$\begin{aligned} V(.) \ \ = \ \ & - \langle X_2, X_8\rangle - \langle A_2X_2, X_6\rangle - \langle A_3X_2, X_5\rangle - \langle A_4X_2, X_4\rangle \\ & - \int_0^1 \langle H_1(\sigma X), X\rangle \, d\sigma - ... - \int_0^1 \langle H_n(\sigma X), X\rangle \, d\sigma \\ & + \langle X_3, X_7\rangle + \langle A_2X_3, X_5\rangle + \langle A_3X_3, X_4\rangle - \langle X_4, X_6\rangle \\ & + \tfrac{1}{2} \langle A_4X_3, X_3\rangle - \tfrac{1}{2} \langle A_2X_4, X_4\rangle + \tfrac{1}{2} \langle X_5, X_5\rangle \\ & - \int_0^1 \langle F_6(_2)X_2, X_2\rangle \, d\sigma. \end{aligned}$$

We define a Lyapunov functional $V_1(.) \equiv V_1(X_{1t}, ..., X_{8t})$ by

$$V_1(.) = V sgn a_3 - \sum_{i=1}^{n} \lambda_i \int_{-\tau_i}^{0} \int_{t+s}^{t} \|X_2(\theta)\|^2 \, d\theta ds,$$

where s is a real variable such that the integrals $\int\limits_{-\tau_i}^{0} \int\limits_{t+s}^{t} \|X_2(\theta)\|^2 \, d\theta ds$ are non-negative, and λ_i are certain positive constants to be determined later in the proof.

It follows that

$$V_1(0,0,0,0,0,0,0,0) = 0.$$

Let

$$\bar{\varepsilon} = (0, 0, (\varepsilon_{31}, ..., \varepsilon_{3n}) sgn a_3, 0, 0, 0, (1 + |a_4|)(\varepsilon_{71}, ..., \varepsilon_{7n}), 0).$$

Hence $$\begin{aligned} &\mathrm{V}_1(0, 0, (\varepsilon_{31}, ..., \varepsilon_{3n}) sgn a_3, 0, 0, 0, (1 + |a_4|)(\varepsilon_{71}, ..., \varepsilon_{7n}), 0) \\ &\geq \left(1 + |a_4| + \tfrac{1}{2} a_4 sgn a_3\right) \; \|\bar{\varepsilon}\|^2 > 0 \end{aligned}$$

for all $\bar{\varepsilon}$ $(\neq 0)$ so that every neighborhood of the origin in the $(X_1, ..., X_8)-$ space contains the points $(\xi_1, ..., \xi_8)$ such that $V_1(\xi_1, ..., \xi_8) > 0$.

Let ...

$$(X_1, ..., X_8) = (X_1(t), ..., X_8(t))$$

be an arbitrary solution of (1.5).

The time derivative of the Lyapunov functional $V_1(.)$ with respect to (1.5) results in

$$\begin{aligned}
\dot{V}_1(.) \quad = \quad & \{\langle A_3X_4, X_4\rangle + \langle F_5(.)X_2, X_4\rangle + \langle F_7(.)X_2, X_2\rangle\} sgna_3 \\
& - < sgna_3X_2, \int_{t-\tau_1}^{t} J_{H_1}(X_1(s))X_2(s)ds > \\
& -...- < sgna_3X_2, \int_{t-\tau_n}^{t} J_{H_n}(X_1(s))X_2(s)ds > \\
& - \langle \lambda_1\tau_1X_2, X_2\rangle - ... - \langle \lambda_n\tau_nX_2, X_2\rangle \\
& + \lambda_1 \int_{t-\tau_1}^{t} \|X_2(\theta)\|^2 d\theta + ... + \lambda_n \int_{t-\tau_n}^{t} \|X_2(\theta)\|^2 d\theta \\
& \geq |a_3| \left\| X_4 + 2^{-1}\left|a_3^{-1}\right| F_5(.)X_2 sgna_3 \right\|^2 \\
& + \langle sgna_3F_7(.)X_2, X_2\rangle - < 4^{-1}\left|a_3^{-1}\right| F_5(.)X_2, F_5(.)X_2 > \\
& - < sgna_3X_2, \int_{t-\tau_1}^{t} J_{H_1}(X_1(s))X_2(s)ds > \\
& -...- < sgna_3X_2, \int_{t-\tau_n}^{t} J_{H_n}(X_1(s))X_2(s)ds > \\
& - \langle \lambda_1\tau_1X_2, X_2\rangle - \langle \lambda_2\tau_2X_2, X_2\rangle - ... - \langle \lambda_n\tau_nX_2, X_2\rangle \\
& + \lambda_1 \int_{t-\tau_1}^{t} \|X_2(\theta)\|^2 d\theta + ... + \lambda_n \int_{t-\tau_n}^{t} \|X_2(\theta)\|^2 d\theta.
\end{aligned}$$

Using the assumptions of Theorem 2.1 and the Schwartz inequality, we have

$$\begin{aligned}
& - < sgna_3X_2, \int_{t-\tau_1}^{t} J_{H_1}(X_1(s))X_2(s)ds > \\
& \geq - \|X_2\| \left\| \int_{t-\tau_1}^{t} J_{H_1}(X_1(s))X_2(s)ds \right\| \\
& \geq -\sqrt{n}\delta_1 \|X_2\| \left\| \int_{t-\tau_1}^{t} X_2(s) \right\| ds \\
& \geq -\sqrt{n}\delta_1 \|X_2\| \int_{t-\tau_1}^{t} \|X_2(s)\| ds
\end{aligned}$$

$$\geq -\tfrac{1}{2}\sqrt{n}\delta_1\tau_1 \|X_2\|^2 \; - \tfrac{1}{2}\sqrt{n}\delta_1 \int_{t-\tau_1}^{t} \|X_2(s)\|^2 ds,$$

$$
\begin{aligned}
& - < sgna_3 X_2, \int_{t-\tau_2}^{t} J_{H_2}(X_1(s))X_2(s)ds > \\
& \geq - \|X_2\| \left\| \int_{t-\tau_2}^{t} J_{H_2}(X_1(s))X_2(s)ds \right\| \\
& \geq -\sqrt{n}\delta_2 \|X_2\| \left\| \int_{t-\tau_2}^{t} X_2(s) \right\| ds \\
& \geq -\sqrt{n}\delta_2 \|X_2\| \int_{t-\tau_2}^{t} \|X_2(s)\| ds \\
& \geq -\tfrac{1}{2}\sqrt{n}\delta_2\tau_2 \|X_2\|^2 \ - \tfrac{1}{2}\sqrt{n}\delta_2 \int_{t-\tau_2}^{t} \|X_2(s)\|^2 ds, ...,
\end{aligned}
$$

$$
\begin{aligned}
& - < sgna_3 X_2, \int_{t-\tau_n}^{t} J_{H_n}(X_1(s))X_2(s)ds > \\
& \geq - \|X_2\| \left\| \int_{t-\tau_n}^{t} J_{H_n}(X_1(s))X_2(s)ds \right\| \\
& \geq -\sqrt{n}\delta_n \|X_2\| \left\| \int_{t-\tau_n}^{t} X_2(s) \right\| ds \\
& \geq -\sqrt{n}\delta_n \|X_2\| \int_{t-\tau_n}^{t} \|X_2(s)\| ds \\
& \geq -\tfrac{1}{2}\sqrt{n}\delta_n\tau_n \|X_2\|^2 \ - \tfrac{1}{2}\sqrt{n}\delta_n \int_{t-\tau_n}^{t} \|X_2(s)\|^2 ds
\end{aligned}
$$

so that

$$
\begin{aligned}
\dot{V}_1(.) \ \geq \ & |a_3| \ \left\| X_4 + 2^{-1} \left| a_3^{-1} \right| F_5(.)X_2 sgna_3 \right\|^2 \\
& + \langle sgna_3 F_7(.)X_2, X_2 \rangle - < 4^{-1} \left| a_3^{-1} \right| F_5(.)X_2, F_5(.)X_2 > \\
& -(\lambda_1 + \tfrac{1}{2}\sqrt{n}\delta_1)\tau_1 \|X_2\|^2 - ... - (\lambda_n + \tfrac{1}{2}\sqrt{n}\delta_n)\tau_n \|X_2\|^2 \\
& + (\lambda_1 - \tfrac{1}{2}\sqrt{n}\delta_1) \int_{t-\tau_1}^{t} \|X_2(s)\|^2 ds + ... + (\lambda_n - \tfrac{1}{2}\sqrt{n}\delta_n) \\
& \int_{t-\tau_n}^{t} \|X_2(s)\|^2 ds.
\end{aligned}
$$

Let $\lambda_i = \frac{1}{2}\sqrt{n}\delta_i$ and $\tau = \max \tau_i, \ \ (i = 1, 2, ..., n)$. Hence, we obtain

$$
\begin{aligned}
\dot{V}_1(.) \ \geq \ & \langle sgna_3 F_7(.)X_2, X_2 \rangle - < 4^{-1} \left| a_3^{-1} \right| F_5(.)X_2, F_5(.)X_2 > \\
& -\sqrt{n}\delta_1\tau_1 \|X_2\|^2 - ... - \sqrt{n}\delta_n\tau_n \|X_2\|^2 \\
& \geq \{\delta - \sqrt{n}(\delta_1 + ... + \delta_n)\tau\} \|X_2\|^2 .
\end{aligned}
$$

If $\tau < \frac{\delta}{\sqrt{n}(\delta_1+...+\delta_n)}$, then, for some positive constant k, we have

$$\dot{V}_1(.) \geq k \|X_2\|^2 > 0.$$

Finally, $\dot{V}_1(.) = 0$ for all $t \geq 0$ necessarily implies that $X_2 = 0$. Hence, it follows that

$$X_1 = \xi_1 \text{ (constant vector)}, \quad X_2 = X' = 0, ..., X_8 = X^{(7)} = 0$$

for all $t \geq 0$ so that

$$\geq -\|X_2\| \left\| \int_{t-\tau_1}^{t} J_{H_1}(X_1(s))X_2(s)ds \right\|$$

Therefore, the estimates $\dot{V}_1(.) = 0$ and (1.5) imply $X_1 = X_2 = ... = X_8 = 0$, since $H_i(\xi_1) = 0$ if and only if $\xi_1 = 0$. It now follows that functional V_1 thus has all the requisite Krasovskii [5] properties subject to the conditions of Theorem 2.1. By this discussion, we can conclude that the zero solution of Eq. (1.3) is unstable.
The proof of Theorem 2.1 is completed. □

The second main result of this paper is given by the following theorem.

Theorem 2.2. *Let all the assumptions imposed to $A_1, ..., A_5$, G_6, G_7 and G_8 and the conditions*

$$\lambda_i(A_1) \geq a_1 > 0, \lambda_i(A_2) \leq a_2 < 0, \lambda_i(A_4) \geq a_4 > 0,$$
$$\lambda_i(G_6(X_2)) \leq 0, \langle G_7(X_1, ..., X_2(t-\tau_n)), X_1 \rangle \geq 0$$

and

$$\langle G_8(X_1, ..., X_8(t-\tau_n)), X_1 \rangle > 0, \ (X_1 \neq 0),$$

hold for arbitrary $X_1, X_1(t-\tau_1), ..., X_8(t-\tau_n)$, where a_1, a_2 and a_4 are certain constants.
Then, the zero solution of Eq. (1.4) is unstable.

Proof. Let the Lyapunov function $V_2(.) = V_2(X_1, ..., X_8)$ be defined by

$$\begin{aligned} V_2(.) \quad = \quad & -\langle X_1, X_8\rangle - \langle A_1X_1, X_7\rangle - \langle A_2X_1, X_6\rangle - \langle A_3X_1, X_5\rangle \\ & -\langle A_4X_1, X_4\rangle - \langle A_5X_1, X_3\rangle + \langle X_2, X_7\rangle + \langle A_1X_2, X_6\rangle \\ & +\langle A_2X_2, X_5\rangle + \langle A_3X_2, X_4\rangle + \langle A_4X_2, X_3\rangle + \tfrac{1}{2}\langle A_5X_2, X_2\rangle \\ & -\langle X_3, X_6\rangle - \langle A_1X_3, X_5\rangle - \langle A_2X_3, X_4\rangle - \tfrac{1}{2}\langle A_3X_3, X_3\rangle \\ & +\langle X_4, X_5\rangle + \tfrac{1}{2}\langle A_1X_4, X_4\rangle - < X_1, \int_0^1 G_6(\sigma X_2)X_2 d\sigma > . \end{aligned}$$

It follows that

$$V_2(0, 0, 0, 0, 0, 0, 0, 0) = 0.$$

Let

$$\varepsilon* = (\varepsilon_{41}, ..., \varepsilon_{4n}).$$

Then

$$V_2(0, 0, 0, \varepsilon*, 0, 0, 0, 0) = \frac{1}{2}a_1 \|\varepsilon*\|^2 > 0$$

for $\varepsilon*$ $(\neq 0)$ so that every neighborhood of the origin in the $(X_1, X_2, ..., X_8)-$space contains points $(\rho_1, \rho_2, ..., \rho_8)$ such that $V_2(\rho_1, \rho_2, ..., \rho_8) > 0$.

Let

$$(X_1, ..., X_8) = (X_1(t), ..., X_8(t))$$

be an arbitrary solution of (1.6).

By an elementary calculation along the solutions of (1.6), the time derivative of the function V_2 results in

$$\begin{aligned} \dot{V}_2(.) \quad = \quad & \langle X_5, X_5\rangle - \langle A_2X_4, X_4\rangle + \langle A_4X_3, X_3\rangle \\ & +\langle G_7(X_1, ..., X_8(t-\tau_n)), X_1\rangle \\ & +\langle G_8(X_1, ..., X_8(t-\tau_n)), X_1\rangle \\ & - < X_2, \int_0^1 G_6(\sigma X_2)X_2 d\sigma > . \end{aligned}$$

Using the assumptions of Theorem 2.2, $\lambda_i(A_2) \leq a_2 < 0$, $\lambda_i(A_4) \geq a_4 > 0$, $\lambda_i(G_6(x_2)) \leq 0$, $X_1G_7(.) \geq 0$ and $X_1G_8(.) > 0$, $(X_1 \neq 0)$, it follows that

$$\begin{aligned} \dot{V}_2(.) \quad \geq \quad & \|X_5\|^2 - a_2\|X_4\|^2 + a_4\|X_3\|^2 \\ & +\langle G_7(X_1, ..., X_8(t-r)), X_1\rangle > \\ & +\langle G_8(X_1, ..., X_8(t-r)), X_1\rangle \\ & - < X_2, \int_0^1 G_6(\sigma X_2)X_2 d\sigma > 0. \end{aligned}$$

Thus, if the assumptions of Theorem 2.2 hold, then $\dot{V}_2(.)$ is positive semi definite.

On the other hand, $\dot{V}_2 = 0$ for all $t \geq 0$ necessarily implies that $X_1 = 0$ and therefore also that

$$X_1 = X = 0, X_2 = X' = 0, ..., X_8 = X^{(7)} = 0$$

for all $t \geq 0$ so that

$$X_1 = X_2 = ... = X_8 = 0, (t \geq 0).$$

On the other hand, noting $\dot{V}_2(.) = 0$ and (1.6), we obtain $X_1 = X_2 = ... = X_8 = 0$ since $G_8(\rho_1, ..., \rho_8(t-\tau_n)) = 0$ and $G_7(\rho_1, ..., \rho_2(t-\tau_n)) = 0$, if and only if $\rho_1 = \rho_2 = 0$. It now follows that the Lyapunov function $V_2(.)$ thus satisfies all the requisite Krasovskii [5] properties subject to the conditions of Theorem 2. By the above discussion, we conclude that the zero solution of Eq. (1.6) is unstable.
The proof of Theorem 2 is completed. □

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