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Connected edge monophonic number of a graph

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Abstract

For a connected graph G of order n, a set S of vertices is called an edge monophonic set of G if every edge of G lies on a monophonic path joining some pair of vertices in S, and the edge monophonic number $m_e(G)$ is the minimum cardinality of an edge monophonic set. An edge monophonic set S of G is a connected edge monophonic set if the subgraph induced by S is connected, and the connected edge monophonic number $m_{ce}(G)$ is the minimum cardinality of a connected edge monophonic set of G. Graphs of order n with connected edge monophonic number 2, 3 or n are characterized. It is proved that there is no non-complete graph G of order $n \geq 3$ with $m_e(G) = 3$ and $m_{ce}(G) = 3$. It is shown that for integers k, l and n with $4 \leq k \leq l \leq n$, there exists a connected graph G of order n such that $m_e(G) = k$ and $m_{ce}(G) = l$. Also, for integers j, k and l with $4 \leq j \leq k \leq l$, there exists a connected graph G such that $m_e(G) = j, m_{ce}(G) = k$ and $g_{ce}(G) = l$, where $g_{ce}(G)$ is the connected edge geodetic number of a graph G.

Key Words: *Monophonic path, edge monophonic number, connected edge monophonic number, connected edge geodetic number.*

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1. Introduction

By a *graph* $G = (V, E)$ we mean a finite undirected connected graph without loops or multiple edges. The *order* and *size* of G are denoted by n and m respectively. For basic graph theoretic terminology we refer to [4]. For vertices x and y in a connected graph G, the *distance* $d(x, y)$ is the length of a shortest x-y path in G. It is known that the distance is a metric on the vertex set of G. An x-y path of length $d(x, y)$ is called an *x-y geodesic*. A vertex v is said to lie on an x-y geodesic P if v is a vertex of P including the vertices x and y. The *neighborhood* of a vertex v is the set $N(v)$ consisting of all vertices u which are adjacent with v. A vertex v is an *extreme vertex* of G if the subgraph induced by its neighbors is complete.

A vertex v is a *semi-extreme vertex* of G if the subgraph induced by its neighbors has a full degree vertex in $N(v)$. In particular, every extreme vertex is a semi-extreme vertex and a semi-extreme vertex need not an extreme vertex. For the graph G in Figure 2.1, v_1 and v_3 are an extreme vertices as well as semi-extreme vertices. Also v_2 is a semi-extreme vertex and not an extreme vertex of G.

A set S of vertices is a *geodetic set* of G if every vertex of G lies on a geodesic joining some pair of vertices in S, and the minimum cardinality of a geodetic set is the *geodetic number* $g(G)$. A geodetic set of cardinality $g(G)$ is called a g-set of G. The geodetic number of a graph was introduced in [1, 5] and further studied in [2, 3, 6]. It was shown in [5] that determining the geodetic number of a graph is an NP-hard problem. A set S of vertices in G is called an *edge geodetic set* of G if every edge of G lies on a geodesic joining some pair of vertices in S, and the minimum cardinality of an edge geodetic set is the *edge geodetic number* $g_e(G)$ of G. An edge geodetic set of cardinality $g_e(G)$ is called a g_e-set of G. An edge geodetic set S of G is called a connected edge geodetic set of G if the subgraph induced by S is connected, and the minimum cardinality of a connected edge geodetic set is the *connected edge geodetic number* $g_{ce}(G)$ of G. A connected edge geodetic set of cardinality $g_{ce}(G)$ is called a g_{ce}-set of G. The edge geodetic number of a graph was introduced and studied in [8, 9].

A *chord* of a path $u_1, u_2, \ldots, u_k$ in G is an edge u_iu_j with $j \geq i + 2$. A u-v path P is called a *monophonic path* if it is a chordless path. A set S of vertices is a *monophonic set* if every vertex of G lies on a monophonic path joining some pair of vertices in S, and the minimum cardinality of a monophonic set is the *monophonic number* $m(G)$ of G. A monophonic set of cardinality $m(G)$ is called an m-set of G. A set S of vertices in G is called an *edge monophonic set* of G if every edge of G lies on a monophonic path joining some pair of vertices in S, and the minimum cardinality of an edge monophonic set is the *edge monophonic number* $m_e(G)$ of G. An edge monophonic set of cardinality $m_e(G)$ is called an m_e-set of G. The edge monophonic number of a graph was introduced and studied in [7].

Theorem 1.1. [7] Every semi-extreme vertex of a connected graph G belongs to each edge monophonic set of G. In particular, if the set S of all semi-extreme vertices of G is an edge monophonic set of G, then S is the unique minimum edge monophonic set of G.

Theorem 1.2. [7] Let G be a connected graph with cut-vertices and S an edge monophonic set of G. If v is a cut-vertex of G, then every component of $G - v$ contains an element of S.

Theorem 1.3. [7] (1) For the complete graph K_n of order $n \geq 2$, $m_e(G) = n$.
(2) For any non-trivial tree T of order n with k endvertices, $m_e(T) = k$.
(3) For any wheel $W_n (n \geq 5)$ of order n, $m_e(W_n) = n - 1$.

2. Connected edge monophonic number of a graph

Definition 2.1. Let G be a connected graph with at least two vertices. A *connected edge monophonic set* of G is an edge monophonic set S such that the subgraph induced by S is connected. The minimum cardinality of a connected edge monophonic set of G is the *connected edge monophonic number* of G and is denoted by $m_{ce}(G)$. A connected edge monophonic set of cardinality $m_{ce}(G)$ is called an m_{ce}-set of G.

Example 2.2. For the graph G given in Figure 2.1, it is easily seen that no 4-element subset of vertices is an edge monophonic set. It is clear that $S = \{v_1, v_2, v_3, v_6, v_8\}$ is an edge monophonic set of G so that $m_e(G) = 5$. It is easily seen that no 6-element subset of vertices is a connected edge monophonic set of G. Since $S = \{v_1, v_2, v_3, v_4, v_5, v_6, v_8\}$ is a connected edge monophonic set of G, we have $m_{ce}(G) = 7$.

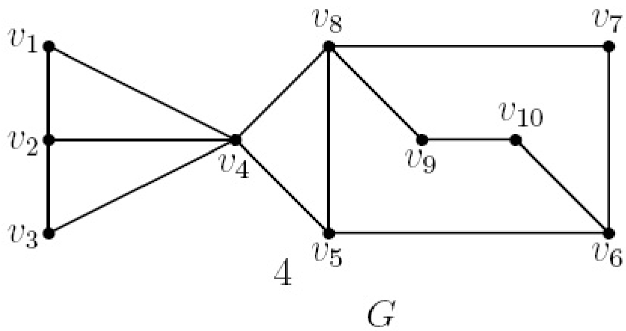


Figure 2.1

Note that $S_1 = \{v_1, v_2, v_3, v_4, v_5, v_6, v_{10}\}$ is also a minimum connected edge monophonic set of G. Thus the edge monophonic number and connected edge monophonic number are different.

Theorem 2.3. For any connected graph G of order n, $2 \leq m_e(G) \leq m_{ce}(G) \leq n$.

Proof. An edge monophonic set needs at least two vertices and so $m_e(G) \geq 2$. Since every connected edge monophonic set is also an edge monophonic set, it follows that $m_e(G) \leq m_{ce}(G)$. Also, since the set of all vertices of G is a connected edge monophonic set of G, $m_{ce}(G) \leq n$. □

We observe that for the complete graph K_2, $m_{ce}(K_2) = m_e(K_2) = 2$ and for the complete graph $K_n (n \geq 3)$, $m_{ce}(G) = m_e(G) = n$. Also, all the inequalities in Theorem 2.3 are strict. For the graph G given in Figure 2.1, $m_e(G) = 5, m_{ce}(G) = 7$ and $n = 10$.

Theorem 2.4. Every semi-extreme vertex of a connected graph G belongs to each connected edge monophonic set of G. In particular, if the set S of all semi-extreme vertices of G is a connected edge monophonic set of G, then S is the unique minimum connected edge monophonic set of G.

Proof. Since every connected edge monophoic set is also an edge monophonic set, the result follows from Theorem 1.1. □

Corollary 2.5. For any connected graph G of order n with k semi-extreme vertices, $max\{2, k\} \leq m_{ce}(G) \leq n$.

Proof. This follows from Theorems 2.3 and 2.4. □

Corollary 2.6. For the complete graph K_n $(n \geq 2)$, $m_{ce}(K_n) = n$.

The converse of Corollary 2.6 need not be true. For the graph G given in Figure 2.2, each vertex is a semi-extreme and $S = \{v_1, v_2, v_3, v_4, v_5, v_6\}$ is an m_{ce}-set of G. Therefore, $m_{ce}(G) = 6$ and G is not a complete graph.

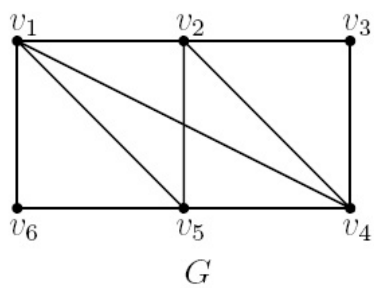


Figure 2.2

Since every connected edge monophonic set is an edge monophonic set, the next theorem follows from Theorem 1.2.

Theorem 2.7. Let G be a connected graph with cut-vertices and S a connected edge monophonic set of G. If v is a cut-vertex of G, then every component of $G - v$ contains an element of S.

Theorem 2.8. Every cut-vertex of a connected graph G belongs to every connected edge monophonic set of G.

Proof. Let G be a connected graph and S a connected edge monophonic set of G. Let v be a cut-vertex of G and $G_1, G_2, \ldots, G_r (r \geq 2)$ the components of $G - v$. By Theorem 2.7, S contains at least one vertex from each $G_i (1 \leq i \leq r)$. Since the subgraph induced by S is connected, it follows that $v \in S$. □

Combining Theorems 2.4 and 2.8, we have the following theorem.

Theorem 2.9. Every semi-extreme vertex and every cut-vertex of a connected graph G belong to each connected edge monophonic set of G.

Since every connected edge geodetic set is a connected edge monophonic set, we have the following theorem.

Theorem 2.10. Every semi-extreme vertex and every cut-vertex of a connected graph G belong to each connected edge geodetic set of G.

Corollary 2.11. For any connected graph G of order n with k semi-extreme vertices and l cut-vertices, $max\{2, k+l\} \leq m_{ce}(G) \leq n$.

Proof. This follows from Theorems 2.3 and 2.9 □

Corollary 2.12. For any tree T of order n, $m_{ce}(T) = n$.

Proof. This follows from Corollary 2.11. □

Theorem 2.13. For the complete bipartite graph $G = K_{r,s}$ $(2 \leq r \leq s)$, $m_{ce}(G) = r + 1$.

Proof. Let $U = \{u_1, u_2, \ldots, u_r\}$ and $W = \{w_1, w_2, \ldots, w_s\}$ be the partite set of G. Let $S = U \cup \{w_1\}$. We prove that S is a minimum connected edge monophonic set of G. Note that any u -v monophonic path in G is of length at most 2. Every edge u_iw_j $(1 \leq i \leq r, 1 \leq j \leq s)$ lies on the monophonic path u_i, w_j, u_k for any $k \neq i$, and so S is a connected edge monophonic set of G. Let T be any set of vertices such that $|T| < |S|$. If $T \subseteq U$ or $T \subseteq W$, then T cannot be a connected edge monophonic set of G. If T is such that T contains vertices from U and W such that $u_i \notin T$ and $w_j \notin T$. Then clearly the edge u_iw_j does not lie on a monophonic path joining two vertices of T so that T is not a connected edge monophonic set. Thus in any case T is not a connected edge monophonic set of G. Hence S is a minimum connected edge monophonic set so that $m_{ce}(G) = |S| = r+1$. □

Theorem 2.14. For any cycle C_n $(n \geq 3)$, $m_{ce}(C_n) = 3$.

Proof. Let $C_n : u_1, u_2, \ldots, u_n, u_1$ be a cycle of order $n \geq 3$. It is clear that no 2-element subset of vertices is a connected edge monophonic set of G. Now, $S = \{u_1, u_2, u_3\}$ is a connected edge monophonic set of G so that $m_{ce}(G) = 3$. □

Theorem 2.15. For any connected graph G of order n, $m_{ce}(G) = 2$ if and only if $G = K_2$.

Proof. If $G = K_2$, then $m_{ce}(G) = 2$. Conversely, let $m_{ce}(G) = 2$. Let $S = \{u, v\}$ be a minimum connected edge monophonic set of G. Then uv is an edge. If $G \neq K_2$, then there exists an edge xy different from uv, and the edge xy does not lie on any u-v monophonic path so that S is not a connected edge monophonic set, which is a contradiction. Thus $G = K_2$. □

A vertex v in graph G is called an *independent vertex* if the subgraph induced by its neighbors contains no edges.

Theorem 2.16. Let G be a non-complete connected graph of order $n \geq 3$. Then $m_{ce}(G) = 3$ if and only if there exist two independent vertices u and w such that $d(u, w) = 2$ and every edge of G lies on a u-w monophonic path.

Proof. Let $m_{ce}(G) = 3$ and let $S = \{u, v, w\}$ be a connected edge monophonic set of G. If the subgraph induced by S is complete, then $G \cong K_3$, which is a contradiction. So assume that u and w are non-adjacent in G. It is clear that $d(u, w) = 2$. Now, we show that u and w are independent vertices of G. Suppose that u is not an independent vertex of G. Then there exist vertices $u_1, u_2 \in N(u)$ such that u_1 and u_2 are adjacent in G. Since S is a connected edge monophonic set, u_1u_2 lies on a u-w monophonic path P. Since u_1 and u_2 are adjacent to u in G, it follows that P is not a monophonic path, which is a contradiction. Thus, u is an independent vertex of G. Similarly, w is an independent vertex of G. Now, since S is a connected edge monophonic set and since the subgraph induced by S is the path $P : u, v, w$, it follows that every edge of G lies on a u-w monophonic path. Conversely, let u and w be two independent vertices of G such that $d(u, w) = 2$ and every edge of G lies on a u-w monophonic path. Since G is non-complete, no 2-element subset of G is a connected edge monophonic set of G. Now, let $P : u, v, w$ be a u-w monophonic path. Then $S = \{u, v, w\}$ is a minimum connected edge monophonic set of G so that $m_{ce}(G) = 3$. □

Corollary 2.17. If G is a non-complete graph of order $n \geq 3$ with $m_{ce}(G) = 3$, then $m_e(G) = 2$.

Corollary 2.18. There is no non-complete graph G of order $n \geq 3$ with $m_e(G) = 3$ and $m_{ce}(G) = 3$.

Corollary 2.19. Let G be any connected graph of order $n \geq 3$. Then $m_e(G) = m_{ce}(G) = 3$ if and only if $G = K_3$.

Theorem 2.20. Let G be a connected graph of order $n \geq 3$. If G contains exactly one vertex v of degree $n-1$, then every vertex of G other than v is a semi-extreme vertex.

Proof. Let v be the unique vertex of degree $n-1$ and let $w \neq v$ be any vertex in G. Then v is adjacent to all the neighbors of w so that $deg_{<N(w)>}(v) = |N(w)| - 1$. Hence w is a semi-extreme vertex of G. □

Theorem 2.21. Let G be a connected graph of order $n \geq 3$. If G contains exactly one vertex v of degree $n-1$ and v is not a cut-vertex of G, then $m_{ce}(G) = n-1$. In fact, $S = V - \{v\}$ is the unique minimum connected edge monophonic set of G.

Proof. Let v be the unique vertex of degree $n-1$. Let $S = V - \{v\}$. Then by Theorem 2.20, S is the set of semi-extreme vertices of G. By Theorem 2.4, every connected edge monophonic set contains S and so $m_{ce}(G) \geq n-1$. Let uv be any edge incident with v. Since v is the only vertex of degree $n-1$, there exists at least one vertex $w \in N(v)$ such that u and w are non-adjacent. Then uv lies on the monophonic path $P : u, v, w$ with $u, w \in S$. Also, any edge xy not incident with v lies on the x-y monophonic path itself. Thus S is an edge monophonic set of G. Since v is not a cut-vertex of G, the subgraph induced by S is connected so that $m_{ce}(G) \leq n-1$. Hence $m_{ce}(G) = n-1$. □

Corollary 2.22. For the wheel $W_n = K_1 + C_{n-1}$ $(n \geq 5)$, $m_{ce}(W_n) = n-1$.

The converse of Theorem 2.21 is not true. For the graph G given in Figure 2.3, $S = \{u_1, u_2, u_3, u_4, u_5\}$ is a minimum connected edge monophonic set of G. Therefore, $m_{ce}(G) = 5 = n-1$ and no vertex has degree $n-1$.

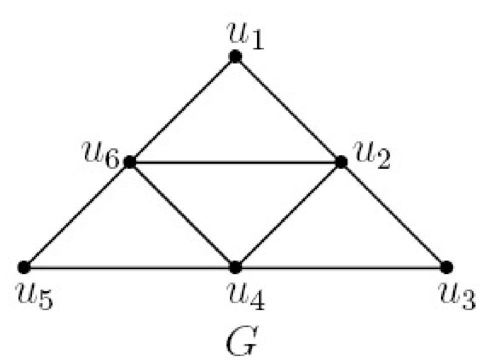


Figure 2.3

Problem 2.23. Characterize graphs G of order n for which $m_{ce}(G) = n - 1$.

Theorem 2.24. Let G be a connected graph of order $n \geq 3$. If G has a cut-vertex of degree $n-1$, then $m_{ce}(G) = n$.

Proof. This follows from Theorems 2.20 and 2.9. □

The converse of Theorem 2.24 is not true. For the graph G given in Figure 2.4, $S = \{v_1, v_3, v_4, v_5, v_6\}$ is the set of all semi-extreme vertices of G and v_2 is the only cut-vertex of G. Hence by Theorem 2.9, $m_{ce}(G) = 6 = n$. However, G has no cut-vertex of degree $n-1$.

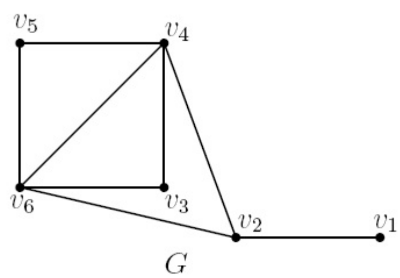


Figure 2.4

Theorem 2.25. Let G be a connected graph of order n. If G has at least two vertices of degree $n-1$, then every vertex of G is a semi-extreme vertex of G.

Proof. Let $u_1, u_2, \ldots, u_l (l \geq 2)$ be the vertices of degree $n-1$. Then each $u_i (1 \leq i \leq l)$ is adjacent to all other vertices in G. Let u be any vertex in G. If $u \neq u_i (1 \leq i \leq l)$, then u is adjacent to each u_i so that $deg_{<N(u)>}(u_i) = |N(u)| - 1$. Hence u is a semi-extreme vertex of G. If $u = u_i (1 \leq i \leq l)$, then u_i is adjacent to u_j, $j \neq i$, and so again $deg_{<N(u)>}(u_j) = |N(u)| - 1$. Hence u is a semi-extreme vertex of G. Thus every vertex of G is a semi-extreme vertex. □

Theorem 2.26. For any graph G of order n with at least two vertices of degree $n-1$, $m_{ce}(G) = n$.

Proof. This follows from Theorems 2.4 and 2.25. □

The converse of Theorem 2.26 is not true. For the graph G given in Figure 2.2, $S = \{v_1, v_2, v_3, v_4, v_5, v_6\}$ is a minimum connected edge monophonic set of G so that $m_{ce}(G) = 6 = n$ and G has no vertex of degree $n - 1$.

Theorem 2.27. Let G be a connected graph of order n. Then $m_{ce}(G) = n$ if and only if every vertex of G is either a cut-vertex or a semi-extreme vertex.

Proof. Let $m_{ce}(G) = n$. Then $S = V$ is the only connected edge monophonic set of G. Suppose that there exists a vertex u such that u is neither a semi-extreme vertex nor a cut-vertex of G. Since u is not a semi-extreme vertex, for each $v \in N(u)$, there exists a vertex $w \in N(u)$ such that $w \neq v$ and vw is not an edge of G. Now, we show that $S = V - \{u\}$ is an edge monophonic set of G. Let vu be any edge of G. Then vu lies on the monophonic path $P : v, u, w$. Also, any edge xy not incident with u lies on the x-y monophonic path itself. Hence S is an edge monophonic set of G. Since u is not a cut-vertex of G, the subgraph induced by S is connected. Therefore, S is a connected edge monophonic set of G so that $m_{ce}(G) \leq n - 1$, which is a contradiction. Hence every vertex of G is either a semi-extreme vertex or a cut-vertex. The converse follows from Theorem 2.9. □

Corollary 2.28. For the graph $G = K_1 + \cup m_j K_j$, where $\sum m_j \geq 2$, of order $n \geq 3$, $m_{ce}(G) = n$.

3. Realisation Results

In view of Corollary 2.18, we have the following realization result.

Theorem 3.1. For integers k, l and n with $4 \leq k \leq l \leq n$, there exists a connected graph G of order n such that $m_e(G) = k$ and $m_{ce}(G) = l$.

Proof. Case 1. $4 \leq k = l = n$. Then, for the compete graph $G = K_n$ of order n, by Theorem 1.3 and Corollary 2.6, we have $m_e(G) = m_{ce}(G) = n$.
Case 2. $4 \leq k < l = n$. Let G be a tree of order n with k endvertices. Then by Theorem 1.3 and Corollary 2.12, $m_e(G) = k$ and $m_{ce}(G) = n = l$.
Case 3. $4 \leq k < l < n$. Let $P_{l-k+3} : u_1, u_2, \ldots, u_{l-k+3}$ be a path of order $l - k + 3$. Let H be the graph formed by taking $n - l + 1$ new vertices $w_1, w_2, \ldots, w_{n-l+1}$, and joining each $w_i (1 \leq i \leq n - l + 1)$ with the vertices u_1 and u_3 in P_{l-k+3}; and also joining the vertex w_1 to the vertex u_2 in

P_{l-k+3}. Now, let G be the graph obtained from H by adding $k-4$ new vertices $y_1, y_2, \ldots, y_{k-4}$ and joining each $y_i (1 \leq i \leq k-4)$ with the vertices u_2 and w_1 in H. The graph G has order n and is shown in Figure 3.1.

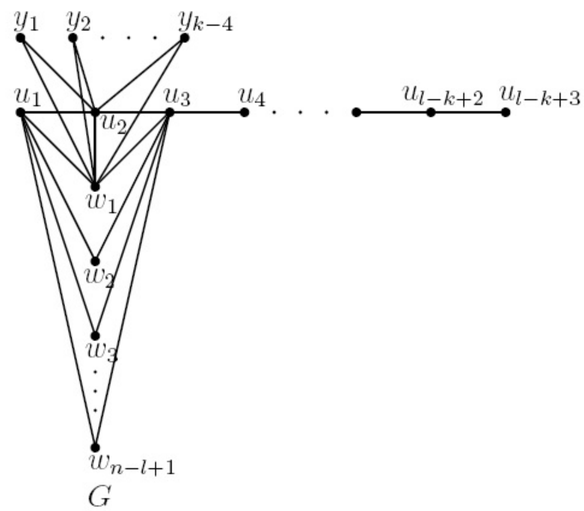


Figure 3.1

Let $S = \{w_1, y_1, y_2, \ldots, y_{k-4}, u_2, u_{l-k+3}\}$ be the set of semi-extreme vertices of G. Then S is not an edge monophonic set of G. Since $S \cup \{u_1\}$ is an edge monophonic set of G, by Theorem 1.1, $m_e(G) = k$. Now, $T = S \cup \{u_3, u_4, \ldots, u_{l-k+2}\}$ is the set of semi-extreme vertices and cut-vertices of G. It is clear that T is not a connected edge monophonic set of G. Since $T \cup \{u_1\}$ is a connected edge monophonic set of G, by Theorem 2.9, $m_{ce}(G) = l$.

Case 4. $4 \leq k = l < n$. First let $n = k+1$. Let G be a wheel with $k+1$ vertices. Then by Theorem 1.3 and Corollary 2.22, we have $m_e(G) = m_{ce}(G) = k$. Next if $n > k+1$, then we construct a graph G as follows : Let $P_3 : u_1, u_2, u_3$ be a path of order 3. Let H be the graph formed by taking $n-k+1$ new vertices $w_1, w_2, \ldots, w_{n-k+1}$, and joining each $w_i (1 \leq i \leq n-k+1)$ with the vertices u_1 and u_3 in P_3; and also joining the vertex w_1 to the vertex u_2 in P_3. Now, let G be the graph obtained from H by adding $k-4$ new vertices $y_1, y_2, \ldots, y_{k-4}$ and joining each $y_i (1 \leq i \leq k-4)$ with the vertices u_2 and w_1 in H. The graph G has order n and is shown in Figure 3.2. Let $S = \{w_1, y_1, y_2, \ldots, y_{k-4}, u_2\}$ be the set of semi-extreme vertices of G. Then for any vertex y in G, $S \cup \{y\}$ is not an edge monophonic set of G. Since $S \cup \{u_1, u_3\}$ is an edge monophonic set as well as connected edge monophonic set of G, it follows from Theorems 1.1 and 2.4 that $m_e(G) = m_{ce}(G) = k$. □

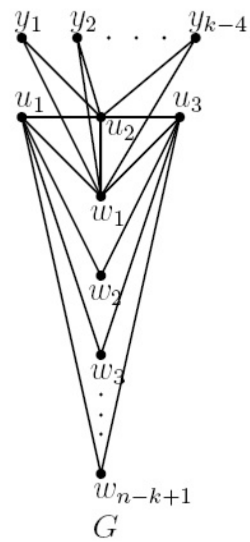


Figure 3.2

Theorem 3.2. For integers k, l and n with $k < l \leq n$ and $k = 2$, 3, there exists a connected graph G of order n such that $m_e(G) = k$ and $m_{ce}(G) = l$.

Proof. Case 1. $k = 2$. If $l = n$, then let G be a path P_n of order n. Hence by Theorems 1.3 and 2.9, $m_e(G) = 2, m_{ce}(G) = n = l$. If $l < n$, then we construct a graph G as follows: Let $P_l : u_1, u_2, \ldots, u_l$ be a path of order l. Let G be the graph obtained from P_l by adding $n - l$ new vertices $w_1, w_2, \ldots, w_{n-l}$ and joining each $w_i (1 \leq i \leq n - l)$ with u_1 and u_3 in P_l. The graph G has order n and is shown in Figure 3.3. If $l > 3$, then u_l is the only semi-extreme vertex of G. Therefore, u_l belongs to every edge monophonic set of G. Since $S = \{u_1, u_l\}$ is an edge monophonic set of G, it follows from Theorem 1.1 that $m_e(G) = 2 = k$. Let $S_1 = \{u_3, u_4, \ldots, u_{l-1}, u_l\}$ be the set of semi-extreme vertices and cut-vertices of G. Then, for any vertex $y \notin S_1$, $S_1 \cup \{y\}$ is not a connected edge monophonic set of G. It is clear that $S_1 \cup \{u_1, u_2\}$ is a connected edge monophonic set of G so that, by Theorem 2.9, $m_{ce}(G) = l$. If $l = 3$, then $\{u_1, u_3\}$ is an edge monophonic set of G so that $m_e(G) = 2 = k$. It is clear that no 2-element subset of vertices is a connected edge monophonic set of G. Since $\{u_1, u_2, u_3\}$ is a connected edge monophonic set, it follows that $m_{ce}(G) = 3 = l$.

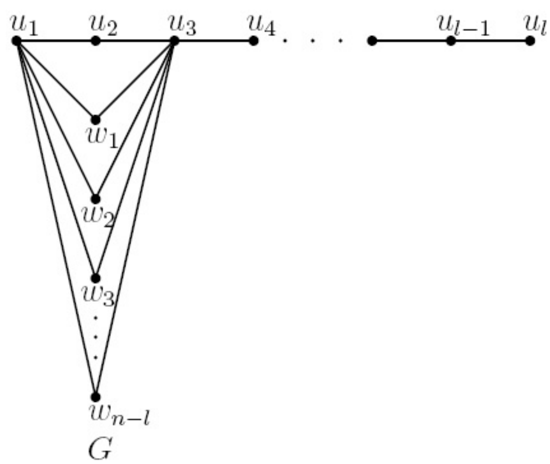


Figure 3.3

Case 2. $k = 3$. If $l = n$, then let G be a tree of order n with three endvertices. Then, by Theorems 1.3 and 2.9, $m_e(G) = 3 = k$ and $m_{ce}(G) = n = l$. If $l < n$, then we construct a graph G as follows: Let H be a graph obtained from the cycle $C_4 : v_1, v_2, v_3, v_4, v_1$ of order 4 and the path $P_{l-3} : u_1, u_2, \ldots, u_{l-3}$ of order $l - 3 \geq 1$ by joining u_1 in P_{l-3} with each v_1, v_2 and v_3 in C_4. Let G be the graph obtained from H by adding $n-l-1$ new vertices $w_1, w_2, \ldots, w_{n-l-1}$ and joining each $w_i (1 \leq i \leq n-l-1)$ with u_1 and v_4 in H. The graph G has order n and is shown in Figure 3.4.

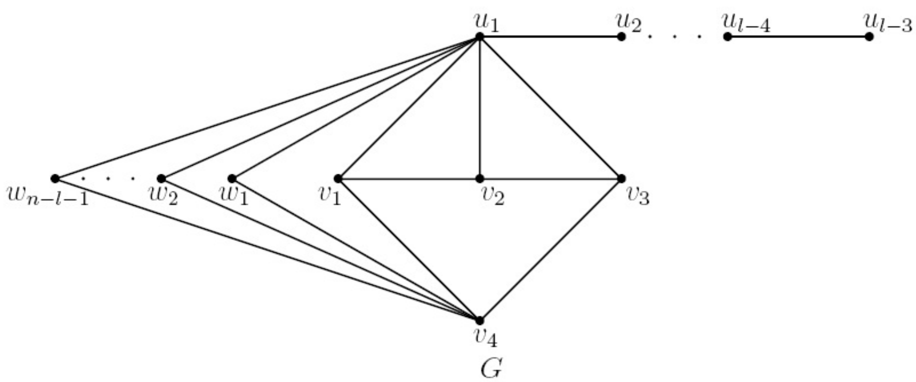


Figure 3.4

If $l > 4$, then $S_1 = \{v_2, u_{l-3}\}$ is the set of semi-extreme vertices of G. It is clear that $S_1 \cup \{v_4\}$ is an edge monophonic set so that by Theorem 1.1, $m_e(G) = 3 = k$. Let $S_2 = S_1 \cup \{u_1, u_2, \ldots, u_{l-4}\}$ be the set of semi-extreme vertices and cut-vertices of G. Then for any vertex $y \notin S_2$, $S_2 \cup \{y\}$ is not a connected edge monophonic set of G. It is easily seen that $S_2 \cup \{v_1, v_4\}$ is a connected edge monophonic set of G and so by Theorem 2.9, $m_{ce}(G) = l$. If $l = 4$, then $S_3 = \{v_2, u_1\}$ is the set of semi-extreme vertices of G. Since $S_3 \cup \{v_4\}$ is an edge monophonic set of G, by Theorem 1.1, $m_e(G) = 3 = k$. It is also easily verified that no 3-element subset of vertices is a connected edge monophonic set of G. Since $S_3 \cup \{v_1, v_4\}$ is a connected edge monophonic set of G, by Theorem 2.4, $m_{ce}(G) = 4 = l$. □

Theorem 3.3. If j, k and l are integers such that $4 \leq j \leq k \leq l$, then there exists a connected graph G with $m_e(G) = j, m_{ce}(G) = k$ and $g_{ce}(G) = l$.

Proof. Case 1. $4 \leq j = k = l$. Let $G = K_j$ be the complete graph of order j. Then by Theorems 1.3, 2.6 and 2.10, $m_e(G) = m_{ce}(G) = g_{ce}(G) = j$.
Case 2. $4 \leq j < k = l$. Let G be a tree of order k with j endvertices. Then by Theorem 1.3, $m_e(G) = j$ and by Theorems 2.12 and 2.10, $m_{ce}(G) = g_{ce}(G) = k$.

Case 3. $4 \leq j < k < l$. Let $P_{k-j+3} : u_1, u_2, \ldots, u_{k-j+3}$ be a path of order $k-j+3$. Take $(l-k)$ copies of K_2 with vertex set $F_i = \{v_i, w_i\}(1 \leq i \leq l-k)$; and also take $j-3$ new vertices $y_1, y_2, \ldots, y_{j-3}$. Let G be the graph obtained by joining u_1 with each $v_i(1 \leq i \leq l-k)$; u_3 with each $w_i(1 \leq i \leq l-k)$; each $y_i(2 \leq i \leq j-3)$ with u_2 and y_1; and also y_1 with u_1, u_2, u_3 in P_{k-j+3}. The graph G is shown in Figure 3.5.

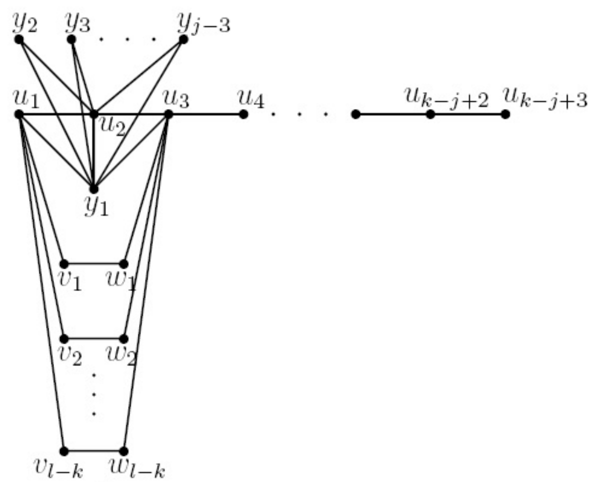
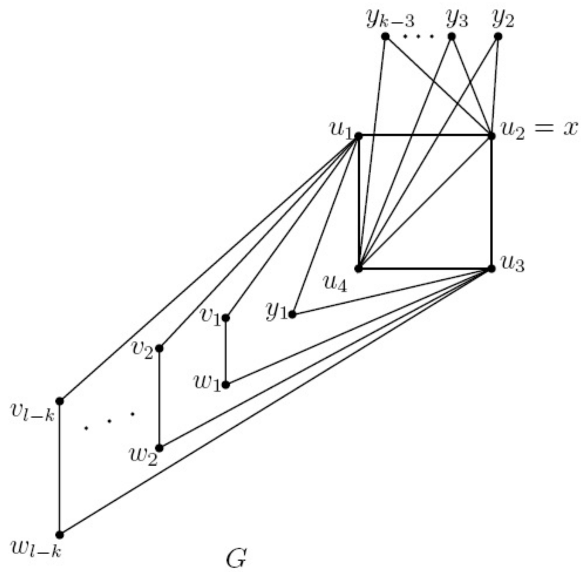


Figure 3.5

Let $S = \{u_2, u_{k-j+3}, y_1, y_2, \ldots, y_{j-3}\}$ be the set of semi-extreme vertices of G. Since S is not an edge monophonic set of G and since $S \cup \{u_1\}$ is an edge monophonic set of G, by Theorem 1.1, we have $m_e(G) = j$. Let $T = S \cup \{u_3, u_4, \ldots, u_{k-j+2}\}$ be the set of semi-extreme vertices and cut-vertices of G. It is clear that T is not a connected edge monophonic set of G. Since $T \cup \{u_1\}$ is a connected edge monophonic set of G, by Theorem 2.9, $m_{ce}(G) = k$. Also, it is easily seen that T is not a connected edge geodetic set of G. Now, we observe that at least one of v_i and $w_i(1 \leq i \leq l-k)$ must belong to every connected edge geodetic set of G. Then, by Theorem 2.10, $T' = T \cup \{u_1, v_1, v_2, \ldots, v_{l-k}\}$ is a minimum connected edge geodetic set of G so that $g_{ce}(G) = l$.

Case 4. $4 \leq j = k < l$. Let $C_4 : u_1, u_2, u_3, u_4, u_1$ be a cycle of order 4. Let H be the graph obtained from the cycle C_4 by taking $k-3$ new vertices $y_1, y_2, \ldots, y_{k-3}$ and joining y_1 with u_1 and u_3, and joining each

$y_i(2 \leq i \leq k-3)$ with both u_2 and u_4, and also joining u_2 and u_4. Now, let G be the graph obtained from H by taking $(l-k)$ copies of K_2 with vertex set $F_i = \{v_i, w_i\}(1 \leq i \leq l-k)$ and joining u_1 with each $v_i(1 \leq i \leq l-k)$ in H and u_3 with each $w_i(1 \leq i \leq l-k)$ in H. The graph G is shown in Figure 3.6.



G

Figure 3.6

Let $S = \{u_2, u_4, y_2, y_3, \ldots, y_{k-3}\}$ be the set of semi-extreme vertices of G. It is clear that S is not an edge monophonic set of G. Also, for any $y \notin S$, $S \cup \{y\}$ is not an edge monophonic set of G. Since $S' = S \cup \{u_1, u_3\}$ is an edge monophonic set as well as a connected edge monophonic set, it follows from Theorems 1.1 and 2.4 that $m_e(G) = m_{ce}(G) = k$. Also, S is not a connected edge geodetic set of G. Now, we observe that at least one vertex of v_i and $w_i(1 \leq i \leq l-k)$ must belong to every connected edge geodetic set of G. Let $T = S \cup \{v_1, v_2, \ldots, v_{l-k}\}$. Then for any $y \notin T$, $T \cup \{y\}$ is not a connected edge geodetic set of G. It follows that $T \cup \{u_1, u_3\}$ is a minimum connected edge geodetic set of G so that, by Theorem 2.10, $g_{ce}(G) = l$. □

Theorem 3.4. For integers j, k and l with $j < k \leq l$ and $j = 2, 3$, there exists a connected graph G such that $m_e(G) = j, m_{ce}(G) = k$ and

$g_{ce}(G) = l$.

Proof. Case 1. $j = 2$. If $k = l$, then let G be a path P_l of order l. Then by Theorems 1.3, 2.9 and 2.10, $m_e(G) = 2, m_{ce}(G) = l$ and $g_{ce}(G) = l$. If $k < l$, then we construct a graph G as follows: Let $P_k : u_1, u_2, \ldots, u_k$ be a path of order k. Let G be the graph obtained by taking $(l-k)$ copies of K_2 with vertex set $F_i = \{v_i, w_i\}(1 \leq i \leq l-k)$ and joining u_1 with each $v_i(1 \leq i \leq l-k)$; and also joining u_3 with each $w_i(1 \leq i \leq l-k)$. The graph G is shown in Figure 3.7.

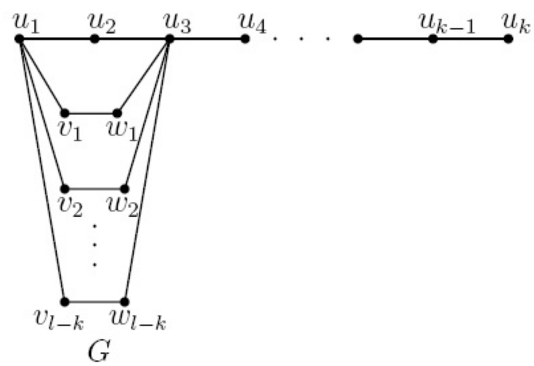


Figure 3.7

If $k > 3$, then u_k is the only semi-extreme vertex of G. Therefore, u_k belongs to every edge monophonic set of G. Since $S = \{u_1, u_k\}$ is an edge monophonic set of G, it follows from Theorem 1.1 that $m_e(G) = 2 = j$. Let $S_1 = \{u_3, u_4, \ldots, u_{k-1}, u_k\}$ be the set of semi-extreme vertices and cut-vertices of G. Then for any vertex $y \notin S_1$, $S_1 \cup \{y\}$ is not a connected edge monophonic set of G. It is clear that $S_1 \cup \{u_1, u_2\}$ is a connected edge monophonic set and so by Theorem 2.9, $m_{ce}(G) = k$. Also, it is clear that S_1 is not a connected edge geodetic set of G. Now, we observe that at least one of v_i and $w_i(1 \leq i \leq l-k)$ must belong to every connected edge geodetic set of G. Let $S_2 = S_1 \cup \{v_1, v_2, \ldots, v_{l-k}\}$. Then for any vertex $y \notin S_2$, $S_2 \cup \{y\}$ is not a connected edge geodetic set of G. Since $T = S_2 \cup \{u_1, u_2\}$ is a connected edge geodetic set of G, by Theorem 2.10, $g_{ce}(G) = l$.

If $k = 3$, then $T = \{u_1, u_3\}$ is an edge monophonic set of G and so $m_e(G) = 2 = j$. Also, no 2-element subset of vertices is a connected edge monophonic set of G. It is clear that $T_1 = \{u_1, u_2, u_3\}$ is a connected edge monophonic set of G so that $m_{ce}(G) = 3 = k$. Now, we observe that at least one of v_i and $w_i (1 \leq i \leq l-k)$ must belong to every connected edge geodetic set of G. Let $T_2 = \{v_1, v_2, \ldots, v_{l-3}\}$. Then for $x, y \notin T_2$, $T_2 \cup \{x, y\}$ is not a connected edge geodetic set of G. Since $T_2 \cup \{u_1, u_2, u_3\}$ is a connected edge geodetic set of G, it follows that $g_{ce}(G) = l$.

Case 2. $j = 3$. If $k = l$, then let G be a tree of order l with three endvertices. Then, by Theorems 1.3, 2.12 and 2.10, $m_e(G) = 3, m_{ce}(G) = l$ and $g_{ce}(G) = l$. If $k < l$, then we construct a graph G as follows: Let H be a graph obtained from the cycle $C_{2l-2k+4} : v_1, v_2, \ldots, v_{2l-2k+4}, v_1$ of order $2l - 2k + 4$, and the path $P_{k-3} : u_1, u_2, \ldots, u_{k-3}$ of order $k - 3$ by joining u_1 in P_{k-3} with v_1, v_2, v_3 in $C_{2l-2k+4}$. The graph G is shown in Figure 3.8. If $k > 4$, then $S = \{v_2, u_{k-3}\}$ is the set of semi-extreme vertices of G. Since S is not an edge monophonic set of G and since $S \cup \{v_4\}$ is an edge monophonic set of G, by Theorem 1.1, we have $m_e(G) = 3 = j$. Let $T = S \cup \{u_1, ..., u_{k-4}\}$ be the set of semi-extreme vertices and cut-vertices of G. It is clear that for any vertex $y \notin T$, $T \cup \{y\}$ is not a connected edge monophonic set of G. Since $T \cup \{v_1, v_3\}$ is a connected edge monophonic set of G, by Theorem 2.9, $m_{ce}(G) = k$. It is clear that T is not a connected edge geodetic set of G and it is easily seen that $T_1 = T \cup \{v_3, v_4, ..., v_{l-k+4}\}$ is a minimum connected edge geodetic set of G and so by Theorem 2.10, $g_{ce}(G) = |T_1| = l$.

If $k = 4$, then $S_1 = \{u_1, u_2\}$ is the set of semi-extreme vertices of G. Since S_1 is not an edge monophonic set and since $S_2 = S_1 \cup \{v_4\}$ is an edge monophonic set of G, by Theorem 1.1, we have $m_e(G) = 3 = j$. It is clear that for any vertex $y \notin S_1$, $S_1 \cup \{y\}$ is not a connected edge monophonic set of G. Since $S_3 = S_1 \cup \{v_1, v_3\}$ is a connected edge monophonic set of G, by Theorem 2.9, $m_{ce}(G) = 4 = k$. Also, it is clear that S_1 is not a connected edge geodetic set of G and it is easily seen that $S_4 = S_1 \cup \{v_3, v_4, \ldots, v_l\}$ is a minimum connected edge geodetic set of G and so by Theorem 2.10, $g_ce(G) = |T_1| = l$

□

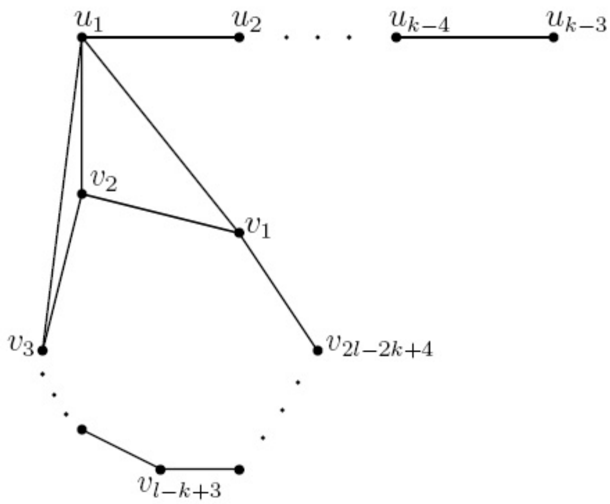


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Figure 3.8

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