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On the generating matrices of the k-Fibonacci numbers

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Abstract

In this paper we define some tridiagonal matrices depending of a parameter from which we will find the k-Fibonacci numbers. And from the cofactor matrix of one of these matrices we will prove some formulas for the k-Fibonacci numbers differently to the traditional form. Finally, we will study the eigenvalues of these tridiagonal matrices.

Keyword : *k-Fibonacci numbers, Cofactor matrix, Eigenvalues.*

AMS : *11B39, 34L16.*

1. Introduction

The generalization of the Fibonacci sequence has been treated by some authors as e.g. Hoggat V.E. [6] and Horadam A.F. [7].

One of these generalizations has been found by Falcon S. and Plaza A. to study the method of triangulation 4TLE [1] and that we define below.

We define the k-Fibonacci numbers [1, 2, 3] by mean of the recurrence relation $F_{k,n+1} = k\,F_{k,n} + F_{k,n-1}$ for $n \geq 1$ with the initial conditions $F_{k,0} = 0$ and $F_{k,1} = 1$.

The recurrence equation of this formula is $r^2 - k \cdot r - 1 = 0$ whose solutions are $\sigma_1 = \frac{k+\sqrt{k^2+4}}{2}$ and $\sigma_2 = \frac{k-\sqrt{k^2+4}}{2}$.

The Binet formula for these numbers is $F_{k,n} = \dfrac{\sigma_1^n - \sigma_2^n}{\sigma_1 - \sigma_2}$

From the definition of the k-Fibonacci numbers, the first of them are presented in Table **??**.

First k-Fibonacci numbers

$\mathrm{F}_{k,0} = 0$
$\mathrm{F}_{k,1} = 1$
$\mathrm{F}_{k,2} = k$
$\mathrm{F}_{k,3} = k^2 + 1$
$\mathrm{F}_{k,4} = k^3 + 2k$
$\mathrm{F}_{k,5} = k^4 + 3k^2 + 1$
$\mathrm{F}_{k,6} = k^5 + 4k^3 + 3k$

For $k = 1$, the classical Fibonacci sequence $\{0, 1, 1, 2, 3, 5, 8, \ldots\}$ is obtained and for $k = 2$ it is the Pell sequence $\{0, 1, 2, 5, 12, 29, \ldots\}$

2. Tridiagonal matrices and the k-Fibonacci numbers

In this section we extend the matrices defined in [4] and applied them to the k-Fibonacci numbers in order to prove some formulas differently to the traditional form.

2.1. The determinant of a special kind of tridiagonal matrices

Let us consider the n-by-n tridiagonal matrices M_n,

$$M_n = \begin{pmatrix} a & b & & & & & & \\ c & d & e & & & & & \\ & c & d & e & & & & \\ & & c & d & e & & & \\ & & & & \ddots & \ddots & \ddots & \\ & & & & & c & d & e \\ & & & & & & c & d \end{pmatrix}$$

Solving the sequence of determinants, we find

$$\begin{aligned} |M_1| &= a \\ |M_2| &= d \cdot |M_1| - b\dot{c} \\ |M_3| &= d \cdot |M_2| - c\dot{e} \cdot |M_1| \\ |M_4| &= d \cdot |M_3| - c\dot{e} \cdot |M_2| \\ \ldots & \end{aligned}$$

And, in general,

$$|M_{n+1}| = d \cdot |M_n| - c\dot{e} \cdot |M_{n-1}| \tag{2.1}$$

2.2. Some tridiagonal matrices and the k–Fibonacci numbers

- If $a = d = k$, $b = e = 1$, and $c = -1$, the matrices M_n are transformed in the tridiagonal matrices

$$H_n(k) = \begin{pmatrix} k & 1 & & & & & \\ -1 & k & 1 & & & & \\ & -1 & k & 1 & & & \\ & & & \ddots & \ddots & \ddots & \\ & & & & -1 & k & 1 \\ & & & & & -1 & k \end{pmatrix}$$

  In this case, and taking into account Table **??**, the above formulas are transformed in

$$\begin{aligned} |H_1(k)| &= k = F_{k,2} \\ |H_2(k)| &= k \cdot k - 1(-1) = k^2 + 1 = F_{k,3} \\ |H_3(k)| &= k(k^2 + 1) - (-1)1k = k^3 + 2k = F_{k,4} \end{aligned}$$

  and Formula (2.1) is $|H_n(k)| = F_{k,n+1}$ for $n \geq 1$.

The k-Fibonacci numbers can also be obtained from the symmetric tridiagonal matrices

$$H'_n(k) = \begin{pmatrix} k & i & & & & & \\ i & k & i & & & & \\ & i & k & i & & & \\ & & & \ddots & \ddots & \ddots & & \\ & & & & i & k & i \\ & & & & & i & k \end{pmatrix}$$

where i is the imaginary unit, i.e. $i^2 = -1$.

- If $a = k^2 + 1$, $b = c = e = 1$, $d = k^2 + 2$, the tridiagonal matrices obtained are

$$O_n(k) = \begin{pmatrix} k^2+1 & 1 & & & & & \\ 1 & k^2+2 & 1 & & & & \\ & 1 & k^2+2 & 1 & & & \\ & & & \ddots & \ddots & \ddots & \\ & & & & 1 & k^2+2 & 1 \\ & & & & & 1 & k^2+2 \end{pmatrix}$$

In this case, it is $|O_n(k)| = F_{k,2n+1}$ for $n \geq 1$. So, with $|O_0(k)| = F_{k,1} = 1$, the sequence of these determinants is the sequence of odd k–Fibonacci numbers $\{1, k^2 + 1, k^4 + 3k^2 + 1\}$.

- Finally, if $a = k$, $b = 0$, $c = 1$, $d = k^2 + 2$, and $e = 1$, for $n \geq 1$ we obtain the even k–Fibonacci numbers from the determinant of the matrices

$$E_n(k) = \begin{pmatrix} k & 0 & & & & & \\ 1 & k^2+2 & 1 & & & & \\ & 1 & k^2+2 & 1 & & & \\ & & & \ddots & \ddots & \ddots & \\ & & & & 1 & k^2+2 & 1 \\ & & & & & 1 & k^2+2 \end{pmatrix}$$

because $|E_n(k)| = F_{k,2n}$ for $n \geq 1$ with $|E_0(k)| = F_{k,0} = 0$.

3. Cofactor matrices of the generating matrices of the k-Fibonacci numbers

The following definitions are well-known: [9]

If A is a square matrix, then the minor of its entry a_{ij}, also known as the (i, j) minor of A, is denoted by M_{ij} and is defined to be the determinant of the submatrix obtained by removing from A its $i - th$ row and $j - th$ column.

It follows $C_{ij} = (-1)^{i+j} M_{ij}$ and C_{ij} called the cofactor of a_{ij}, also referred to as the (i, j) cofactor of A.

Define the cofactor matrix of A, as the $n \times n$ matrix C whose (i,j) entry is the (i,j) cofactor of A.

Finally, the inverse matrix of A is $A^{-1} = \frac{1}{|A|}C^T$, where $|A|$ is the determinant of the matrix A (assuming non zero) and C^T is the transpose of the cofactor matrix C or adjugate matrix of A.

On the other hand, let us consider the n-by-n nonsingular tridiagonal matrix

$$T = \begin{pmatrix} a_1 & b_1 & & & \\ c_1 & a_2 & b_2 & & \\ & c_2 & \ddots & \ddots & \\ & & \ddots & \ddots & b_{n-1} \\ & & & c_{n-1} & a_n \end{pmatrix} \tag{3.1}$$

In [8], Usmani gave an elegant and concise formula for the inverse of the tridiagonal matrix $T^{-1} = (t_{i,j})$:

$$t_{i,j} = \begin{cases} (-1)^{i+j}\dfrac{1}{\theta_n} b_i \cdots b_{j-1}\theta_{i-1}\phi_{j+1} & if \quad i \leq j \\ (-1)^{i+j}\dfrac{1}{\theta_n} c_j \cdots c_{i-1}\theta_{j-1}\phi_{i+1} & if \quad i > j \end{cases} \tag{3.2}$$

where

- θ_i verify the recurrence relation $\theta_i = a_i\theta_{i-1} - b_{i-1}c_{i-1}\theta_{i-2}$ for $i = 2, \dots n$

  with the initial conditions $\theta_0 = 1$ and $\theta_1 = a_1$.

  Formula (2.1) is one special case of this one.
- ϕ_i verify the recurrence relation

  $\phi_i = a_i\phi_{i+1} - b_i c_i \phi_{i+2}$ for $i = n-1, \dots 1$

  with the initial conditions $\phi_{n+1} = 1$ and $\phi_n = a_n$

Observe that $\theta_n = det(T)$.

3.1. Cofactor matrix of $H_n(k)$

For the matrix $H_n(k)$, it is $a_i = k,\ b_i = 1,\ c_i = -1,\ \theta_i = F_{k,i+1}$ and $\phi_j = F_{k,n-j+2}$. Consequently,

$$((H_n(k))^{-1})_{i,j} = \begin{cases} (-1)^{i+j}\dfrac{1}{F_{k,n+1}} F_{k,i} \cdot F_{k,n-j+1} & if \quad i \leq j \\ \dfrac{1}{F_{k,n+1}} F_{k,j} \cdot F_{k,n-i+1} & if \quad i > j \end{cases}$$

We will work with the cofactor matrix whose entries are

$$c_{i,j}(H_n(k)) = \begin{cases} (-1)^{i+j} F_{k,j} F_{k,n-i+1} & if \quad i \geq j \\ F_{k,i} F_{k,n-j+1} & if \quad i < j \end{cases}$$

So, $c_{j,i}(H_n(k)) = (-1)^{i+j} c_{i,j}(H_n(k))$ if $i > j$.

In this form, the cofactor matrix of $H_n(k)$ for $n \geq 2$ is
$C_{n-1}(k) =$

$$\begin{pmatrix} F_{k,n} & F_{k,n-1} & F_{k,n-2} & F_{k,n-3} & \cdots & F_{k,2} & F_{k,1} \\ -F_{k,n-1} & F_{k,2}F_{k,n-1} & F_{k,2}F_{k,n-2} & F_{k,2}F_{k,n-3} & \cdots & F_{k,2}F_{k,2} & F_{k,2} \\ F_{k,n-2} & -F_{k,2}F_{k,n-2} & F_{k,3}F_{k,n-2} & F_{k,3}F_{k,n-3} & \cdots & F_{k,3}F_{k,2} & F_{k,3} \\ -F_{k,n-3} & F_{k,2}F_{k,n-3} & -F_{k,3}F_{k,n-3} & F_{k,4}F_{k,n-3} & \cdots & F_{k,4}F_{k,2} & F_{k,4} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ F_{k,2} & -F_{k,2}F_{k,2} & F_{k,3}F_{k,2} & -F_{k,4}F_{k,2} & \cdots & F_{k,n-1}F_{k,2} & F_{k,n-1} \\ -F_{k,1} & F_{k,2} & -F_{k,3} & F_{k,4} & \cdots & -F_{k,n-1} & F_{k,n} \end{pmatrix}$$

On the other hand, taking into account the inverse matrix $A^{-1} = \frac{1}{|A|} Adj(A)$, it is easy to prove $|Adj(A)| = |A|^{n-1}$.

So, $|C_{n-1}(k)| = F_{k,n+1}^{n-1}$.

In this form, for $n = 2, 3, 4, \ldots$, it is

$$\begin{aligned}
C_1(k) &= \begin{vmatrix} F_{k,2} & F_{k,1} \\ -F_{k,1} & F_{k,2} \end{vmatrix} = F_{k,3} \to F_{k,2}^2 + F_{k,1}^2 = F_{k,3} \\
C_2(k) &= \begin{vmatrix} F_{k,3} & F_{k,2} & F_{k,1} \\ -F_{k,2} & F_{k,2}F_{k,2} & F_{k,2} \\ F_{k,1} & -F_{k,2} & F_{k,3} \end{vmatrix} = F_{k,4}^2 \\
&\to F_{k,2}^2(F_{k,3}^2 + 2F_{k,3} + 1) = F_{k,4}^2 \to \left(\frac{F_{k,3} - F_{k,1}}{k}\right)^2 (F_{k,3} + F_{k,1})^2 = F_{k,4}^2 \\
&\to F_{k,3}^2 - F_{k,1}^2 = kF_{k,4} \\
C_3(k) &= \begin{vmatrix} F_{k,4} & F_{k,3} & F_{k,2} & F_{k,1} \\ -F_{k,3} & F_{k,2}F_{k,3} & F_{k,2}F_{k,2} & F_{k,2} \\ F_{k,2} & -F_{k,2}F_{k,2} & F_{k,3}F_{k,2} & F_{k,3} \\ -F_{k,1} & F_{k,2} & -F_{k,3} & F_{k,4} \end{vmatrix} = F_{k,5}^3 \\
&\to (F_{k,2}^2 + F_{k,3}^2)F_{k,5}^2 = F_{k,5}^3 \to F_{k,2}^2 + F_{k,3}^2 = F_{k,5} \\
C_4(k) &= F_{k,6}^4 \to F_{k,3}^2(F_{k,2} + F_{k,4})^2 F_{k,6}^2 = F_{k,6}^4 \\
&\to \left(\frac{F_{k,4} - F_{k,2}}{k}\right)^2 (F_{k,4} + F_{k,2})^2 = F_{k,6}^2 \to F_{k,4}^2 - F_{k,2}^2 = kF_{k,6}
\end{aligned}$$

$\ldots$

Generalizing these results, and taking into account $F_{k,n} = \frac{F_{k,n+1} - F_{k,n-1}}{k}$, we find the following two formulas for the k-Fibonacci numbers according to that n is odd or even [2]: $F_{k,n+1}^2 + F_{k,n}^2 = F_{k,2n+1}$ and $F_{k,n+1}^2 - F_{k,n-1}^2 = k \cdot F_{k,2n}$

3.2. Cofactor matrix of $O_n(k)$

To apply formula (3.2) to the matrices $O_n(k)$, we must take into account that

$$\begin{aligned} a_1 &= k^2+1 \\ a_i &= k^2+2,\ i \geq 2 \\ b_i &= c_i = 1,\ i \geq 1 \\ \theta_i &= F_{k,2i-1},\ i \geq 1 \\ \phi_j &= \frac{1}{k} F_{k,2(n-j+2)},\ j \geq 1 \end{aligned}$$

and consequently the cofactor of the (i,j) entry of these matrices is
$c_{i,j}(O_n(k)) = (-1)^{i+j}\frac{1}{k}F_{k,2j-1}F_{k,2(n-i+1)} \quad if \quad i \geq j$

$c_{j,i}(O_n(k)) = c_{i,j}(O_n(k))$ for $j > i$

3.3. Cofactor matrix of $E_n(k)$

For the matrices $E_n(k)$ it is

$$\begin{aligned} a_1 &= k = F_{k,2} \\ a_i &= k^2+2,\ i \geq 2 \\ b_1 &= 0 \\ b_{i+1} &= c_i = 1,\ i \geq 1 \\ \theta_i &= F_{k,2(i+1)},\ i \geq 1 \\ \phi_j &= \frac{1}{k} F_{k,2(n-j+2)},\ j \geq 1 \end{aligned}$$

and consequently the cofactor of the (i,j) entry of these matrices is

$$c_{1,j}(E_n(k)) = (-1)^{j+1}\frac{1}{k}F_{k,2(n-j+1)}$$

$\mathrm{c}_{i,j}(E_n(k)) = (-1)^{i+j}\frac{1}{k}F_{k,2j}F_{k,2(n-i+1)}, \qquad if \quad i \geq j, \quad i > 1$
$\mathrm{c}_{j,i}(E_n(k)) = c_{i,j}(E_n(k)), \qquad if \quad j > i > 1$

4. Eigenvalues

This section is dedicated to the study of the eigenvalues of the matrices $H_n(k)$, $O_n(k)$ and $E_n(k)$.

4.1. Eigenvalues of the matrices $H_n(k)$

The matrix (3.1) has entries in the diagonals $a_1, \ldots, a_n,\ b_1, \ldots, b_{n-1},\ c_1, \ldots, c_{n-1}$.

It is well-known the eigenvalues of the matrix (3.1) are
$\lambda_r = a + 2\sqrt{b \cdot c}\ \cos\left(\dfrac{r\pi}{n+1}\right)$ for $r = 1, 2, \ldots, n$.

Consequently, the eigenvalues of the matrix $H_n(k)$ where $a = k,\ b = 1,\ c = -1$, are $\lambda_r = k + 2i\ \cos\left(\dfrac{r\pi}{n+1}\right)$

If n is odd, then the matrix $H_n(k)$ has one unique real eigenvalue corresponding to $r = \frac{n+1}{2}$.
If n is even, no one eigenvalue is real.

So, the sequence of spectra of the tridiagonal matrices $H_n(k)$ for $n = 1, 2, \ldots$ is

$$\begin{aligned}
\Sigma_1 &= \{k\} \\
\Sigma_2 &= \{k \pm i\} \\
\Sigma_3 &= \left\{k, k \pm i\sqrt{2}\right\} \\
\Sigma_4 &= \left\{k \pm \frac{1+\sqrt{5}}{2}i, k \pm \frac{1-\sqrt{5}}{2}i\right\} = \{k \pm \phi i, k \pm (-\phi)^{-1} i\} \\
\Sigma_5 &= \left\{k, k \pm i, k \pm i\sqrt{3}\right\} \\
&\ldots
\end{aligned}$$

where $\phi = \frac{1+\sqrt{5}}{2}$ is the Golden Ratio.

All the roots λ_r lie on the segment $\Re(\lambda_j) = k,\ -2 < \Im(\lambda_j) < 2$.

It is verified $\displaystyle\sum_{j=1}^{n} \lambda_j = nk$ and $\displaystyle\prod_{j=1}^{n} \lambda_j = F_{k,n+1}$.

Moreover, taking into account the product of all eigenvalues is the determinant of the matrix $H_n(k)$ and as $|H_n(k)| = F_{k,n+1}$, it is verified that

$$F_{k,n+1} = \prod_{j=1}^{n} \left(k + 2i\ \cos\left(\frac{\pi j}{n+1}\right)\right)$$

4.2. Eigenvalues of the matrices $O_n(k)$

Matrices $O_n(k)$ are symmetric, so all its eigenvalues are real.

4.2.1. Theorem

If λ_i is an eigenvalue of the matrix $O_n(k)$ for a fixed value k, then $\lambda_i + 2k + 1$ is eigenvalue of the matrix $O_n(k+1)$.

Proof. If λ_i is an eigenvalue of the matrix $O_n(k)$, then it is

$$|O_n(k) - \lambda_i I_n| = \begin{vmatrix} k^2+1-\lambda_i & 1 & & & & \\ 1 & k^2+2-\lambda_i & 1 & & & \\ & 1 & k^2+2-\lambda_i & 1 & & \\ & & & \ddots & \ddots & \ddots \end{vmatrix} =$$

$$\begin{vmatrix} (k+1)^2+1-(\lambda_i+2k+1) & 1 & & & \\ 1 & (k+1)^2+2-(\lambda_i+2k+1) & 1 & & \\ & 1 & (k+1)^2+2-(\lambda_i+2k+1) & 1 & \\ & & & \ddots & \ddots \end{vmatrix} =$$

$|O_n(k+1) - (\lambda_i + 2k+1)I_n|$

Consequently, only it is necessary to find the eigenvalues of the matrix $O_n(1)$ for $n = 2, 3, \ldots$ and then, if λ_j is an eigenvalue of $O_n(1)$, then

$$\lambda'_j = \lambda_j + k^2 - 1 \tag{4.1}$$

is an eigenvalue of the matrix $O_n(k)$.

Now we will study the spectra of these matrices.

First, we present the spectra of these matrices for $k = 1$ and $n = 2, 3, 4, 5, 6, 7, 8$ obtained with the help of MATHEMATICA:

$\Sigma_2 = \{1.381966, 3.618034\}$
$\Sigma_3 = \{1.198062, 2.554958, 4.246980\}$
$\Sigma_4 = \{1.120615, 2.000000, 3.347296, 4.532089\}$
$\Sigma_5 = \{1.081014, 1.690279, 2.715370, 3.830830, 4.682507\}$
$\Sigma_6 = \{1.058116, 1.502979, 2.290790, 3.241073, 4.136129, 4.770912\}$
$\Sigma_7 = \{1.043705, 1.381966, 2.000000, 2.790943, 3.618034, 4.338261, 4.827091\}$
$\Sigma_8 = \{1.034054, 1.299566, 1.794731, 2.452674, 3.184537, 3.891477, 4.478018, 4.864944\}$

Evidently,

$$\sum \lambda_j = 3n - 1 \tag{4.2}$$

and

$$\prod \lambda_j = F_{2n+1} \tag{4.3}$$

Below are the minimum and maximum eigenvalues of these spectra:

	$n=2$	$n=3$	$n=4$	$n=5$	$n=6$	$n=7$	$n=8$
minima	1.381966	1.198062	1.120615	1.081014	1.058116	1.043705	1.034054
Maxima	3.618034	4.246980	4.532089	4.682507	4.770912	4.827091	4.864944

In the first case, we can see this sequence is decreasing and converge to 1, and consequently, $\lim_{n\to\infty} min\, \lambda(O_n(k)) = k$.

In the same form, the sequence of Maxima is increasing and converge to 5, so we can say $\lim_{n\to\infty} Max\,\lambda(O_n(k)) = k^2+4$.

Finally, if $k \neq 1$, then, taking into account Formula (4.1), the formulas (4.2) and (4.3) are transformed into

$$\begin{aligned}\sum_{j=1}^{n} \lambda_j(k) &= \sum(\lambda_i(1)+k^2-1) = nk^2+2n-1 \\ \prod_{j=1}^{n} \lambda_j(k) &= F_{k,2n+1}\end{aligned}$$

4.3. Eigenvalues of the matrices $E_n(k)$

Finally, we say a matrix is positive if all the entries are real and nonnegative. If a matrix is tridiagonal and positive, then all the eigenvalues are real [5]. So, taking into account matrix $E_n(k)$ is tridiagonal and positive, all its eigenvalues are real.

Following the same process that for the matrices $O_n(k)$, we can prove that the first eigenvalue is k and the others verify $\lambda_i(k) = \lambda_i(1)+k^2-1$.

Moreover, $\sum_{j=1}^{n} \lambda_j(k) = (n-1)(k^2+2)+k$ and $\prod_{j=1}^{n} \lambda_j(k) = F_{k,2n}$

The sequence of spectra of the matrices $E_n(1)$ is

$\Sigma_2 = \{1,3\}$
$\Sigma_3 = \{1,2,4\}$
$\Sigma_4 = \{1,1.585786,3,4.414214\}$
$\Sigma_5 = \{1,1.381966,2.381966,3.618034,4.618034\}$
$\Sigma_6 = \{1,1.267949,2,3,4,4.732051\}$
$\sigma_7 = \{1,1.198062,1.753020,2.554958,3.445042,4.246980,4.801938\}$

The sequence of minima eigenvalue converges to 1 (to k in general) and the sequience of maxima converges to 5 (to k^2+4 in the general case).

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