Proyecciones Journal of Mathematics
Vol. 32, N$^o$ 4, pp. 321-332, December 2013.
Universidad Católica del Norte
Antofagasta - Chile
DOI: 10.4067/S0716-09172013000400002

# Multiplication and Composition operators on $w_p(f)$

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*Received : June 2012. Accepted : October 2013*

**Abstract**

*In this paper we characterize the boundedness, closed range, invertibility of the multiplication operators acting on sequence spaces $w_p(f)$ defined by a modulus function. We also make an efforts to study some properties of composition operators on these spaces.*

**Subjclass [2000] :** *Primary 47B20, Secondary 47B38.*

**Keywords :** *Modulus function, multiplication operator, composition operator, closed range, invertibility.*

## 1. Introduction and Preliminaries

A modulus function is a function $f : [0, \infty) \to [0, \infty)$ such that

1. $f(x) = 0$ if and only if $x = 0$;
2. $f(x + y) \leq f(x) + f(y)$ for all $x \geq 0$, $y \geq 0$;
3. $f$ is increasing;
4. $f$ is continuous from right at 0.

It follows that $f$ must be continuous everywhere on $[0, \infty)$. The modulus function may be bounded or unbounded. For example, if we take $f(x) = \frac{x}{x+1}$, then $f(x)$ is bounded. If $f(x) = x^p$, $0 < p < 1$, then the modulus $f(x)$ is unbounded. Subsequently, modulus function has been discussed in ([2], [7], [9]) and many others.

For any sequence $x$, write

$$d_{mn} = d_{mn}(x) = \frac{1}{m+1} \sum_{i=0}^{m} x_{n+i}.$$

G. G. Lorentz [6] proved that

$$\hat{c} = \Big\{ x : \lim_{m \to \infty} d_{mn}(x) \text{ exists uniformly in } n \Big\}.$$

Khan M. A. [3] extend the definition of $d_{mn}$ to $m = -1$ by taking $d_{-1,n} = x_{n-1}$, then write for $m, n \geq 0$

$$t_{mn} = t_{mn}(x) = d_{mn}(x) - d_{m-1,n}(x).$$

A straight forward calculation then show that

$$t_{mn} = \frac{1}{m(m+1)} \sum_{v=1}^{m} v(x_{n+v} - x_{n+v-1}).$$

If $f$ is a modulus function and homogeneous of degree 1, then we define a sequence space as

$$w_p(f) = \Big\{ x : \sup_n \sum_m m^{p-1} f(|t_{mn}(x)|^p) < \infty \Big\}.$$

The space $w_p(f)$ with the norm

$$||x|| = \sup_n \Big\{ \sum_{m=1}^{\infty} m^{p-1} f(|t_{mn}(x)|^p) \Big\}^{1/p}, \quad p \geq 1 \text{ for } x \in w_p(f)$$

is a Banach space.

Let $v : \mathbf{N} \to \mathbf{N}$ and $u : \mathbf{N} \to \mathbf{C}$ be two mappings.
Then the bounded linear transformations

$$T_v : w_p(f) \to w_p(f)$$

and

$$M_u : w_p(f) \to w_p(f)$$

defined by $(T_v h)(x) = h(v(x))$ and $(M_u h)(x) = u(x)h(x)$ are called composition and multiplication operators respectively. By $B(w_p(f))$, we denote the set of all bounded linear operators from $w_p(f)$ into itself and $[z(u)]$ denote, the set $\{n \in \mathbf{N} : u(n) = 0\}$. For more details about the study of multiplication and composition operators see ([1], [4], [5], [8], [10], [11]).

In this paper we study multiplication and composition operators acting on sequence spaces $w_p(f)$ defined by a modulus function.

## 2. Multiplication operators acting on sequence spaces defined by a modulus function

In this section we characterize multiplication operators acting on $w_p(f)$.

**Theorem 2.1.** *Let $M_u : w_p(f) \to w_p(f)$ be a linear transformation. Then $M_u$ is a bounded operator if and only if there exists $M > 0$ such that*

$$f\Big(|u(m)t_{mn}(x)|^p\Big) \leq M f\Big(|t_{mn}(x)|\Big)^p$$

*for all $m \in N$.*

**Proof.** Suppose that the condition of the theorem is true. For $x \in w_p(f)$, we have

$$\sup_n \sum_{m=1}^{\infty} m^{p-1} f\Big(|u(m)t_{mn}(x)|^p\Big) \leq M \sup_n \sum_{m=1}^{\infty} m^{p-1} f\Big(|t_{mn}(x)|^p\Big) < \infty.$$

Thus $M_u x \in w_p(f)$. Further,

$$\begin{aligned} ||M_u(x)|| &= \sup_n \Big\{ \sum_{m=1}^{\infty} m^{p-1} f\Big(|u(m)t_{mn}(x)|^p\Big)\Big\}^{1/p} \\ &\leq M \sup_n \Big\{ \sum_{m=1}^{\infty} m^{p-1} f\Big(|t_{mn}(x)|^p\Big)\Big\}^{1/p} \\ &\leq M||x||. \end{aligned}$$

This proves the continuity of $M_u$ at the origin and hence everywhere in view of linearity of $M_u$.

Conversely, if the condition of the theorem were false, then for every integer $k > 0$ there exists $n_k \in N$ and $y_k = y \in \mathbf{R}^+$ such that

$$f\Big(|u(n_k)t_{mn}(y_k)|^p\Big) > kf\Big(|t_{mn}(y_k)|^p\Big)$$

Let $g_k = t_{mn}(y_k)\chi_{\{n_k\}}$. Then

$$\begin{aligned} k||g_k|| &= k\Big\|t_{mn}(y_k)\chi_{\{n_k\}}\Big\| \\ &= k\Big(\sup_n \sum_{m=1}^{\infty} m^{p-1} f\Big(|t_{mn}(y_k)|\Big)^p\Big)^{1/p} \end{aligned}$$

$$\begin{aligned} &< \Big(\sup_n \sum_{m=1}^{\infty} m^{p-1} f\Big(|u(n_k)t_{mn}(y_k)|\Big)^p\Big)^{1/p} \\ &= \Big(\sup_n \sum_{m=1}^{\infty} m^{p-1} f|M_u g_k|^p\Big)^{1/p} \\ &= \|M_u g_k\|. \end{aligned}$$

This proves that $M_u$ is not bounded. Hence the condition must be true.

**Theorem 2.2.** *Let $AM_u = M_u A$. Then $A$ is a multiplication operator.*

**Proof.** Let $V = Ae$. Then

$$Ae_n = AM_{e_n}e = M_{e_n}Ae = M_{e_n}V = e_nV = Ve_n = M_Ve_n.$$

We now prove that $V$ induces a multiplication operator. If $V$ does not induce a bounded operator, then for every $k \in N$, there exists $n_k \in N$ such that

$$f\Big(|V(n_k)t_{mn}(y_k)|^p\Big) > mf\Big(|t_{mn}(y_k)|^p\Big).$$

Let $g_k = t_{mn}(y_k)e_{n_k}$. Then

$$\begin{aligned}
k||g_k|| &= k||t_{mn}(y_k)e_{n_k}|| \\
&= k\Big(\sup_n \sum_{m=1}^{\infty} m^{p-1} f\Big(|t_{mn}(y_k)|\Big)^p\Big)^{1/p} \\
&< \Big(\sup_n \sum_{m=1}^{\infty} m^{p-1} f\Big(|V(n_k)t_{mn}(y_k)|\Big)^p\Big)^{1/p} \\
&= \Big(\sup_n \sum_{m=1}^{\infty} m^{p-1} f|Ag_k|^p\Big)^{1/p} \\
&= ||Ag_k||,
\end{aligned}$$

which contradicts the continuity of $A$. Hence $A$ must be a bounded operator and $A = M_V$.

**Theorem 2.3.** *Let $M_u \in B(w_p(f))$. Then $M_u$ is invertible if and only if there exists $\epsilon > 0$ such that*

$$f\Big(|u(k)t_{mn}(y)|^p\Big) \geq \epsilon f\Big(|t_{mn}(y)|^p\Big), \ \ \forall p \in \mathbf{N} \ \ \textit{and} \ \ y \in \mathbf{R}^+.$$

**Proof.** We first assume that there exists $\epsilon > 0$ such that

$$f\Big(|u(k)t_{mn}(y)|^p\Big) \geq \epsilon f\Big(|t_{mn}(y)|^p\Big), \ \ \forall p \in \mathbf{N} \ \ \text{and} \ \ y \in \mathbf{R}^+.$$

Now

$$\epsilon f\Big[\frac{|t_{mn}(y)|^p}{|u(k)|^p}\Big] \quad \leq \quad f\Big[|u(k)|^p.\frac{|t_{mn}(y)|^p}{|u(k)|^p}\Big]$$

$=$ f$\Big($—t$_{mn}(y)|^p\Big)$or

$$f\Big[|\frac{1}{|u(k)|^p}t_{mn}(y)|^p\Big] \leq \frac{1}{\epsilon}f\Big(|t_{mn}(y)|^p\Big), \ \ \forall p \in \mathbf{N}.$$

This proves that $M_V$ is a bounded operator, where $V = \frac{1}{u}$. Clearly $M_V$ is inverse of $M_u$.

Conversely, suppose that $M_u$ is invertible with $M_V$ as its inverse. Clearly $V = \frac{1}{u}$. Hence by continuity of $M_V$, there exists $M > 0$ such that

$$f\Big(|V(k)t_{mn}(y)|^p\Big) \leq Mf\Big(|t_{mn}(y)|^p\Big), \ \ \forall k \in \mathbf{N} \ \text{ and } \ y \in \mathbf{R}^+.$$

Or equivalently

$$f\Big[|\frac{1}{|u(k)|}t_{mn}(y)|^p\Big] \leq Mf\Big(|t_{mn}(y)|^p\Big).$$

Taking

$$|t_{mn}(y)| = |u(k)t_{mn}(y)|,$$

we get

$$f\Big(|t_{mn}(y)|^p\Big) \leq Mf\Big(|u(k)t_{mn}(y)|^p\Big)$$

or

$$f\Big(|u(k)t_{mn}(y)|^p\Big) \geq \frac{1}{M}f(|t_{mn}(y)|^p) \ \ \forall k \in \mathbf{N}.$$

Taking $\epsilon = \frac{1}{M}$, we get

$$f\Big(|u(k)t_{mn}(y)|^p\Big) \geq \epsilon f(|t_{mn}(y)|^p).$$

Hence the condition must be true.

**Theorem 2.4.** *Let $M_u \in B(w_p(f))$. Then $M_u$ is Fredholm if and only if*

*(i)* $[Z(u)]$ *is a finite set*
*(ii) there exists* $\epsilon > 0$ *such that*

$$f\Big(|u(k)t_{mn}(y)|^p\Big) \geq \epsilon f\Big(|t_{mn}(x)|^p\Big) \ \ \forall\, m \in [Z(u)]^{'}.$$

**Proof.** If $[Z(u)]$ is a finite set, then $\ker M_u$ is finite dimensional. From the condition (ii), $M_u$ has closed range.

Moreover $\dim(w_p(f)/\mathrm{ran} M_u)$ is finite. This proves that $M_u$ is Fredholm.

The converse of the theorem is obvious.

**Corollary 2.5.** *Let* $M_u \in B(w_p(f))$*. Then* $M_u$ *has closed range if and only if there exists* $\delta > 0$ *such that*

$$f\Big(|u(k)t_{mn}(y)|^p\Big) \geq \delta f\Big(|t_{mn}(y)|^p\Big), \quad \forall\, k \in [Z(u)]^{'} \ \textit{and}\, y \in \mathbf{R}^+.$$

**Proof.** Assume that the condition of the theorem is true. Let $h \in \overline{\mathrm{ran} M_u}$.

Then there exists a sequence $\{h_n\}$ such that $M_u h_n \to h$ that is $||M_u h_n - M_u h|| \to 0$ as $n \to \infty$. Now $\{M_u h_n\}$ is a Cauchy sequence. Therefore for every $\epsilon > 0$ there exists $n_0 \in N$ such that

$$||M_u t_{mn} h_n - M_u t_{mn} h_k|| < \epsilon \ \ \forall n, k \geq n_0.$$

Now

$$\begin{aligned} \delta \sup_{n\in[Z(u)]^{'}} \sum_{m=1}^{\infty} m^{p-1} f\Big(|t_{mn}(h_n - h_k)|^p\Big) &\leq \sup_{n\in[Z(u)]^{'}} \sum_{m=1}^{\infty} m^{p-1} f\Big(|u(m)t_{mn}(h_n - h_k)|^p\Big) \\ &< \epsilon \quad \forall\, n, k \geq n_0.(1) \end{aligned}$$

Define

$$\tilde{h_n}(k) = \begin{cases} h_n(k), & \text{if } m \in [Z(u)]^{'} \\ 0, & \text{elsewhere.} \end{cases}$$

Then from (1) it follows that $\{\tilde{h_n}\}$ is a Cauchy sequence in $w_p(f)$. But $w_p(f)$ is complete.

Therefore there exists $\tilde{h} \in w_p(f)$ such that $\tilde{h_n} \to \tilde{h}$. Hence by continuity of $M_u$, we get $M_u h_n = M_u \tilde{h_n} \to M_u \tilde{h}$. Hence $h = M_u \tilde{h}$ so that $h \in \mathrm{ran} M_u$. Thus $M_u$ has closed range.

Conversely, if the condition of the theorem were false, then for every positive integer $k$ there exists $n_k \in N$ and $y_k \in \mathbf{R}^+$ such that

$$f\Big(|u(n_k)t_{mn}(y_k)|^p\Big) < 1/kf\Big(|t_{mn}y_k|^p\Big).$$

Let $g_k = t_{mn}y_k\chi_{\{n_k\}}$.

Then

$$||M_u g_k|| \;=\; \sup_n \Big\{ \sum_{m=1}^{\infty} m^{p-1} f(|u.g_k|^p) \Big\}^{1/p}$$

$$= \sup_n \Big\{ \sum_{m=1}^{\infty} m^{p-1} f(|u(n_k)t_{mn}(y_k)|^p) \Big\}^{1/p}$$

$$\leq 1/k \sup_n \Big\{ \sum_{m=1}^{\infty} m^{p-1} f(|t_{mn}(y_k)|^p) \Big\}^{1/p}$$

$$= 1/\mathrm{k}\ \|g_k\| .$$

This proves that $M_u$ is not bounded away from zero so that $M_u$ does not have closed range.

## 3. Composition operators acting on sequence spaces defined by a modulus function

In this section we study some properties of composition operators on $w_p(f)$.

**Theorem 3.1.** *Let $T_v : w_p(f) \to w_p(f)$ be a linear transformation. Then $T_v$ is a bounded operator if there exists $M > 0$ such that*

$$\sum_{k \in v^{-1}(n)} m^{p-1} f\Big(|t_{mk}(x)|^p\Big) \leq M m^{p-1} f\Big(|t_{mn}(x)|^p\Big).$$

**Proof.** Suppose that the condition of the theorem is true. If $x \in w_p(f)$, then

$$\sup_n \sum_{m=1}^{\infty} \sum_{k \in v^{-1}(n)} m^{p-1} f\Big(|t_{mk}(x)|^p\Big) \quad \leq \quad M \sup_n \sum_{m=1}^{\infty} m^{p-1} f\Big(|t_{mn}(x)|^p\Big)$$

$< \ \infty$,which shows that $T_v x \in w_p(f)$. Further,

$$||T_v x||_f \quad = \quad \sup_n \Big\{ \sum_{m=1}^{\infty} m^{p-1} f\Big(|t_{mn}(x \circ v(k))|^p\Big) \Big\}^{1/p}$$

$$= \ \sup_n \Big\{ \sum_{m=1}^{\infty} \sum_{k \in v^{-1}(n)} m^{p-1} f\Big(|t_{mk} x|^p\Big) \Big\}^{1/p}$$
$$\leq M \sup_n \Big\{ \sum_{m=1}^{\infty} m^{p-1} f\Big(|t_{mn}(x)|^p\Big) \Big\}^{1/p}$$
$$\leq M ||x||_f . (2)$$

The continuity of $T_v$ at origin follows from the inequality (2). Since $T_v$ is linear, so it is continuous everywhere.

**Theorem 3.2.** *Let $T_v \in B(w_p(f))$. Then $T_v$ has closed range if there exists $\delta > 0$ such that*

$$\sum_{k \in v^{-1}(n)} m^{p-1} f\Big(|t_{mk}(x)|^p\Big) \geq \delta m^{p-1} f\Big(|t_{mn}(x)|^p\Big) \ \textit{for every} \ m \in \mathbf{N}. (3)$$

**Proof.** We assume that the condition (3) is true. We have to show that $T_v$ has closed range. Let $x \in \overline{\mathrm{ran} T_v}$ and let $\{x^i\}$ be a sequence in $w_p(f)$ such that $T_v x^n \to x$. Then for every $\epsilon > 0$ there exists positive integer $n_0$ such that

$$||T_v x^i - T_v x^j|| < \epsilon \ \ \forall \, i, j \geq n_0.$$

Equivalently,

$$\epsilon \quad > \quad \sup_n \Big\{ \sum_{m=1}^{\infty} \sum_{k \in v^{-1}(n)} m^{p-1} f\Big(|t_{mk}(x^i \circ v(k) - x^j \circ v(k))|^p\Big) \Big\}^{1/p}$$

$$\begin{aligned} &\geq \delta \sup_n \Big\{ \sum_{m=1}^{\infty} m^{p-1} f\Big(|t_{mn}(x^i - x^j)|^p\Big) \Big\}^{1/p} \\ &= \delta ||x^i - x^j||, \ \ \forall\, i, j \geq n_0(4) \end{aligned}$$

from (4) it follows that $\{x^i\}$ is a Cauchy sequence in $w_p(f)$. In view of completeness of $w_p(f)$, there exists $y \in w_p(f)$ such that $x^i \to y$. From the continuity of $T_v$, $T_v x^i \to T_v y$. Hence $x = T_v y$ so that $x \in \text{ran} T_v$. Hence ran $T_v$ is closed.

**Theorem 3.3.** *Let $T_v \in B(w_p(f))$. Then $T_v$ is an isometry if*

$$\sum_{k \in v^{-1}(n)} m^{p-1} f\Big(|t_{mk}(x)|^p\Big) = m^{p-1} f\Big(|t_{mn}(x)|^p\Big).$$

**Proof.** If the condition of the theorem is satisfied, then for every $x \in w_p(f)$, we have

$$\begin{aligned} ||T_v x|| &= \sup_n \Big\{ \sum_{m=1}^{\infty} \sum_{k \in v^{-1}(m)} m^{p-1} f\Big(|t_{mk} x|^p\Big) \Big\}^{1/p} \\ &= \sup_n \Big\{ \sum_{m=1}^{\infty} m^{p-1} f\Big(|t_{mn}(x)|^p\Big) \Big\}^{1/p} \\ &= ||x||. \end{aligned}$$

Hence $T_v$ is an isometry.

**Theorem 3.4.** Let $T_v \in B(w_p(f))$. If $T_v$ is an isometry, then

$$\sup_n \sum_{k \in v^{-1}(m)} k^{p-1} f\Big(|t_{nk}(x)|^p\Big) = \sup_n m^{p-1} f\Big(|t_{mn} x|^p\Big).$$

**Proof.** The proof is trivial.

## References

[1] M. B. Abrahmse, Multiplication operators, Hilbert space operators. *Lecture notes in Mathematics*, 693 : pp. 17-36, (1978).

[2] T. Bilgen, On statistical convergence. *An. Univ. Timisoara Ser. Math. Inform.*, 32 : pp. 3-7, (1994).

[3] M. A. Khan, Some sequence spaces with an index defined by a modulus function. *Thai J. Math.*, 2 : 259-264, (2004).

[4] B. S. Komal and Kuldip Raj, Multiplication operators induced by operator valued maps. *Int. J. Contemp. Math. Sci.*, Vol.3 : pp. 667-673, (2008).

[5] B. S. Komal and P. S. Singh, Composition operators on the space of entire functions. *Kodai Math. J.*, 14 : pp. 463-469, (1991).

[6] G. G. Lorentz, A contribution to the theory of divergent series. *Acta. Math.*, 80 : pp. 167-190, (1948).

[7] E. Malkowsky and E. Savas, Some $\lambda$-sequence spaces defined by a modulus. *Arch. Math.*, 36 : pp. 219-228, (2000).

[8] Kuldip Raj, B. S. Komal and Vinay Khosla, Composition operators on sequence spaces of entire functions. *Int. Electron. J. Pure Appl. Math.*, 1 : pp. 469-474, (2010).

[9] E. Savas, On some generalized sequence spaces defined by a modulus. *Indian J. pure Appl. Math.*, 30 : pp. 459-464, (1999).

[10] R. K. Singh and J. S. Manhas, Composition operators on function spaces. *North-Holland*, (1993).

[11] H. Takagi and K. Yokouchi, Multiplication and composition operators between $L^p$-spaces. *Contemp. Math.*, 232 : pp. 321-338, (1999).

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