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Titchmarsh's Theorem for the Dunkl transform in the space $L^2(\mathbf{R}^d, w_k(x)dx)$

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Abstract

Using a generalized spherical mean operator, we obtain a generalization of Titchmarsh's theorem for the Dunkl transform for functions satisfying the (ψ, α, β)-Dunkl Lipschitz condition in $L^2(\mathbf{R}^d, w_k(x)dx)$.

Keywords : *Dunkl operator, Dunkl transform, generalized spherical mean operator.*

Mathematics Subject Classification : *47B48; 33C52.*

1. Intoduction and preliminaries

In [10], E. C. Titchmarsh characterized the set of functions in $L^2(\mathbf{R})$ satisfying the Cauchy Lipschitz condition for the Fourier transform, namely we have

Theorem 1.1. *Let $\alpha \in (0,1)$ and assume that $f \in L^2(\mathbf{R})$. Then the following are equivalents*

1. $\|f(x+h)-f(x)\|_{L^2(\mathbf{R})} = O(h^\alpha)$ *as* $h \longrightarrow 0$,
2. $\int_{|\lambda|\geq r} |F(\lambda)|^2 d\lambda = O(r^{-2\alpha})$ *as* $r \longrightarrow +\infty$,

where F stands for the Fourier transform of f.

The main aim of this paper is to establish a generalization of Theorem 1.1 in the Dunkl transform setting by means of the generalized spherical mean operator.

In this paper we consider the Dunkl operators $T_j,\ \ j=1,2,...,d$, which are the differential-difference operators introduced by C.F. Dunkl in [3]. These operators are very important in pure mathematics and in physics.

In the first we collect some notations and results on Dunkl operators and the Dunkl kernel (see [3], [4], [6]).

We consider $\mathbf{R}^d$ with the Euclidean scalar product $\langle .,. \rangle$ and $|x| = \sqrt{\langle x,x\rangle}$. For $\alpha \in \mathbf{R}^d\backslash\{0\}$, let σ_α be the reflection in the hyperplane $H_\alpha \subset \mathbf{R}^d$ orthogonal to α.
i.e.,

$$\sigma_\alpha(x) = x - 2\frac{\langle \alpha, x\rangle}{|x|^2}\alpha.$$

A finite set $R \subset \mathbf{R}^d\backslash\{0\}$ is called a root system, if $R \cap \mathbf{R}.\alpha = \{\alpha, -\alpha\}$ and $\sigma_\alpha R = R$ for all $\alpha \in R$. For a given root system R the reflection σ_α, $\alpha \in R$, generate a finite group $W \subset O(d)$, the reflection group associated with R. We fix $\beta \in \mathbf{R}^d\backslash \cup_{\alpha\in R} H_\alpha$ and define a positive root system $R_+ = \{\alpha \in R/\ \ \langle \alpha, \beta\rangle > 0\}$.

A function $k : R \longrightarrow \mathbf{C}$ on a root system R is called a multiplicity function, if it is invariant under the action of the associated reflection group W.

If one regards k as a function on the corresponding reflections, this means that k is constant on the conjugacy classes of reflections in W.

We consider the weight function

$$w_k(x) = \prod_{\alpha \in R_+} |\langle \alpha, x \rangle|^{2k(\alpha)},$$

where w_k is W-invariant and homogeneous of degree 2γ where

$$\gamma = \sum_{\alpha \in R_+} k(\alpha).$$

We let η be the normalized surface measure on the unit sphere $\mathbf{S}^{d-1}$ in $\mathbf{R}^d$ and set

$$d\eta_k(y) = w_k(y) d\eta(y).$$

Then η_k is a W-invariant measure on $\mathbf{S}^{d-1}$, we let $d_k = \eta_k(\mathbf{S}^{d-1})$.

Introduced by C.F. Dunkl in [3] the Dunkl operators T_j, $1 \leq j \leq d$, on $\mathbf{R}^d$ associated with the reflection group W and the multiplicity function k are the first-order differential-difference operators given by

$$T_j f(x) = \frac{\partial f}{\partial x_j}(x) + \sum_{\alpha \in R_+} k(\alpha) \alpha_j \frac{f(x) - f(\sigma_\alpha(x))}{\langle \alpha, x \rangle}, \quad f \in C^1(\mathbf{R}^d),$$

where $\alpha_j = \langle \alpha, e_j \rangle$; $(e_1, ..., e_d)$ being the canonical basis of $\mathbf{R}^d$ and $C^1(\mathbf{R}^d)$ is the space of functions of class C^1 on $\mathbf{R}^d$.

The Dunkl kernel E_k on $\mathbf{R}^d \times \mathbf{R}^d$ has been introduced by C.F. Dunkl in [5]. For $y \in \mathbf{R}^d$ the function $x \longmapsto E_k(x, y)$ can be viewed as the solution on $\mathbf{R}^d$ of the following initial problem

$$\begin{cases} T_j u(x,y) = y_j u(x,y) & for \quad 1 \leq j \leq d \\ u(0,y) = 1 & for\ all \quad y \in \mathbf{R}^d \end{cases}$$

This kernel has unique holomorphic extension to $\mathbf{C}^d \times \mathbf{C}^d$.

M. Rösler has proved in [8] the following integral representation for the Dunkl kernel

$$E_k(x,z) = \int_{\mathbf{R}^d} e^{\langle y,z\rangle} d\mu_x(y),\ x \in \mathbf{R}^d,\ z \in \mathbf{C}^d,$$

where μ_x is a probability measure on $\mathbf{R}^d$ with support in the closed ball $B(0,|x|)$ of center 0 and raduis $|x|$.

Proposition 1.2. *[6]: Let $z, w \in \mathbf{C}^d$ and $\lambda \in \mathbf{C}$. Then*

1. $E_k(z,0) = 1,$
2. $E_k(z,w) = E_k(w,z),$
3. $E_k(\lambda z, w) = E_k(z, \lambda w),$
4. *For all $\nu = (\nu_1, ..., \nu_d) \in \mathbf{N^d}$, $x \in \mathbf{R}^d$, $z \in \mathbf{C}^d$, we have*

$$|D_z^\nu E_k(x,z)| \leq |x|^{|\nu|} exp(|x||Rez|),$$

where

$$D_z^\nu = \frac{\partial^{|\nu|}}{\partial z_1^{\nu_1} .....\partial z_d^{\nu_d}}; \quad |\nu| = \nu_1 + ... + \nu_d.$$

In particulier

$$|D_z^\nu E_k(ix,z)| \leq |x|^{|\nu|},$$

for all $x, z \in \mathbf{R}^d$.

The Dunkl transform is defined for $f \in L_k^1(\mathbf{R}^d) = L^1(\mathbf{R}^d, w_k(x)dx)$ by

$$\widehat{f}(\xi) = c_k^{-1} \int_{\mathbf{R}^d} f(x) E_k(-i\xi, x) w_k(x) dx,$$

where the constant c_k is given by

$$c_k = \int_{\mathbf{R}^d} e^{-\frac{|z|^2}{2}} w_k(z) dz.$$

According to [4, 6, 9] we have the following results:

1. When both f and $\widehat{f}$ are in $L^1_k(\mathbf{R}^d)$, we have the inversion formula

$$f(x) = \int_{\mathbf{R}^d} \widehat{f}(\xi) E_k(ix,\xi) w_k(\xi) d\xi, \quad x \in \mathbf{R}^d,$$

2. (Plancherel's theorem) The Dunkl transform on $S(\mathbf{R}^d)$, the space of Schwartz functions, extends uniquely to an isometric isomorphism on $L^2_k(\mathbf{R}^d)$.

K. Trimèche has introduced [11] the Dunkl translation operators τ_x, $x \in \mathbf{R}^d$. For $f \in L^2_k(\mathbf{R}^d)$ and we have

$$(\widehat{\tau_x(f)})(\xi) = E_k(ix,\xi)\widehat{f}(\xi),$$

and

$$\tau_x(f)(y) = c_k^{-1} \int_{\mathbf{R}^d} \widehat{f}(\xi) E_k(ix,\xi) E_k(iy,\xi) w_k(\xi) d\xi.$$

The generalized spherical mean operator for $f \in L^2_k(\mathbf{R}^d)$ is defined by

$$M_h f(x) = \frac{1}{d_k} \int_{\mathbf{S}^{d-1}} \tau_x(hy) d\eta_k(y), \ x \in \mathbf{R}^d, h > 0.$$

From [7], we have $M_h f \in L^2_k(\mathbf{R}^d)$ whenever $f \in L^2_k(\mathbf{R}^d)$ and

$$\|M_h f\|_{L^2_k} \leq \|f\|_{L^2_k}.$$

For $p \geq -\frac{1}{2}$, we introduce the normalized Bessel functuion j_p defined by

$$j_p(z) = \Gamma(p+1) \sum_{n=0}^{\infty} \frac{(-1)^n (z/2)^{2n}}{n!\Gamma(n+p+1)}, \ z \in \mathbf{C},$$

where Γ is the gamma-function.

Lemma 1.3. *[1] The following inequalities are fulfilled*

1. $|j_p(x)| \leq 1$,
2. $1 - j_p(x) = O(x^2); \ 0 \leq x \leq 1.$

Lemma 1.4. *The following inequality is true*

$$|1 - j_p(x)| \geq c,$$

with $|x| \geq 1$, *where* $c > 0$ *is a certain constant.*

(Analog of lemma 2.9 in [2])

Proposition 1.5. *Let* $f \in L_k^2(\mathbf{R}^d)$. *Then*

$$(\widehat{M_h f})(\xi) = j_{\gamma+\frac{d}{2}-1}(h|\xi|)\widehat{f}(\xi).$$

(See [7])

For any function $f(x) \in L_k^2(\mathbf{R}^d)$ we define differences of the order m ($m \in \{1, 2, ...\}$) with a step $h > 0$.

$$\Delta_h^m f(x) = (M_h - I)^m f(x), \tag{1.1}$$

here I is the unit operator.

Lemma 1.6. *Let* $f \in L_k^2(\mathbf{R}^d)$. *Then*

$$\|\Delta_h^m f(x)\|_{L_k^2}^2 = \int_{\mathbf{R}^d} |1 - j_{\gamma+\frac{d}{2}-1}(h|\xi|)|^{2m} |\widehat{f}(\xi)|^2 w_k(\xi) d\xi.$$

From formula (1.1) and proposition 1.5, we have

$$(\widehat{\Delta_h^m f})(\xi) = (j_{\gamma+\frac{d}{2}-1}(h|\xi|) - 1)^m \widehat{f}(\xi).$$

By Parseval's identity, we obtain

$$\|\Delta_h^m f(x)\|_{L_k^2}^2 = \int_{\mathbf{R}^d} |1 - j_{\gamma+\frac{d}{2}-1}(h|\xi|)|^{2m} |\widehat{f}(\xi)|^2 w_k(\xi) d\xi.$$

The lemma is proved

2. Main Results

In this section we give the main result of this paper. We need first to define (ψ, α, β)-Dunkl Lipschitz class.

Definition 2.1. *Let $\alpha > 0$ and $\beta > 0$. A function $f \in L^2_k(\mathbf{R}^d)$ is said to be in the (ψ, α, β)-Dunkl Lipschitz class, denoted by $Lip(\psi, \alpha, \beta)$; if:*

$$\|\Delta_h^m f(x)\|_{L^2_k} = O(h^\alpha \psi(h^\beta)) \ as \ h \longrightarrow 0; \ m \in \{1, 2, ...\},$$

where $\psi(t)$ is a continuous increasing function on $[0, \infty)$, $\psi(0) = 0$ and $\psi(ts) = \psi(t)\psi(s)$ for all $t, s \in [0, \infty)$ and this function verify

$$\int_0^{1/h} r^{2m-2\alpha-1}\psi(r^{-2\beta})dr = O(h^{2\alpha-2m}\psi(h^{2\beta})) \ as \ h \longrightarrow 0.$$

Theorem 2.2. *Let $f \in L^2_k(\mathbf{R}^d)$. Then the following are equivalents*

1. *$f \in Lip(\psi, \alpha, \beta)$,*
2. *$\int_{|\xi|\geq s} |\widehat{f}(\xi)|^2 w_k(\xi) d\xi = O(s^{-2\alpha}\psi(s^{-2\beta}))$ as $s \longrightarrow +\infty$.*

1) $\Longrightarrow$ 2) Assume that $f \in Lip(\psi, \alpha, \beta)$. Then we have

$$\|\Delta_h^m f(x)\|_{L^2_k} = O(h^\alpha \psi(h^\beta)) \ as \ h \longrightarrow 0,$$

From lemma 1.6, we have

$$\|\Delta_h^m f(x)\|^2_{L^2_k} = \int_{\mathbf{R}^d} |1 - j_{\gamma+\frac{d}{2}-1}(h|\xi|)|^{2m} |\widehat{f}(\xi)|^2 w_k(\xi) d\xi.$$

If $|\xi| \in [\frac{1}{h}, \frac{2}{h}]$ then $h|\xi| \geq 1$ and lemma 1.4 implies that

$$1 \leq \frac{1}{c^{2m}} |1 - j_{\gamma+\frac{d}{2}-1}(h|\xi|)|^{2m}.$$

Then

$$\begin{aligned}
\int_{\frac{1}{h}\leq|\xi|\leq\frac{2}{h}} |\widehat{f}(\xi)|^2 w_k(\xi) d\xi &\leq \frac{1}{c^{2m}} \int_{\frac{1}{h}\leq|\xi|\leq\frac{2}{h}} |1 - j_{\gamma+\frac{d}{2}-1}(h|\xi|)|^{2m} |\widehat{f}(\xi)|^2 w_k(\xi) d\xi \\
&\leq \frac{1}{c^{2m}} \int_{\mathbf{R}^d} |1 - j_{\gamma+\frac{d}{2}-1}(h|\xi|)|^{2m} |\widehat{f}(\xi)|^2 w_k(\xi) d\xi \\
&\leq Ch^{2\alpha}(\psi(h^\beta))^2 = Ch^{2\alpha}\psi(h^{2\beta}).
\end{aligned}$$

Therefore

$$\int_{s\leq|\xi|\leq 2s} |\widehat{f}(\xi)|^2 w_k(\xi) d\xi \leq Cs^{-2\alpha}\psi(s^{-2\beta}).$$

Furthermore, we have
$\int_{|\xi|\geq s}|\widehat{f}(\xi)|^2w_k(\xi)d\xi$
$=\left(\int_{s\leq|\xi|\leq 2s}+\int_{2s\leq|\xi|\leq 4s}+\int_{4s\leq|\xi|\leq 8s}+.....\right)|\widehat{f}(\xi)|^2w_k(\xi)d\xi$
$\leq C\left(s^{-2\alpha}\psi(s^{-2\beta})+(2s)^{-2\alpha}\psi((2s)^{-2\beta})+(2^2s)^{-2\alpha}\psi((2^2s)^{-2\beta})+......\right)$
$\leq C\left(s^{-2\alpha}\psi(s^{-2\beta})+2^{-2\alpha}s^{-2\alpha}\psi(s^{-2\beta})+(2^{-2\alpha})^2s^{-2\alpha}\psi(s^{-2\beta})+....\right)$
$\leq Cs^{-2\alpha}\psi(s^{-2\beta})(1+2^{-2\alpha}+(2^{-2\alpha})^2+(2^{-2\alpha})^3+.....)$
$\leq K_\alpha s^{-2\alpha}\psi(s^{-2\beta})$,
where $K_\alpha=C(1-2^{-2\alpha})^{-1}$.

This proves that

$$\int_{|\xi|\geq s}|\widehat{f}(\xi)|^2w_k(\xi)d\xi=O(s^{-2\alpha}\psi(s^{-2\beta}))\ as\ s\longrightarrow+\infty$$

2) $\Longrightarrow$ 1) Suppose now that

$$\int_{|\xi|\geq s}|\widehat{f}(\xi)|^2w_k(\xi)d\xi=O(s^{-2\alpha}\psi(s^{-2\beta}))\ as\ s\longrightarrow+\infty.$$

We have to show that

$$\int_0^\infty r^{2\gamma+d-1}|1-j_{\gamma+\frac{d}{2}-1}(hr)|^{2m}\phi(r)dr=O(h^{2\alpha}\psi(h^{2\beta})),$$

where we have set

$$\phi(r)=\int_{\mathbf{S}^{d-1}}|\widehat{f}(ry)|^2w_k(y)dy.$$

We write

$$I_1=\int_0^{1/h}r^{2\gamma+d-1}|1-j_{\gamma+\frac{d}{2}-1}(hr)|^{2m}\phi(r)dr,$$

and

$$I_2=\int_{1/h}^\infty r^{2\gamma+d-1}|1-j_{\gamma+\frac{d}{2}-1}(hr)|^{2m}\phi(r)dr.$$

Firstly, from (1) in lemma 1.3 we see that

$$I_2\leq 4^m\int_{1/h}^\infty r^{2\gamma+d-1}\phi(r)dr=O(h^{2\alpha}\psi(h^{2\beta}))\ as\ h\longrightarrow 0.$$

Set

$$g(r) = \int_r^\infty x^{2\gamma+d-1}\phi(x)dx.$$

From (2) in lemma 1.3, an integration by parts yields

$$\begin{aligned}
I_1 &\leq -C_1 h^{2m}\int_0^{1/h} r^{2m} g'(r)dr \\
&\leq -C_1 g(1/h) + 2mC_1 h^{2m}\int_0^{1/h} r^{2m-1}g(r)dr \\
&\leq 2mC_1 h^{2m}\int_0^{1/h} r^{2m-1}r^{-2\alpha}\psi(r^{-2\beta})dr \\
&\leq 2mC_1 h^{2m}\int_0^{1/h} r^{2m-2\alpha-1}\psi(r^{-2\beta})dr \\
&\leq C_2 h^{2m}h^{2\alpha-2m}\psi(h^{2\beta}) \\
&\leq C_2 h^{2\alpha}\psi(h^{2\beta}),
\end{aligned}$$

where C_1 and C_2 are positive constants, and this ends the proof

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