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Computing the Field of Moduli of the KFT family*

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Abstract

The computation of the field of moduli of a given closed Riemann surface is in general a very difficult task. In this note we consider the family of closed Riemann surfaces of genus three admitting the symmetric group in four letters as a group of conformal automorphisms and we provide the computations of the corresponding field of moduli.

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1. Introduction

There is a natural one to one correspondence between birational isomorphism classes of non-singular irreducible projective complex algebraic curves and conformal classes of closed Riemann surfaces. If C_1 and C_2 are non-singular complex irreducible projective algebraic curves, we denote by the symbol $C_1 \cong C_2$ to indicate that they are birationally equivalent (that is, the corresponding closed Riemann surfaces are conformally equivalent). We denote by $\mathrm{Gal}(\mathbf{C}/\mathbf{Q})$ the group of field automorphisms of $\mathbf{C}$ and, if $\mathbf{K}$ is a subfield of $\mathbf{C}$, then we denote by $\mathrm{Gal}(\mathbf{C}/\mathbf{K})$ the subgroup of $\mathrm{Gal}(\mathbf{C}/\mathbf{Q})$ formed by those elements acting as the identity on $\mathbf{K}$. It is known that the fixed subfield of $\mathrm{Gal}(\mathbf{C}/\mathbf{K})$ still $\mathbf{K}$ (since $\mathbf{C}$ is algebraically closed of characteristic zero). If $P \in \mathbf{C}[x_0, ..., x_n]$ is any polynomial and if $\sigma \in \mathrm{Gal}(\mathbf{C}/\mathbf{Q})$, then $P^\sigma \in \mathbf{C}[x_0, ..., x_n]$ will denote the polynomial obtained by applying σ to each of the coefficients of P.

Let $C \subset \mathbf{P}^n$ be a non-singular irreducible projective complex algebraic curve, say defined by the homogeneous polynomials $P_1, ..., P_r \in \mathbf{C}[x_0, ..., x_n]$. If $\sigma \in \mathrm{Gal}(\mathbf{C}/\mathbf{Q})$, then the polynomials $P_1^\sigma, ..., P_r^\sigma$ define a new non-singular irreducible projective complex algebraic curve C^σ. The *field of moduli* of C, denoted by $\mathcal{M}(C)$, is the fixed subfield of the group $G_C = \{\sigma \in \mathrm{Gal}(\mathbf{C}/\mathbf{Q}) : C^\sigma \cong C\}$. Notice from the definition that if $C \cong \widehat{C}$ and $\sigma \in \mathrm{Gal}(\mathbf{C}/\mathbf{Q})$ then $C^\sigma \cong \widehat{C}^\sigma$; in particular $G_C = G_{\widehat{C}}$. In this way, if S is a closed Riemann surface and C is any non-singular irreducible projective algebraic curve defining S, then we may set $G_S = G_C$ and define the *field of moduli* of S as $\mathcal{M}(S) = \mathcal{M}(C)$.

A *field of definition* of a closed Riemann surface S is a subfield $\mathbf{K}$ of $\mathbf{C}$ for which it is possible to find a non-singular irreducible projective complex algebraic curve C, whose Riemann surface structure is conformally equivalent to S, defined by homogeneous polynomials with coefficients in $\mathbf{K}$; it is said that S is *definable over* $\mathbf{K}$. If the closed Riemann surface S is definable over $\mathbf{K}$, say by the algebraic curve C, and $\sigma \in \mathrm{Gal}(\mathbf{C}/\mathbf{K})$, then $C^\sigma = C$; so it follows that $\mathcal{M}(S) < \mathbf{K}$, that is, every field of definition of S contains its field of moduli. By results of Koizumi [13] it is also known that $\mathcal{M}(S)$ is equal to the intersection of all the fields of definition of S and, by results of Hammer-Herrlich [8], S is always definable over a finite extension of its field of moduli.

If the genus of S is zero, then S is conformally equivalent to the Riemann sphere $\widehat{\mathbf{C}}$; so it can be defined over its field of moduli $\mathbf{Q}$. If S has genus one, then it can be described by a curve of the form (Legendre normal form)

$E_\lambda : y^2 = x(x-1)(x-\lambda)$, where $\lambda \in \mathbf{C} - \{0,1\}$. A direct consequence of the fact that any two conformal automorphisms of order two with fixed points (necessarily four fixed points) of E_λ are conjugate in the group of conformal automorphisms, is that the field of moduli of E_λ is $\mathbf{Q}(j(\lambda))$, where j is the classical j-function. It is also known that E_λ can be defined over $\mathbf{Q}(j(\lambda))$ [17, Chapter III, Prop. 1.4].

Let us assume, from now on, that S has genus at least two and let $\mathrm{Aut}(S)$ be its full group of conformal automorphisms. It is a well known fact that $|\mathrm{Aut}(S)| \leq 84(g-1)$ (Hurwitz's bound) [11]. In this case, $\mathcal{M}(S)$ is not in general a field of definition of S, as it is shown by explicit examples provided by Earle [4] and Shimura [16] in the case of hyperelliptic case and by the author [9] in the non-hyperelliptic case. Sufficient conditions for S to be definable over its field of moduli are given by Weil's Galois descent theorem [19]. If $\mathrm{Aut}(S)$ is trivial, then (as a direct consequence of Weil's Galois descent theorem) S can be defined over its field of moduli. Unfortunately, Weil's conditions are in general very difficult to check if $\mathrm{Aut}(S)$ is non-trivial. But, in the particular case that $S/\mathrm{Aut}(S)$ has signature of the form $(0; a, b, c)$ (one says that S is quasiplatonic), Wolfart [21] proved that S can be defined over its field of moduli (which is a number field by Belyi's theorem [1]). The computation of the field of moduli of S is in general a difficult task. Moreover, if we have computed explicitly the field of moduli, to determine if S can be defined over it is also a difficult problem (except for some simple cases). Even, if we already have explicitly the field of moduli and we know that the surface can be defined over it, it is a very hard problem to compute an algebraic curve defined over it that represents the surface.

In this paper we work out the family of closed Riemann surfaces of genus three admitting the symmetric group in four letters S_4 as a group of conformal automorphisms. It is well known that, up to conformal equivalence, there is only one such hyperelliptic surface; which is described by the hyperelliptic curve $C : y^2 = x^8 + 14x^4 + 1$; so it is already defined over its field of moduli $\mathbf{Q}$. In the non-hyperelliptic case, there are exactly two conformal classes for which the full group of conformal automorphisms is bigger than S_4: Fermat's curve $F : x^4 + y^4 + z^4 = 0$ and Klein's curve $K : x^3y + y^3z + z^3x = 0$. It is well known that $|\mathrm{Aut}(F)| = 96$ and that $|\mathrm{Aut}(K)| = 168$ and that $F/\mathrm{Aut}(F)$ has signature $(0; 2, 4, 8)$ and that $K/\mathrm{Aut}(K)$ has signature $(0; 2, 3, 7)$. Again, these quasiplatonic surfaces are defined over their field of moduli $\mathbf{Q}$. In [15] there is provided a pencil of non-singular quartic curves C_λ, where $\lambda \in \mathcal{P} = \mathbf{C} - \{-2, -1, 2\}$, called the

KFT family. Each non-hyperelliptic Riemann surface of genus three admitting S_4 as a group of conformal automorphisms is represented by a quartic in the KFT family and, conversely, every member of the KFT family is of such type of Riemann surfaces. For three especial values of $\lambda \in \mathcal{P}$ the quartics correspond to Klein's quartic and Fermat's quartic. For the rest of the parameters, they correspond to closed Riemann surfaces of genus three with full group of automorphisms isomorphic to S_4; the quotient orbifold of signature $(0; 2, 2, 2, 3)$. In Section 2 we will compute the field of moduli of each memeber of the KFT family and will notice that the quartics provided in the KFT family, with the exception of Klein's quartic, are already defined over them.

Recently, we have noticed the paper [6] on which the KFT family (and also other families of genus three curves) has been considered and their results may also be used to compute the corresponding fields of moduli.

2. The field of moduli of the KFT Family

In this section we consider the family of closed Riemann surfaces of genus three admitting the symmetric group S_4 as a group of conformal automorphisms and we provide the field of moduli of these surfaces and explicit equations in these fields.

2.1. The hyperelliptic case

As the hyperelliptic involution is in the center of the group of conformal automorphisms of a hyperelliptic Riemann surface, it is not difficult to see that there is exactly one, up to biholomorphisms, hyperelliptic Riemann surface S_0 of genus 3 with a group of conformal automorphisms isomorphic to S_4. If we quotient S_0 by the hyperelliptic involution, then we obtain that the 8 cone points of order 2 should be invariant under the action of a group of Möbius transformations isomorphic to S_4. This permits to see that $\mathrm{Aut}(S_0) = S_4 \oplus \mathbf{Z}/2\mathbf{Z}$ and that $S_0/\mathrm{Aut}(S_0)$ has signature $(0; 2, 4, 6)$. Using the above information, one can see that S_0 can be represented by the algebraic curve $C : y^2 = x^8 + 14x^4 + 1$, that is, S_0 can be defined over $\mathbf{Q}$.

2.2. The non-hyperelliptic case

A well known fact is the topological rigidity property on the action of the group S_4 as group of conformal automorphisms of closed non-hyperelliptic Riemann surfaces of genus $g = 3$.

Theorem 1 (Broughton 2). *If (S, H) and (R, K) are so that S and R are non-hyperelliptic Riemann surfaces of genus* 3 *and $H \cong K \cong S_4$ are respective group of conformal automorphisms, then there is an orientation preserving homeomorphism between the surfaces conjugating the groups.*

In Section 3 we provide a simple proof, based on Fuchsian groups, of Theorem 1 as a matter of completeness.

Remark 2. *The hyperelliptic Riemann surface $C : y^2 = x^8 + 14x^4 + 1$ admits two different subgroups H_1 and H_2 inside* $\mathrm{Aut}(C)$ *with $H_j \cong S_4$ so that C/H_1 has signature $(0; 2, 4, 6)$ and C/H_2 has signature $(0; 2, 2, 2, 3)$. If (S, H) is so that S is non-hyperelliptic Riemann surface and $S_4 \cong H <$ $\mathrm{Aut}(S)$, then there is a an orientation preserving homeomorphism $f : S \to C$ so that $H_2 = fHf^{-1}$. A description of these Riemann surfaces, from the point of view of Schottky uniformizations, can be found in [10].*

Let S be a non-hyperelliptic closed Riemann surface of genus $g = 3$ and let $S_4 \cong H < \mathrm{Aut}(S)$. As a consequence of the Riemann-Hurwitz formula [5], the orbifold S/H has signature $(0; 2, 2, 2, 3)$. It follows that the locus, in the moduli space of genus three Riemann surfaces, of the classes of Riemann surfaces admitting the non-hyperelliptic action of S_4 is one-complex dimensional.

As a consequence of Singerman's list [18] of maximal Fuchsian groups, one has that either $\mathrm{Aut}(S) = H$ or $S/\mathrm{Aut}(S)$ has signature of the form $(0; a, b, c)$.

If $S/\mathrm{Aut}(S)$ has signature of the form $(0; a, b, c)$, then S is quasiplatonic and so it can be defined over its field of moduli [21]. The set of these quasiplatonic surfaces form a finite subset up to conformal equivalence. Apart from the hyperelliptic case, there are only other two such Riemann surfaces; Fermat's curve $F : x^4 + y^4 + z^4 = 0$ and Klein's curve $K : x^3y + y^3z + z^3x = 0$. It is well known that $|\mathrm{Aut}(F)| = 96$ and that $|\mathrm{Aut}(K)| = 168$ and that $F/\mathrm{Aut}(F)$ has signature $(0; 2, 4, 8)$ and that $K/\mathrm{Aut}(K)$ has signature $(0; 2, 3, 7)$. All of these quasiplatonic surfaces are defined over their field of moduli, that is, over $\mathbf{Q}$. If $\mathrm{Aut}(S) = H$, the generic situation, then we will see that S can be defined over its field of moduli and we will in fact compute it.

2.2.1. The KFT family

It is well known that the canonical embedding of a non-hyperelliptic Riemann surface of genus 3 is a non-singular projective algebraic curves of degree 4 (a quartic) in the complex projective plane $\mathbf{P}^2$. A description of such quartics for the family of non-hyperelliptic Riemann surfaces of genus 3 admitting S_4 as group of conformal automorphisms has been done in [15]; called the KFT family. A study of such a family from the point of view of idempotents has been done in [7]. This family has been studied in [3, 14, 20]. The quartics in the KFT family are of the form [15]

$$C_\lambda = \left\{x^4 + y^4 + z^4 + \lambda(x^2y^2 + y^2z^2 + z^2x^2) = 0\right\} \subset \mathbf{P}^2,$$

where $\lambda \in \mathcal{P} = \mathbf{C} - \{-2, -1, 2\}$. The curves C_λ, where $\lambda \in \{\pm 2, -1\}$ are singular quartics.

The group $H \cong S_4$, for each member of of the KFT family, is generated by the transformations

$$A([x : y : z]) = [y : -x : -z], \quad B([x : y : z]) = [x : z : y].$$

As a consequence of Theorem 1, every non-hyperelliptic Riemann surface of genus 3 admitting a group of conformal automorphisms isomorphic to S_4 is represented by one of the curves in the KFT family. Conversely, every curve C_λ, with $\lambda \in \mathcal{P}$, is a closed Riemann surface of genus 3 admitting the group H as a group of conformal automorphisms.

Remark 3. *The quartic C_0 corresponds to Fermat's curve $x^4+y^4+z^4 = 0$, for which $|\mathrm{Aut}(C_0)| = 96$ and $C_0/\mathrm{Aut}(C_0)$ has signature $(0; 2, 3, 8)$. The quartics $C_{3\alpha} \cong C_{3\overline{\alpha}}$, where $\alpha = (-1+i\sqrt{7})/2$, correspond to Klein's quartic and $C_{3\alpha}/\mathrm{Aut}(C_{3\alpha})$ has signature $(0; 2, 3, 7)$. An extra automorphism of order 7 of this quartic is given by $C([x : y : z]) = [-x + y + \overline{\alpha}z : \alpha(x + y) : -x + y - \overline{\alpha}z]$.*

As for each $\sigma \in \mathrm{Gal}(\mathbf{C}/\mathbf{Q})$ one has that $C_\lambda^\sigma = C_{\sigma(\lambda)}$, it follows that the orbits under the action of $\mathrm{Gal}(\mathbf{C}/\mathbf{Q})$ of such a family are given as:

1. the orbit of C_π (containing exactly all curves of the form C_λ, where λ is transcendental); and
2. the orbits of the curves $C_{\lambda_1}, \ldots, C_{\lambda_n}, \ldots$, where $\lambda_1, \ldots, \lambda_n, \ldots \in \overline{\mathbf{Q}}$ is a maximal collection of algebraic numbers non-equivalent under the absolute Galois group $\mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$

2.2.2. Equivalence of curves

In order to find explicitly the field of moduli $\mathcal{M}(C_\lambda)$, we first need to provide conditions on λ_1 and λ_2 for C_{λ_1} and C_{λ_2} to be conformally equivalent.

Let $\mathbf{G} = \langle \eta(z) = z/(z-1) \rangle \cong \mathbf{Z}_2$ and let $F(z) = z^2/(z-1)$. The map F is a regular branched cover with $\mathbf{G}$ as Deck group.

As $\lambda \in \mathcal{P}$, we have that $\lambda^2 - \lambda - 2 \neq 0$. In this way, for each simply-connected subset D of $\mathcal{P}$ we may choose one of the branchs of $\sqrt{\lambda^2 - \lambda - 2}$ to get an analytic map $f(\lambda) = \sqrt{\lambda^2 - \lambda - 2}$ defined over D.

Let us fix $\lambda \in \mathcal{P}$ and let us consider the map $Q : C_\lambda \to \widehat{\mathbf{C}}$ defined as

$$Q([x:y:z]) = \frac{x^2y^2z^2}{(x^2+y^2+z^2)^3}.$$

As the polynomials $x^2y^2z^2$ and $x^2+y^2+z^2$ are invariant under A and B, we obtain that $Q \circ A = Q = Q \circ B$. It follows from Bezout's theorem that Q has degree 24. In particular, $Q : C_\lambda \to \widehat{\mathbf{C}}$ is a regular branched cover with H as Deck group of cover transformations.

If we fix one of the two values of $\sqrt{\lambda^2 - \lambda - 2}$, then the branch values of Q are given by the points ∞ (of order 3), 0 (of order 2) and the following two points (each of order 2)

$$l_1(\lambda) = \frac{(2+\lambda)\left(\lambda + \sqrt{\lambda^2-\lambda-2}\right)^2}{\left(2-\lambda-2\sqrt{\lambda^2-\lambda-2}\right)^3}$$

$$l_2(\lambda) = \frac{(2+\lambda)\left(\lambda - \sqrt{\lambda^2-\lambda-2}\right)^2}{\left(2-\lambda+2\sqrt{\lambda^2-\lambda-2}\right)^3}$$

Notice that the branch value 0 is the projection under Q of the fixed points of the conjugates of A^2 and that the branch values $l_1(\lambda)$ and $l_2(\lambda)$ are the projections of the fixed points of those elements of order two conjugate to B.

Let us consider the Möbius transformation T_λ so that $T_\lambda(0) = 0$, $T_\lambda(l_1(\lambda)) = \infty$ and $T_\lambda(l_2(\lambda)) = 1$, that is,

$$T_\lambda(z) = \left(\frac{l_2(\lambda) - l_1(\lambda)}{l_2(\lambda)}\right) \frac{z}{z - l_1(\lambda)}.$$

The map $\pi_\lambda = T_\lambda \circ Q$ defines a regular branched cover with H as Deck group of cover transformations whose branch values of order 2 are ∞, 0 and 1 and the one of order 3 is

$$\mu(\lambda) = \frac{4(2+\lambda)^2\left(\sqrt{\lambda^2-\lambda-2}\right)^3}{\left(\lambda-\sqrt{\lambda^2-\lambda-2}\right)^2\left(\lambda-2+2\sqrt{\lambda^2-\lambda-2}\right)^3} \in \mathbf{C} - \{0,1\}.$$

Remark 4. *If we change to the other value of $\sqrt{\lambda^2-\lambda-2}$ the roles of $l_1(\lambda)$ and $l_2(\lambda)$ get interchanged. In this case, the Möbius transformation that fixes 0 and sends $l_1(\lambda)$ to ∞ and $l_2(\lambda))$ to 1 is given by*

$$\widehat{T}(z)=\eta(T_\lambda(z))=\left(\frac{l_1(\lambda)-l_2(\lambda)}{l_1(\lambda)}\right)\frac{z}{z-l_2(\lambda)}$$

and the branch values of the cover map $\widehat{T}\circ Q_\lambda : C_\lambda \to \widehat{\mathbf{C}}$ are ∞, 0 and 1 (of order 2) and the one of order 3 is $\eta(\mu(\lambda))=\mu(\lambda)/(\mu(\lambda)-1)$. In this way, to each $\lambda\in\mathcal{P}$ we have associated the two values $\mu(\lambda)$ and $\eta(\mu(\lambda))$ depending on the choice of $\sqrt{\lambda^2-\lambda-2}$. Let us also notice that

$$\mu(\lambda)+\eta(\mu(\lambda))=\mu(\lambda)^2/(\mu(\lambda)-1)=F(\mu(\lambda))=G(\lambda)=\frac{16(1+\lambda)^3}{27(2+\lambda)}$$

is well defined over all $\mathcal{P}$.

Let $\mathcal{P}_0$ be the subset of $\mathcal{P}$ consisting of those values for which $\mathrm{Aut}(C_\lambda)\neq H$. Then $\mathcal{P}_0$ is a set of isolated points. Notice that if $C_{\lambda_1}\cong C_{\lambda_2}$, then $\lambda_1\in\mathcal{P}_0$ if and only if $\lambda_2\in\mathcal{P}_0$. Parts (1) and (3) of the next result was also obtained, by a different method, in [6].

Theorem 5.

1. *If $\lambda_1,\lambda_2\in\mathcal{P}-\mathcal{P}_0$, then C_{λ_1} and C_{λ_2} are conformally equivalent if and only if $\lambda_1=\lambda_2$.*

2. *If $\lambda_1,\lambda_2\in\mathcal{P}-\mathcal{P}_0$, then C_{λ_1} and C_{λ_2} are anticonformally equivalent if and only if $\lambda_1=\overline{\lambda_2}$.*

3. *$\mathcal{P}_0=\{0,3(-1\pm i\sqrt{7})/2\}$. The curves $C_{3(-1-i\sqrt{7})/2}$ and $C_{3(-1+i\sqrt{7})/2}$ are equivalent to Klein's curve and the curve C_0 is Fermat's curve.*

Proof. Given any two points $\lambda_1,\lambda_2\in\mathcal{P}$, we may consider a simply connected domain $D\subset\mathcal{P}$ containing the points λ_1 and λ_2. Once this is done, we make a choice for a analytic branch of $\sqrt{\lambda^2-\lambda-2}$ in D. Using such a choice, we have fixed the choices of $\pi_\lambda(z)$ and of $\mu(\lambda)$ for $\lambda\in D$ (both are analytic on the parameter $\lambda\in D$).

Case(1)

We assume $\lambda_1, \lambda_2 \in D - \mathcal{P}_0$. If $C_{\lambda_1} \cong C_{\lambda_2}$, then there is a conformal homeomorphism $f : C_{\lambda_1} \to C_{\lambda_2}$. As $\mathrm{Aut}(C_{\lambda_1}) = H = \mathrm{Aut}(C_{\lambda_2})$, it follows that there is a Möbius transformation M so that $\pi_{\lambda_2} \circ f = M \circ \pi_{\lambda_1}$. Moreover, $M(0) = 0$, $M(\mu(\lambda_1)) = \mu(\lambda_2)$ and $M(\{1, \infty\}) = \{1, \infty)\}$. It follows that either $M(z) = z$ (in which case $\mu(\lambda_2) = \mu(\lambda_1)$) or $M(z) = \eta(z)$ (in which case $\mu(\lambda_2) = \eta(\mu(\lambda_1))$). We have obtained that necessarily $G(\lambda_1) = G(\lambda_2)$. In this way, we obtain that

$$\lambda_2 \in \left\{ \lambda_1, -\left(\frac{6 + 5\lambda_1 + \lambda_1^2 - (1 + \lambda_1)\sqrt{\lambda_1^2 - 4}}{2(2 + \lambda_1)} \right), \right.$$

$$\left. -\left(\frac{6 + 5\lambda_1 + \lambda_1^2 + (1 + \lambda_1)\sqrt{\lambda_1^2 - 4}}{2(2 + \lambda_1)} \right) \right.$$

Notice that if $\lambda_1 = -5/2$, then $\lambda_2 = \lambda_1$ as $\lambda_2 \neq -1$. We assume now on that $\lambda_1 \neq -5/2$.

Let us consider the Riemann orbifolds $\mathcal{O}_j = C_{\lambda_j}/\langle AB \rangle$ which has signature $(1; 3, 3)$. It was obtained in [15] that the j-invariant of the underlying Riemann surface structure T_j of $\mathcal{O}_j$ is

$$j_3(\lambda) = \frac{(16\lambda^2 + 48\lambda + 33)^3}{108(1 + \lambda)(2 + \lambda)}$$

Similarly, we may consider the orbifolds (all of them of genus one) obtained by quotient C_{λ_j} by the cyclic groups $\langle A \rangle$, $\langle B \rangle$ and $\langle A^2 \rangle$. The corresponding j-invariantes are

$$j_4(\lambda) = \frac{(\lambda^2 + 18\lambda + 33)^3}{108(1 + \lambda)^4(2 + \lambda)}$$

$$j_2(\lambda) = \frac{-(\lambda^2 - 12\lambda - 12)^3}{108(1 + \lambda)(2 + \lambda)^4}$$

$$j_{2,2}(\lambda) = \frac{4(\lambda^2 + 3\lambda + 3)^3}{27(1 + \lambda)^2(2 + \lambda)^2}$$

Next we make a comparison of $j_2, j_3, j_4, j_{2,2}$ for the three above possible values for λ_2 and those for λ_1 (this can be done with any computational software) and we obtain that the only possibility is $\lambda_2 = \lambda_1$ (as $\lambda_1 \neq -5/2$)

Case(2)

The anticonformal situation is worked in a similar fashion as the previous case.

Case (3)

Now we assume that $\lambda_1, \lambda_2 \in D \cap \mathcal{P}_0$. We know that C_{λ_1} is conformally equivalent to either Klein's curve (which is given by $\lambda = 3(-1 + i\sqrt{7})/2$ [15]) or Fermat's curve (given with $\lambda = 0$). As each of them can be defined over $\mathbf{R}$, each of them is conformally equivalent to their conjugates, that is, $C_{\lambda_j} \cong C_{\overline{\lambda_j}}$, for $j = 1, 2$.

If $C_{\lambda_1} \cong C_{\lambda_2}$, then there exist sequences $\lambda_{1,n}, \lambda_{2,n} \in D - \mathcal{P}_0$ so that $\lambda_{1,n} \to \lambda_1$ and $\lambda_{2,n} \to \lambda_2$ as $n \to +\infty$ and so that $C_{\lambda_{1,n}} \cong C_{\lambda_{2,n}}$. By the previous case, we have that $\lambda_{2,n} = \lambda_{1,n}$. It follows that $\lambda_2 = \lambda_1$. $\square$

It follows from Theorem 5 that the locus in $\mathcal{M}_3$ (the moduli space of genus 3) of the classes of non-hyperelliptic Riemann surfaces admitting S_4 as a group of conformal automorphisms is given by the set $\mathcal{P}$ after identification of the points $3(-1-i\sqrt{7})/2$ with its conjugate $3(-1+i\sqrt{7})/2$.

Corollary 6. *The normalization of the locus in moduli space of genus* 3 *consisting of classes of non-hyperelliptic Riemann surfaces admitting the symmetric group* S_4 *as group of conformal automorphisms is isomorphic to the* $\mathcal{P} = \mathbf{C} - \{-2, -1, 2\}$. *The puncture corresponding to the point* -2 *corresponds to the hyperelliptic curve admitting* S_4 *as a group of conformal automorphisms.*

2.2.3. Fields of moduli

The following result states that, except for the Klein curve, the KFT family provides equations on the corresponding field of moduli.

Corollary 7.

1. *If* $\lambda \in \mathcal{P} - \mathcal{P}_0$, *then* $\mathcal{M}(C_\lambda) = \mathbf{Q}(\lambda)$.
2. *If* $\lambda \in \mathcal{P}_0$, *then* $\mathcal{M}(C_\lambda) = \mathbf{Q}$.

Proof. If $\lambda \in \mathcal{P}_0$, then C_λ is either Klein's curve or Fermat's curve, both of them can be defined over $\mathbf{Q}$.

Let $\lambda \in \mathcal{P} - \mathcal{P}_0$. If $\sigma \in \mathrm{Gal}(\mathbf{C}/\mathbf{Q})$, then, as $C_\lambda^\sigma = C_{\sigma(\lambda)}$, it follows from Theorem 5 that $C_\lambda^\sigma \cong C_\lambda$ if and only if $\sigma(\lambda) = \lambda$. $\square$

3. Fuchsian uniformization of the KFT family

For each $\lambda_1 \in \mathcal{P}$ there are three values for $\lambda \in \mathcal{P}$ so that $G(\lambda) = G(\lambda_1)$; one of them being clearly λ_1 and the others two are given by

$$\lambda_2 = -\left(\frac{6 + 5\lambda_1 + \lambda_1^2 - (1+\lambda_1)\sqrt{\lambda_1^2 - 4}}{2(2+\lambda_1)}\right)$$

$$\lambda_3 = -\left(\frac{6 + 5\lambda_1 + \lambda_1^2 + (1+\lambda_1)\sqrt{\lambda_1^2 - 4}}{2(2+\lambda_1)}\right).$$

We have that in $\mathcal{P}$ there is no solution for $\lambda_1 = \lambda_3$ or for $\lambda_2 = \lambda_3$ and there is exactly one solution for $\lambda_1 = \lambda_2$, this being for $\lambda_1 = -5/2$. Notice that in this case $\lambda_3 = 2 \notin \mathcal{P}$ corresponds to the hyperelliptic curve admitting S_4 as a group of conformal automorphisms.

The curves C_{λ_1}, C_{λ_2} and C_{λ_3} can also be seen as follows from a Fuchsian uniformization's point of view. Let us consider the orbifold of signature $(0; 2, 2, 2, 3)$ whose cone points of order 2 are given by ∞, 0, 1 and the cone point of order 3 is $\mu(\lambda_1) = \mu_1$ (once we have fixed a value for $\sqrt{\lambda^2 - \lambda - 2}$). Let us consider a Fuchsian group

$$\Gamma = \langle x_1, x_2, x_3 : x_1^2 = x_2^2 = x_3^2 = (x_1x_2x_3)^3 = 1\rangle$$

acting on the unit disc $\mathbf{D}$ and a universal branched cover $P : \mathbf{D} \to \widehat{\mathbf{C}}$ with Γ as Deck group of covering transformations so that the fixed point of x_3 projects by P to 0, the fixed point of x_1 projects to ∞ and the fixed point of x_2 projects to 1. The fixed point of $x_1x_2x_3$ projects to μ_1.

As a consequence of results due to L. Keen [12] there is a fundamental domain for Γ given by a suitable hyperbolic triangle Δ_1, say with sides s_{11}, s_{12} and s_{13} counted in counterclockwise order, so that x_j is an involution with fixed point at the middle side of s_{1j}.

In order to find the torsion free normal subgroups F of Γ so that $\Gamma/F \cong S_4$, up to inner conjugation in Γ, we only need to find all possible different surjective homomorphisms $\Theta : \Gamma \to S_4$ with torsion free kernel up to post-composition with automorphisms of S_4.

Since $S_4 = \langle A, B : A^4 = B^2 = (BA)^3 = 1 \rangle$, up to post-composition by a suitable automorphism of S_4, we may assume that $\Theta(x_1x_2x_3) = (BA)^{-1}$ and $\Theta(x_3) = A^2$. As there is no non-trivial automorphism of S_4 that fixes A^2 and BA, we have that we cannot post-compose with other non-trivial automorphisms of S_4 without destroying the above choices.

Now, in order for the kernel of Θ to be torsion free, we need to have that $\Theta(x_1)$ and $\Theta(x_2)$ are order two elements of S_4 so that $\Theta(x_1)\Theta(x_2) = A^{-1}BA^2$.

By direct inspection one obtains that the only possible choices are given by:

1. $\Theta(x_1) = B$, $\Theta(x_2) = ABA^{-1}$;
2. $\Theta(x_1) = ABA^{-1}$, $\Theta(x_2) = (BA)B(BA)^{-1}$;
3. $\Theta(x_1) = (BA)B(BA)^{-1}$, $\Theta(x_2) = B$.

Each of the above three choices provides a Fuchsian group F as desired (they can be computed with GAP). These three Fuchsian groups provide the uniformization of C_{λ_1}, C_{λ_2} and C_{λ_3}.

If we consider the elements $y_1 = x_2$, $y_2 = (x_1x_2x_3)^{-1}x_1(x_1x_2x_3)$ and $y_3 = x_3$, then we have the relations $y_1^2 = y_2^2 = y_3^2 = (y_1y_2y_3)^3 = 1$ and that $\Gamma = \langle y_1, y_2, y_3 \rangle$.

Again, by the results in [12], there is another fundamental domain for Γ given by a suitable hyperbolic triangle Δ_2, say with sides s_{21}, s_{22} and s_{23} counted in counterclockwise order, so that y_j is an involution with fixed point at the middle side of s_{2j}. This permits to see that there is an orientation preserving self-homeomorphism $h_1 : : \mathbf{D} \to \mathbf{D}$ that $h_1 \circ x_j \circ h_1^{-1} = y_j$, for each $j = 1, 2, 3$; in particular, h_1 self-conjugates Γ into itself. Next we observe that if $\Theta(x_1) = B$, $\Theta(x_2) = ABA^{-1}$ and $\Theta(x_3) = A^2$, then $\Theta(y_1) = ABA^{-1}$, $\Theta(y_2) = (BA)B(BA)^{-1}$ and $\Theta(y_3) = A^2$. In this way, both Fuchsian groups obtained in (1) and (2) are conjugated by the orientation preserving homeomorphism h_1.

If we consider the elements $z_1 = x_1x_2x_1$, $z_2 = x_1$ and $z_3 = x_3$, then we have the relations $z_1^2 = z_2^2 = z_3^2 = (z_1z_2z_3)^3 = 1$ and that $\Gamma = \langle z_1, z_2, z_3 \rangle$. Again, as a consequence of [12] a fundamental domain for Γ is given by a suitable hyperbolic triangle Δ_3, say with sides s_{31}, s_{32} and s_{33} counted in counterclockwise order, so that z_j is an involution with fixed point at the middle side of s_{3j}. This permits to see that there is an orientation preserving self-homeomorphism $h_2 : \mathbf{D} \to \mathbf{D}$ that $h_2 \circ x_j \circ h_2^{-1} = z_j$, for each $j = 1, 2, 3$; in particular, h_2 self-conjugates Γ into itself. Next we observe that if $\Theta(x_1) = B$, $\Theta(x_2) = ABA^{-1}$ and $\Theta(x_3) = A^2$, then $\Theta(y_1) = (BA)B(BA)^{-1}$, $\Theta(y_2) = B$ and $\Theta(y_3) = A^2$. In this way, both Fuchsian groups obtained in (1) and (3) are conjugated by the orientation preserving homeomorphism h_2.

All of the above also provides a proof of Theorem 1.

Remark 8. *Let us denote the internal angles of the triangle Δ_1 by θ_1, θ_2 and θ_3, so that θ_1 is the angle between the sides s_{11} and s_{12}, θ_2 is the angle between s_{12} and s_{13} and θ_3 is the angle between s_{13} and s_{11}. Clearly, $\theta_1 + \theta_2 + \theta_3 = 2\pi/3$. In the particular case when $\theta_1/2 = \theta_2 = \theta_3 = \pi/6$, there is a conformal automorphism $U : \mathbf{D} \to \mathbf{D}$ of order 4 with the same fixed points as for x_3 (so $U^2 = x_3$). The image under U of the triangle Δ_1 is a new triangle, say Δ_4, whose sides, counted in counterclockwise order are $s_{41} = U(s_{12})$, $s_{42} = U(s_{11})$ and $s_{43} = s_{13}$. Let $w_1 = x_2$, $w_2 = x_3x_1x_3$ and $w_3 = x_3$. Then w_j is an involution with a fixed point in the middle of the side s_{4j}.*

If

$$(\Theta(x_1), \Theta(x_2), \Theta(x_3)) = (B, ABA^{-1}, A^2),$$

then

$$(\Theta(w_1), \Theta(w_2), \Theta(w_3)) = (ABA^{-1}, A^2BA^2, A^2),$$

so

$$(A^{-1}\Theta(w_1)A, A^{-1}\Theta(w_2)A, A^{-1}\Theta(w_3)A) = (B, ABA^{-1}, A^2).$$

If

$$(\Theta(x_1), \Theta(x_2), \Theta(x_3)) = (ABA^{-1}, (BA)B(BA)^{-1}, A^2),$$

then

$$(\Theta(w_1), \Theta(w_2), \Theta(w_3)) = ((BA)B(BA)^{-1}, A^{-1}BA, A^2),$$

so

$$(A^2\Theta(w_1)A^2, A^2\Theta(w_2)A^2, A^2\Theta(w_3)A^2) = ((BA)B(BA)^{-1}, ABA^{-1}, A^2).$$

As a consequence, this is the case corresponding to $G(\lambda) = 4$. The curve $C_{-5/2}$ is uniformized by any of the two Fuchsian groups appearing in (2) and (3) and a hyperelliptic one is uniformized by the one appearing in (1).

References

[1] G. V. Belyi. On Galois extensions of a maximal cyclotomic field. *Izv. Akad. Nauk SSSR Ser. Mat.* 43, pp. 267-276, 479, (1979).

[2] S. A. Broughton. Classifying finite group actions on surfaces of low genus, *J. Pure Applied Algebra* 69, pp. 233-270, (1990).

[3] I. Dolgachev and V. Kanev. Polar covariants of plane cubics and quartics. *Advances in Math.* 98, pp. 216-301, (1992).

[4] C. J. Earle. On the moduli of closed Riemann surfaces with symmetries. *Advances in the Theory of Riemann Surfaces* (1971) 119-130. Ed. L.V. Ahlfors et al. (Princeton Univ. Press, Princeton).

[5] H. Farkas and I. Kra. *Riemann Surfaces*. Second edition. Graduate Texts in Mathematics **71**. Springer-Verlag, New York, (1992).

[6] Y. Fuertes and M. Streit. Genus 3 normal coverings of the Riemann sphere branched over 4 points. *Rev. Mat. Iberoamericana* 22 No. 2, pp. 413-454, (2006).

[7] M. Guizatullin. Bialgebra and geometry of plane quartics, Preprint Max-Planck-Institute für Mathematik 46, (2000).

[8] H. Hammer and F. Herrlich. A Remark on the Moduli Field of a Curve. *Arch. Math.* 81, pp. 5-10, (2003).

[9] R. A. Hidalgo. Non-hyperelliptic Riemann surfaces with real field of moduli but not definable over the reals. *Archiv der Mathematik* 93, pp. 219-222, (2009).

[10] R. A. Hidalgo. Schottky uniformizations of genus 3 and 4 reflecting $\mathcal{S}_4$. *Journal of the London Math. Soc.* 72, No. 1, pp. 185-204, (2005).

[11] A. Hurwitz. Über algebraische Gebilde mit eindeutigen Transformationen in sich. *Math. Ann.* 41, pp. 403-442, (1893).

[12] L. Keen. Canonical polygons for finitely generated Fuchsian groups. *Acta Mathematica* **115** No.1, pp. 1-16, (1966).

[13] S. Koizumi. The fields of moduli for polarized abelian varieties and for curves. *Nagoya Math. J.* 48, pp. 37-55, (1972).

[14] J. Ries. Splittable jacobian varieties. *Contemporary Mathematics*, 136, pp. 305-326, (1992).

[15] R. E. Rodríguez and V. González-Aguilera. Fermat's Quartic Curve, Klein's Curve and the Tetrahedron. *In Contemporary Mathematics*, 201, pp. 43-62, (1997).

[16] G. Shimura. On the field of rationality for an abelian variety. *Nagoya Math. J.* 45, 167-178, (1972).

[17] J. Silverman. *The Arithmetic of Elliptic Curves.* GTM, Springer-Verlag, (1986).

[18] D. Singerman. Finitely Maximal Fuchsian Groups. *J. London Math. Soc.* 6, No. 2, pp. 29-38, (1972).

[19] A. Weil. The field of definition of a variety. *Amer. J. Math.*, 78, pp. 509-524, (1956).

[20] A. Wiman. Über die hyperelliptischen curven und diejenigen vom geschlechte p=3 welche eindeutige transformationen in sich zulassen. Bilhang till Kongl Svenska Veteskjaps Handliger Stockholm, 21, pp. 1-23, (1985).

[21] J. Wolfart. *ABC* for polynomials, dessins d'enfants and uniformization a survey. Elementare und analytische Zahlentheorie, 313–345, Schr. Wiss. Ges. Johann Wolfgang Goethe Univ. Frankfurt am Main, 20, Franz Steiner Verlag Stuttgart, Stuttgart, (2006).

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