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Hardy-Type Spaces and its Dual

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Abstract

In this paper we defined a new Hardy-type spaces using atoms on homogeneous spaces which we call $H^{\varphi,q}$*. Also we prove that under certain conditions* $BMO_{\varphi}^{(p)}$ *is the dual of* $H^{\varphi,q}$.

Keywords : *BMO, Dual space, Hardy space, Space of homogeneous type.*

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1. Introduction

The Hardy space H^p were first studied on the unit disk in the complex plane. In their 1968 paper Duren, Romberg and Shield (see [4]) make the following definitions and comments about H^p. For $0 < p \leq \infty$, H^p is the linear space of functions $f(z)$ analytic in $|z| < 1$ such that

$$M_p(r, f) = \left(\frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^p d\theta \right)^{\frac{1}{p}}, \quad 0 < p < \infty$$

or

$$M_\infty(r, f) = \max_{0 \leq \theta < 2\pi} |f(re^{i\theta})|$$

remains bounded as $r \to 1$. If $1 \leq p \leq \infty$, H^p is a Banach space under the norm

$$\|f\|_p = \lim_{r \to 1} M_p(r, f).$$

For $0 < p < 1$, this is not a norm, but H^p is still a complete metric space with a translation invariant metric

$$d(f, g) = \|f - g\|_p^p.$$

A linear functional φ on H^p is bounded ($\varphi \in (H^p)^*$) if

$$\|\varphi\| = \sup_{\|f\|_p = 1} |\varphi(f)| < \infty.$$

It is easily verify that $(H^p)^*$ is a Banach space. Duren, Romberg and Shield (see [4]) were the first to study the linear space structure of the H^p space with $0 < p < 1$. These H^p spaces are not Banach spaces and are not locally convex.

They may be regarded as closed subspaces of L_p for $0 < p < 1$; however,it is interesting to note that although there are no continuous linear functionals on L_p for $0 < p < 1$, there are many on H^p. Duren, Romberg and Shield (see [4]) prove for $1/2 < p < 1$, that $(H^p)^* = \Lambda_\alpha$ the Lipschitz space of order $\alpha = \frac{1}{p} - 1$. For $p \leq \frac{1}{2}$, the results are similar. Even though H^p is not locally convex, there are still enough linear functionals to distinguish elements. For example as noted in [4], $g(z) = (1 - \xi z)^{-1}$ generates the functional $\varphi(f) = f(\xi)$.

Later, the study of H^p spaces was extended to $H^p(\mathbf{R}^n)$. The results were highly specialized to $\mathbf{R}^n$ until Latter (see [5]), Coifman and Weiss (see [3]) defined $H^p(\mathbf{R}^n)$ using the notion of an atom and proved that the atomic $H^p(\mathbf{R}^n)$ space were equivalent to the original $H^p(\mathbf{R}^n)$. Roughly speaking, an atom is a "building block" function which is supported on a ball, has zero integral and has a bounded average.

By thinking of the H^p spaces in terms of atoms Coifman and Weiss (see [2]) were able to prove that the dual of H^p is again a Lipschitz space of order $\alpha = \frac{1}{p} - 1$ not only in $\mathbf{R}^n$, but on any homogeneous space $\mathcal{X}$. The H^p space for $0 < p \le 1$ on $\mathbf{R}^n$ were first characterized in terms of atoms by Coifman (see [3]) and Latter (see [6]). Coifman and Weiss (see [2]) then used this characterization to define $H^p(\mathcal{X})$, where $\mathcal{X}$ is a homogeneous space.

In this paper, we extend the work of Coifman and Weiss (see [3]) by defining new Hardy-type spaces using atoms on homogeneous space which we call $H^{\varphi,q}$. The main result of this paper is the following.

Theorem 1.1. Suppose φ and w are related by

$$w^{-1}(t) = \frac{t}{\varphi\left(\frac{1}{t}\right)}$$

or equivalent by

$$\varphi(t) = \frac{1}{tw^{-1}(\frac{1}{t})}.$$

Suppose also that $\dfrac{\varphi(t)}{t}$ is a decreasing function of t and that $\dfrac{\varphi(t)}{t^\epsilon}$ is an increasing function for some $0 < \epsilon < 1$. Let $1 \le q < \infty$, and let p be conjugate of q. Then the dual of $H^{\varphi,q}$ is BMO^p_φ.

2. Atoms

We begin by defining atoms. The idea for the relationship between w and φ functions come from Janson's paper (see [5]). Throughout this paper, we will assume that the measure μ is a regular measure.

Definition 2.1. A measurable function a is said to be a (φ, q) atom if it satisfies:

1. The support of a is contained in a ball $B(x_0, r)$,
2. $\int a d\mu = 0$,
3. $\left(\frac{1}{\mu(B)} \int_B |a|^q d\mu\right)^{\frac{1}{q}} \le w^{-1}\left(\frac{1}{\mu(B)}\right)$,

where w and φ are related by

$$w^{-1}(t) = \frac{1}{\varphi\left(\frac{1}{t}\right)} \quad or \quad \varphi(t) = \frac{1}{tw^{-1}(\frac{1}{t})}.$$

Note that

$$w^{-1}\left(\frac{1}{\mu(B)}\right) = \frac{1}{\mu(B)\varphi(\mu(B))}$$

and that (3) can be written as

$$\|a\|_q \leq \mu(B)\left(w^{-1}\left(\frac{1}{\mu(B)}\right)\right)^q,$$

where $B = B(x_0, r)$.

3. Spaces of Homogeneous type

Let us begin by recalling the notion of space of homogeneous type.

Definition 3.1. A quasimetric d on a set $\mathcal{X}$ is a function $d : \mathcal{X} \times \mathcal{X} \to [0, \infty)$ with the following properties:

1. $d(x, y) = 0$ if and only if $x = y$.
2. $d(x, y) = d(y, x)$ for all $x, y \in \mathcal{X}$.
3. There exists a constant K such that

$$d(x, y) \leq K\left[d(x, z) + d(z, y)\right],$$

   for all $x, y, z \in \mathcal{X}$.

A quasimetric defines a topology in which the balls

$$B(x, r) = \{y \in \mathcal{X} : d(x, y) < r\}$$

form a base. These balls may be not open in general; anyway, given a quasimetric d, is easy to construct an equivalent quasimetric d' such that the d'-quasimetric balls are open (the existence of d' has been proved by using topological arguments in [7]). So we can assume that the quasimetric balls are open. A general method of constructing families $\{B(x, \delta)\}$ is in terms of a quasimetric.

Example 3.1. A space of homogeneous type $(\mathcal{X}, d, \mu)$ is a set $\mathcal{X}$ with a quasimetric d and a Borel measure μ finite on bounded sets such that, for some absolute positive constant A the following doubling property holds

$$\mu\left(B(x, 2r)\right) \leq A\mu\left(B(x, r)\right)$$

for all $x \in \mathcal{X}$ and $r > 0$.

Next, we are ready to give some example of a space of homogeneous type.

Example 3.2. Let $\mathcal{X} \subset \mathbf{R}^n$, $\mathcal{X} = \{0\} \cup \{x : |x| = 1\}$, put in $\mathcal{X}$ the euclidean distance and the following measure μ: μ is the usual surface measure on $\{x : |x| = 1\}$ and $\mu\left(\{0\}\right) = 1$. Then μ is doubling so that $(\mathcal{X}, d, \mu)$ is a homogeneous space.

Example 3.3. In $\mathbf{R}^n$, let C_k $(k = 1, 2, \cdots)$ be the point $(k^k + 1/2, 0, \cdots, 0)$, for $k \geq 2$, let B_k be the ball $B(C_k, 1/2)$ and $B_1 = B(0, 1/2)$. Let $\mathcal{X} = \cup_{k=1}^{\infty} B_k$ with the euclidean distance and the measure μ such that $\mu\left(B_k\right) = 2^k$ and on each ball B_k, μ is uniformly distributed.
Claim 1. μ satisfies the doubling condition. Let $B_r = B(P, r)$ with $P = (P_1, \ldots, P_n)$ *and* $r > 0$.
Case 1. Assume for some k, $B_k \subset B_r$ and let $k_0 = \max\{k : B_k \subset B_r\}$. Then certainly $P_1 + r \leq b_{k_0+1} = (k_0 + 1)^{k_0+1} + 1$ and $\mu(B_r) \geq 2^{k_0}$. But, then

$$\begin{aligned} P_1 + 2r &\leq 2\left((k_0 + 1)^{k_0+1} + 1\right) \\ &\leq (k_0 + 2)^{k_0+2} = a_{k_0+2}. \end{aligned}$$

Therefore $B_{2r} \subset B_{a_{k_0+2}}(0) \equiv B_0$. But

$$\mu(B_0) = \sum_{k=0}^{k_0+1} 2^k \leq 2^{k_0+2} \leq 4\mu(B_r).$$

Hence the doubling condition holds with $A = 4$.
Case 2. If for all k, $B_k B_r$, then $r < 1$ so that B_r and B_{2r} intersect only on ball B_k. Then the doubling condition holds.

4. φ-Lipschitz space

We define the φ-Lipschitz space and denoted it by $\mathcal{L}_\varphi$ to be the space for all measurable functions f on $\mathcal{X}$ for which

$$|f(x) - f(y)| \leq C\varphi(\mu(B)),$$

where B is any ball containing both x and y and C is a constant depending only of f.

Let $\gamma(f)$ be the inf over all C for which the above inequality holds. Then if we define

$$\|f\|_\varphi^{\mathcal{L}} = \begin{cases} \gamma(f) & if \quad \mu(\mathcal{X}) = \infty \\ \gamma(f) + \int_{\mathcal{X}} f d\mu & if \quad \mu(\mathcal{X}) = 1, \end{cases}$$

a straightforward argument shows that $\mathcal{L}_\varphi$, with this norm, is a Banach space. To simplify calculations, we assume that if $\mu(\mathcal{X})$ is finite, then $\mu(\mathcal{X}) = 1$. We now define $H^{\varphi,q}$ to be the subspace of $(\mathcal{L}_\varphi)^*$ consisting of those linear functionals admitting an atomic decomposition as follows:

$h \in H^{\varphi,q}$ if h can be written as a sum $h = \sum_{j\in N} \lambda_j a_j$, where a_j is a (φ, q) atom, and $\sum_{j\in N} w(|\lambda_j|) < \infty$. We denote by the symbol $\|h\|_{\varphi,q}$ the quantity (which is not, in general a norm)

$$\|h\|_{\varphi,q} = \inf_{alldescompof \ f} w^{-1}\left(\sum_{j\in N} w(|\lambda_j|)\right).$$

Example. If $\varphi(t) = t^{\frac{1}{p}-1}$, then $w(t) = t^p$ and $H^{\varphi,q}(\mathcal{X}) = H^p(\mathcal{X})$.

5. Functions of Bounded (φ, p) mean Oscillation

In this section, we recall the definition of the space of functions of bounded (φ, p) mean oscillation, $BMO_\varphi^{(p)}(\mathcal{X})$, where $\mathcal{X}$ is a space of homogeneous type. Let φ be a nonnegative function on $[0, \infty)$. A locally μ-integrable function $f : \mathcal{X} \to \mathbf{R}$ is said to belong to the class $BMO_\varphi^{(p)}(\mathcal{X})$, $1 \leq p < \infty$, if

$$\sup \left(\frac{1}{\mu(B)[\varphi(\mu(B))]^p} \int_B |f(x) - f_B|^p d\mu(x)\right)^{\frac{1}{p}} < \infty.$$

Where the sup is taken over all balls $B \subset \mathcal{X}$ and

$$f_B = \frac{1}{\mu(B)} \int_B f(y) d\mu.$$

For more detail on functions of bounded (φ, p) mean oscillation see Castillo, Ramos Fernández and Trousselot [1].

6. Quasi-Concavity

In this section, we study the notion of quasi-concavity, which is the condition that we will need to prove our main result.

Definition 6.1. A non-negative function ϕ is said to be quasi-convex if there exists a convex function A and a constant $C > 1$ such that

$$A(t) \leq \phi(t) \leq CA(t).$$

Definition 6.2. A function ψ is said to be quasi-concave if there exists a constant $C > 1$ and a concave function M such that

$$CM(Ct) \leq \psi(t) \leq M(t).$$

We will use the following Lemmas to prove that the function W as introduced in the definition of a (φ, q) atom is quasi-concave under appropriate conditions on φ.

Lemma 6.1. Suppose that $\dfrac{\varphi(x)}{x}$ is a decreasing function of x, and suppose also that $\dfrac{\varphi(x)}{x^\epsilon}$ is an increasing function for some $0 < \epsilon < 1$. Let

$$\psi(x) = \int_0^x \frac{\varphi(t)}{t} dt.$$

Then ψ is concave, φ is quasi-concave, and $x\psi(x)$ is quasi-convex.

Proof: The derivative

$$\psi'(x) = \frac{\varphi(x)}{x}$$

is decreasing by hypothesis. Therefore, ψ is concave. To show that φ is quasi-concave, first note that $\varphi(x) \leq \psi(x)$ since

$$\psi(x) = \int_0^x \frac{\varphi(t)}{t} dt \geq \int_0^x \frac{\varphi(x)}{x} dt = \varphi(x).$$

To show the other inequality, we estimate $\psi(Cx)$, for $C < 1$ by

$$\begin{aligned} \psi(Cx) &= \int_0^{Cx} \frac{\varphi(t)}{t} dt = \int_0^{Cx} \frac{\varphi(t)}{t^\epsilon t^{1-\epsilon}} dt \leq \frac{\varphi(x)}{x^\epsilon} \int_0^{Cx} t^{\epsilon-1} dt \\ &= \frac{\varphi(x)}{x^\epsilon} \frac{(Cx)^\epsilon}{\epsilon} = \varphi(x) \frac{C^\epsilon}{\epsilon}. \end{aligned}$$

Therefore, we have

$$\frac{\epsilon}{C^{\epsilon}}\psi(x) \le \varphi(x).$$

Next, we choose C by letting $C = \epsilon^{\frac{1}{\epsilon+1}}$.

Since $0 < \epsilon < 1$, C also satisfies $C < 1$ and $C\psi(Cx) \le \varphi(x)$. Thus, we have shown that φ is quasi-concave.

To show that $x\psi(x)$ is quasi-convex, let $g(x) = x\psi(x)$. Note that $\dfrac{g(x)}{x} = \psi(x)$ is increases, so

$$A(x) = \int_0^x \frac{g(u)}{u}du$$

is convex. Also,

$$A(x) \quad = \quad \int_0^x \frac{g(u)}{u}du \le \frac{g(x)}{x}x = g(x),$$

so $A(x) \le g(x)$.

We also have

$$A(2x) \quad = \quad \int_0^{2x} \frac{g(u)}{u}du \ge \int_x^{2x} \frac{g(u)}{u}du \ge \frac{g(x)}{x}x = g(x),$$

thus

$$g(x) \le A(2x) \le 2A(2x),$$

and we have shown that

$$A(x) \le g(x) \le 2A(2x).$$

Therefore, g is quasi-convex, which completes the proof.

Lemma 6.2. 1. φ is quasi-concave if and only if there exists a constant $C < 1$ such that

$$\frac{\varphi(t_1)}{t_1} \ge \frac{C\varphi(Ct_2)}{t_2}$$

for all $0 \le t_1 \le t_2$.

2. ψ is quasi-convex if and only if there exists a $C > 1$ such that

$$\frac{\varphi(t_1)}{t_1} \ge \frac{C\varphi(Ct_2)}{t_2}$$

for all $0 < t_1 \le t_2$.

Proof: of (1) ($\Rightarrow$) since φ is quasi-concave, we have M concave and $C < 1$ such that

$$CM(Ct) \leq \varphi(t) \leq M(t).$$

Now, $\dfrac{M(t)}{t}$ is a non-increasing function of t, so, for all $0 < t_1 \leq t_2$, we have

$$\frac{M(t_1)}{t_1} \geq \frac{M(t_2)}{t_2}.$$

Thus,

$$\begin{aligned}\frac{\varphi(t_1)}{t_1} &\geq \frac{CM(Ct_1)}{t_1} \geq \frac{C^2M(Ct_1)}{Ct_1} \geq \frac{C^2M(Ct_2)}{Ct_2} \\ &= \frac{CM(Ct_2)}{t_2} \geq \frac{C\varphi(Ct_2)}{t_2}.\end{aligned}$$

($\Leftarrow$) Let $C < 1$, $t_1 \leq t_2$, and suppose $\dfrac{\varphi(t_1)}{t_1} \geq \dfrac{C\varphi(Ct_2)}{t_2}$. Consider the function

$$\psi(t) = \frac{1}{C}\int_0^{\frac{t}{C}} \inf_{x<s<\frac{t}{C}} \frac{\varphi(s)}{s} dx.$$

Then ψ is concave by Lemma 6.1. Also, as in the proof of Lemma 1, we have

$$C\psi(t) = \int_0^{\frac{t}{C}} \frac{\varphi(x)}{x} dx = \int_0^{\frac{t}{C}} \frac{\varphi(x)}{x^{\epsilon}x^{1-\epsilon}} dx$$

$\leq \frac{\varphi\left(\frac{t}{C}\right)}{\left(\frac{t}{C}\right)^{\epsilon}} \left(\frac{t}{C}\right)^{\epsilon} = \varphi\left(\frac{t}{C}\right).$

Thus, $C\psi(Ct) \leq \varphi(t)$, which gives us the first inequality in the quasi-concavity definition. For the other inequality, note that since $\dfrac{\varphi(t)}{t}$ decreases, and $C < 1$,

$$\psi(t) = \int_0^{\frac{t}{C}} \frac{\varphi(x)}{x} dx \geq \int_0^{t} \frac{\varphi(x)}{x} dx \geq \varphi(t).$$

Thus, we have shown that $C\psi(Ct) \leq \varphi(t) \leq \psi(t)$, where ψ is concave, proving that φ is quasi-concave.

The proof of (2) is similar to the above proof of (1).

7. Duality

Theorem 1.1 and its proof are modeled on $H^{\varphi,q}$, where $\varphi(t) = t^{1/p-1}$ and $w(t) = t^p$. Clearly, in this case, $\dfrac{\varphi(t)}{t}$ decreases and $\dfrac{\varphi(t)}{t^\epsilon}$ increases for some $0 < \epsilon < 1$. To prove Theorem 1.1, we let L be a bounded linear functional on $H^{\varphi,q}$, and we fix a ball B in $\mathcal{X}$. We show first that L is a bounded linear functional on the subspace

$$L_0^q(B) = \left\{ f \in L^q(B) : \int_B f d\mu = 0 \right\}$$

of $L^q(B)$. Then, using the Hahn-Banach Theorem and the Riesz Representation Theorem, we extend L to $L^q(B)$ with the same norm, and we uniquely represent L by an integral with L^p function g. Using an increasing sequence of balls converging to $\mathcal{X}$, we then find a unique function g such that if $f \in L^q(B)$,

$$Lf = \int_B fgd\mu,$$

for any ball B. Finally, by making a (φ, q) atom from $f - f_B$, we show that $g \in BMO_\varphi^{(p)}$, and we note that by Hölder Inequality, $BMO_\varphi^{(p)} \subset BMO_\varphi$.

To show that any $g \in BMO_\varphi^{(p)}$ defines a bounded linear functional on $H^{\varphi,q}$, we first show for an atom $a \in H^{\varphi,q}$, supported on a ball B,

$$\left| \int_B gad\mu \right| \leq \|g\|_{BM_\varphi^{(p)}}$$

for $h \in H^{\varphi,q}$, we decompose h into a sum of (φ, q)-atoms and we use the quasi-concavity of w to show that

$$\left| \int_B ghd\mu \right| \leq \|g\|_{BMO_\varphi^{(p)}} w^{-1} \left(\frac{2}{C^4} \|h\|_{H^{\varphi,q}} \right).$$

Therefore, g defines a bounded linear functional on $H^{\varphi,q}$ given by

$$L_g(H) = \int ghd\mu$$

and

$$\|L\| \leq C\|g\|_{BMO_\varphi^{(p)}}.$$

This shows that L_g is a bounded linear functional on $H^{\varphi,q}$.

Proof of Theorem 1.1 The proof of this Theorem follows along the same lines as the proof of [2]. Let L be a bounded linear functional on $H^{\varphi,q}$, and let $\|L\|$ be the norm of L. Fix a ball B in $\mathcal{X}$. Let

$$L_0^q(B) = \left\{ f \in L^q(B) : \int_B f d\mu = 0 \right\}.$$

It follows that if $f \in L_0^q(B)$, then

$$a(x) = \frac{[\mu(B)]^{1/q}}{\|f\|_{L^q(B)}} w^{-1}\left(\frac{1}{\mu(B)}\right) f(x)$$

is a (φ, q) atom, since, by (2) of the atomic definition

$$\begin{aligned} \left(\frac{1}{\mu(B)} \int_B |a(x)|^q d\mu(x)\right)^{\frac{1}{q}} &\leq \frac{[\mu(B)]^{1/q}}{[\mu(B)]^{1/q}\|f\|_{L^q}(B)} w^{-1}\left(\frac{1}{\mu(B)}\right) \|f\|_{L^q(B)} \\ &\leq w^{-1}\left(\frac{1}{\mu(B)}\right). \end{aligned}$$

We also have

$$\|f\|_{H^{\varphi,q}} \leq \frac{1}{w^{-1}\left(\frac{1}{\mu(B)}\right)} \frac{\|f\|_{L^q(B)}}{[\mu(B)]^{1/q}}.$$

Hence, Lf is defined and

$$\|Lf\| \leq \|L\| \frac{\|f\|_{L^q(B)}}{[\mu(B)]^{1/q}}.$$

That is, L is a bounded linear functional on $L_0^q(B)$. By the Hahn-Banach Theorem, we can extend L to $L^q(B)$ with the same norm and by the Riesz Representation Theorem, we can conclude that there exists $g \in L^p(B)$ such that $Lf = \int_B fgd\mu$ for all $f \in L_0^q(B)$.

The function g is uniquely determined up to a constant , or, equivalently if $\int_B fgd\mu = 0$ for all $f \in L_0^q(B)$, then it follows that g is a constant. To see this, suppose $\int_B fgd\mu = 0$ for all $f \in L_0^q(B)$. Choose $h \in L^q(B)$.

Since $h - h_B \in L_0^q(B)$, we have

$$0 = \int_B g(h - h_B)d\mu = \int_B (gh - gh_B)d\mu = \int h(g - g_B)d\mu.$$

Since this equality holds for all $h \in L^q(B)$, it must be true that $g(x) = g_B$ a.e. x in B.

Let $\{B_j\}_{j=1}^{\infty}$ be an increasing sequence of balls converging to $\mathcal{X}$, such that $\mu(B_1) > 0$. We obtain a function $\tilde{g}_j$ satisfying

$$Lf = \int_{B_j} f\tilde{g}_j d\mu \tag{7.1}$$

for each j. Now, let

$$g_j = \tilde{g}_j - (\tilde{g}_j)_{B_1}.$$

Then, $\int_{B_1} g_j d\mu = 0$. It remains to show that $g_j|_{B_k} = g_k$ for all $k \le j$. By the above remark, we know that on $B_k \supset B_1$, we have $g_j - g_k = C$. Now, integrate both sides over B_1 to obtain

$$\int_{B_1} (g_j - g_k) d\mu = \int_{B_1} C d\mu,$$

which implies that $0 = \mu(B_1)$. Therefore, $C = 0$, and we conclude that $g_j|_{B_k} = g_k$ for $k \le j$.

We now have a unique function g such that if $f \in L^q(B)$, then

$$Lf = \int_B fg d\mu,$$

which holds for any ball B.

In particular, if a is a (φ, q) atom supported in B, we have

$$\|L\| \ge |La| = \left|\int_B gad\mu\right| = \left|\int_B (g - g_B) ad\mu\right|, \tag{7.2}$$

if f is supported in B and $\|f\|_{L^q} = 1$, then

$$a = \frac{[\mu(B)]^{1/q}}{2} w^{-1}\left(\frac{1}{\mu(B)}\right)(f - f_B)$$

is a (φ, q) atom, since

$$\begin{aligned}
\left(\frac{1}{\mu(B)}\int_B |a(x)|^q d\mu\right)^{\frac{1}{q}} &= \frac{[\mu(B)]^{1/q}}{2[\mu(B)]^{1/q}} w^{-1}\left(\frac{1}{\mu(B)}\right)\left(\int_B |f - f_B|^q d\mu\right)^{\frac{1}{q}} \\
&\le \frac{w^{-1}\left(\frac{1}{\mu(B)}\right)}{2} 2\left(\int_B |f|^q d\mu\right)^{\frac{1}{q}} \\
&\le w^{-1}\left(\frac{1}{\mu(B)}\right).
\end{aligned}$$

Now, using this atom in (7.2) above, and using the fact that $g-g_B$ has mean zero on B, we obtain

$$\left|\int_B (g-g_B)\frac{[\mu(B)]^{1/q}}{2}w^{-1}\left(\frac{1}{\mu(B)}\right)(f-f_B)d\mu\right| \leq \|L\|,$$

which implies that

$$\left|\int_B (g-g_B)\frac{[\mu(B)]^{1/q}}{2}w^{-1}\left(\frac{1}{\mu(B)}\right)fd\mu\right| \leq \|L\|,$$

which in turn implies that

$$\left|\int_B (g-g_B)fd\mu\right| \leq \frac{2\|L\|}{[\mu(B)]^{1/q}w^{-1}\left(\frac{1}{\mu(B)}\right)}.$$

If we now take the supremum of all f supported in B such that $\|f\|_{L^q}=1$, we obtain

$$\begin{aligned}\|g-g_B\|_{L^p} &\leq \frac{2\|L\|}{[\mu(B)]^{1/q}w^{-1}\left(\frac{1}{\mu(B)}\right)} \\ &= \frac{2\|L\|}{[\mu(B)]^{1/q}}\mu(B)\varphi(\mu(B)) \\ &= 2\|L\|[\mu(B)]^{1/q}\varphi(\mu(B)).\end{aligned}$$

Rewriting this inequality, we obtain

$$\left(\frac{1}{\mu(B)[\varphi(\mu(B))]^p}\int_B |g-g_B|^p d\mu\right)^{\frac{1}{p}} \leq 2\|L\|,$$

so $g\in BMO_\varphi^{(p)}$. By Hölder's inequality, we also have

$$\begin{aligned}\frac{1}{\varphi(\mu(B))\mu(B)}\int_B |g(x)-g_B|d\mu(x) &\leq \frac{[\mu(B)]^{1/q}}{\mu(B)\varphi(\mu(B))}\left(\int_B |g-g_B|^p d\mu\right)^{\frac{1}{p}} \\ &\leq \left(\frac{1}{\mu(B)[\varphi(\mu(B))]^p}\int_B |g-g_B|^p d\mu\right)^{\frac{1}{p}}.\end{aligned}$$

So $g\in BMO_\varphi$, also.

We have now shown that

$$(H^{\varphi,q})^* \subset BMO_\varphi^{(p)} \subset BMO_\varphi.$$

Now, suppose that $g \in BMO_\varphi^{(p)}$. We will show that g defines a bounded linear functional on $H^{\varphi,q}$. Let a be a (φ, q) atom. Then

$$\begin{aligned}
\left|\int_B gad\mu\right| &= \left|\int_B (g-g_B)a(u)d\mu\right| \leq \int |g-g_B|a(u)d\mu \\
&\leq \left(\int_B |g-g_B|^p d\mu\right)^{\frac{1}{p}} \left(\int_B |a(u)|^q d\mu\right)^{\frac{1}{q}} \\
&\leq \left(\int_B |g-g_B|^p d\mu\right)^{\frac{1}{p}} [\mu(B)]^{1/q} w^{-1}\left(\frac{1}{\mu(B)}\right) \\
&\leq \frac{[\mu(B)]^{1/q}}{\mu(B)\varphi(\mu(B))}\left(\int_B |g-g_B|^p d\mu\right)^{\frac{1}{p}} \\
&\leq \left(\frac{1}{\mu(B)[\varphi(\mu(B))]^p}\int_B |g-g_B|^p d\mu\right)^{\frac{1}{p}} \\
&\leq \|g\|_{BMO_\varphi^{(p)}}.
\end{aligned}$$

Therefore, we have shown that

$$\left|\int_B gad\mu\right| \leq \|g\|_{BMO_\varphi^{(p)}}. \tag{7.3}$$

Now, let $h \in H^{\varphi,q}$ and let $h = \sum_{j=1}^\infty \alpha_j a_j$ be decomposition of h into (φ, q) atom such that

$$w^{-1}\left(\sum w(|\alpha_j|)\right) \leq (1+C^4)\|h\|_{H^{\varphi,q}},$$

where $C < 1$ is the quasi-concavity constant for w. Since $C < 1$ then

$$\begin{aligned}
\sum |\alpha_j| &\leq w^{-1}\left(\frac{1}{C}\sum w\left(\frac{|\alpha_j|}{C}\right)\right) \\
&\leq w^{-1}\left(\frac{1}{C}\sum w\left(\frac{C|\alpha_j|}{C^2}\right)\right).
\end{aligned}$$

Now, let $t_1 = |\alpha_j|$, and let $t_2 = \dfrac{|\alpha_j|}{C^2}$. Since $C < 1$, we have $t_1 < t_2$ and by Lemma 6.2 implies that

$$C\frac{w(Ct_2)}{t_2} \leq \frac{w(t_1)}{t_1}.$$

Therefore, we have

$$w(Ct_2) \leq \frac{t_2}{t_1}\frac{w(t_1)}{C}$$

which implies that

$$w\left(C\frac{|\alpha_j|}{C^2}\right) \leq \frac{|\alpha_j|}{C^2|\alpha_j|}\frac{w(|\alpha_j|)}{C} = \frac{w(|\alpha_j|)}{C^3}.$$

Thus

$$\begin{aligned}\sum|\alpha_j| &\leq w^{-1}\left(\frac{1}{C}\sum\frac{w(|\alpha_j|)}{C^3}\right) = w^{-1}\left(\frac{1}{C^4}\sum w(|\alpha_j|)\right)\\ &\leq w^{-1}\left(\frac{1}{C^4}(1+C^4)\|h\|_{H^{\varphi,q}}\right)\\ &\leq w^{-1}\left(\frac{2}{C^4}\|h\|_{H^{\varphi,q}}\right).\end{aligned}$$

Now, if $g \in BMO_{\varphi}^{(p)}$, since $C < 1$, we have

$$\left|\int ghd\mu\right| \leq \sum|\alpha_j|\left|\int ga_jd\mu\right|$$

$\leq \|g\|_{BMO_{\varphi}^{(p)}} w^{-1}\left(\frac{2}{C^4}\|h\|_{H^{\varphi,q}}\right).$

Therefore, g defines a bounded linear functional L on $H^{\varphi,q}$ given by

$$L_g(h) = \int ghd\mu,$$

which satisfies

$$\|L_g\| = \sup_{\|h\|_{H^{\varphi,q}=1}} |L_g(h)| \leq w^{-1}\left(\frac{2}{C^4}\right)\|g\|_{BMO_{\varphi}^{(p)}}.$$

Thus,

$$BMO_{\varphi}^{(p)} \subset (H^{\varphi,q})^*$$

and the Theorem is proved.

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