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Some characterization theorems on dominating chromatic partition-covering number of graphs

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Abstract

Let $G = (V, E)$ be a graph of order $n = |V|$ and chromatic number $(\mathcal{G})$. A dominating set D of G is called a dominating chromatic partition-cover or dcc-set, if it intersects every color class of every $\mathcal{X}$-coloring of G. The minimum cardinality of a dcc-set is called the dominating chromatic partition-covering number, denoted $dcc(G)$. The dcc-saturation number equals the minimum integer k such that every vertex $v \in V$ is contained in a dcc-set of cardinality k. This number is denoted by $dccs(G)$. In this paper we study a few properties of these two invariants $dcc(G)$ and $dccs(G)$.

Keywords : *Dominating set, chromatic partition, dominating chromatic partition-covering number.*

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1. Introduction, terminology and definitions

By a graph $G = (V, E)$ we mean a finite, undirected graph without loops or multiple edges, of order $n = |V|$ and size $m = |E|$. For graph theoretic terminology not given here, the reader is referred to Harary [5].

One of the fastest growing areas within graph theory is the study of domination and related subset problems, such as independence, covering and matching. A comprehensive treatment of the fundamentals of domination is given in the book by Haynes et al. [8]. Surveys of several advanced topics in domination can be seen in the book edited by the same authors [7]. Perhaps the most studied area of graph theory is the study of graph colorings, or partitions of either V or E according to certain rules. Topics lying in the intersection of these two areas are starting to appear in the literature, such as partitions of V into dominating sets and the corresponding invariant called the domatic number (cf. Chapter 13 by Zelinka in [7]. This paper is a contribution to this intersection.

We will need several definitions. A set $S \subseteq V$ is called a *dominating set* if every vertex in $V - S$ is adjacent to a vertex in S. The minimum cardinality of a dominating set in G is called the *domination number* and is denoted by $\gamma(G)$; also a dominating set of minimum cardinality is called a γ-set. A set $S \subset V$ is called *independent* if no two vertices in S are adjacent. A k-coloring of G is a partition $\pi = \{V_1, V_2, \ldots, V_k\}$ of V into k independent sets, called *color classes*. The *chromatic number* of G, denoted $(\mathcal{G})$, equals the minimum integer k such that G has a k-coloring. A *chromatic partition* is a partition $\pi = \{V_1, V_2, \ldots, V_k\}$ of V into $(\mathcal{G})$ independent sets, or equivalently, a chromatic partition is a $\mathcal{X}$-*coloring*.

We say that a set $S \subset V$ *covers* a partition π if $S \cap V_i \neq \emptyset$ for every i, $1 \leq i \leq k$. A *chromatic partition cover* is a set S that covers every chromatic partition of G. If S is also a dominating set, then S is called a *dominating chromatic partition-cover* of G, or a *dcc*-set. The minimum cardinality of a *dcc*-set is called the dominating chromatic partition-covering number, denoted $dcc(G)$. The concept of a dominating chromatic-partition cover was first introduced and studied by the present authors in [3], who called $dcc(G)$ the *chromatic transversal domination number*. But since this is not truly a domination number, we have changed the terminology and notation to better reflect the fact these sets are covers (or transversals), but restricted to dominating sets, of the chromatic partitions of a graph. This paper is a continuation of that study.

A vertex $v \in V$ is called $\mathcal{X}$-critical if $(\mathcal{G} - \sqsubseteq) < (\mathcal{G})$. We call this vertex a $\mathcal{X}$-*critical vertex*. If every vertex $v \in V$ is $\mathcal{X}$-critical, then G is called a $\mathcal{X}$-critical graph. It is easy to see, for example, that every complete graph K_n and every cycle C_{2n+1} of odd length is a $\mathcal{X}$-critical graph.

Example 1:

For the above graph G, $S = \{c, d, e\}$ is a *dcc*-set, since S covers every $\mathcal{X}$-partition of G and so $dcc(G) = 3$.

Example 2: Consider any cycle C_n of odd length n. As every vertex v of C_n is a $\mathcal{X}$-critical vertex, it follows that $\{v\}$ is a colour class of some $\mathcal{X}$-partition of C_n. Therefore $dcc(C_n) = n$.

A *dcc*-set S is called minimal if no proper subset of S is also a *dcc*-set. The property of being a *dcc*-set is *super-hereditary* since any superset of a *dcc*-set is also a *dcc*-set. Thus, a set S is a minimal *dcc*-set if and only if for every $v \in S$, $S - \{v\}$ is not a *dcc*-set.

In [1], Acharya introduced the concept of the *domsaturation number* $ds(G)$, that equals the minimum integer k such that every vertex $v \in V$ is contained in a dominating set of cardinality k. Notice that for any graph G, either every vertex $v \in V$ is contained in a γ-set, in which case $ds(G) = \gamma(G)$, or can be added to a γ-set, in which case $ds(G) = \gamma(G) + 1$. Thus, $\gamma(G) \leq ds(G) \leq \gamma(G) + 1$.

A dominating set S of a graph G is called a *global dominating set* if S is also a dominating set of the complement $\overline{G}$ of G. The minimum cardinality of a global dominating set, $\gamma_g(G)$, is called the *global domination number* of G. The *global domsaturation number* $dsg(G)$ equals the minimum integer k such that every vertex v is contained in a global dominating set of cardinalilty k [7].

Finally, The *open neighborhood* $N(v)$ of a vertex $v \in V$ equals the set of vertices adjacent to v in G, that is, $N(v) = \{u|uv \in E\}$. The *closed neighborhood* $N[v]$ of a vertex $v \in V$ is the set $N(v) \cup \{v\}$. Let $S \subseteq V$ be a set of vertices and let $u \in S$. We define $pn[u, S] = N[u] - N[S - u]$ and $pn(u, S) = N(u) - N(S - u)$.

The set of private neighbors of a vertex in $u \in S$ is denoted by $pn[u, S]$. Notice that if a vertex $u \in S$ is not adjacent to any vertex in S, then $u \in pn[u, S]$, in which case we say that u is its *own* private neighbor, while every other private neighbor of u is a vertex in $V - S$.

The following are the important results proved in [3] by the authors.

Theorem 1.1. *([3,9]) A dcc-set S is minimal if and only if for every vertex $u \in S$, at least one of the following holds. (i) $pn[u, S] \neq \emptyset$. (ii) There exists a chromatic partition $\pi = \{V_1, V_2, \ldots, V_k\}$ such that $S \cap V_i = \{u\}$ for some i.*

Theorem 1.2. *([3]) For a connected graph G, $dcc(G) = n$ if and only if G is $\mathcal{X}$-critical.*

Theorem 1.3. *([3]) Let G be a connected bipartite graph of order $n \geq 3$ and vertex bipartition (X, Y) with $|X| \leq |Y|$. Then $dcc(G) = \gamma(G) + 1$ if and only if every vertex in X has at least two leaves as its neighbors in Y.*

Theorem 1.4. *([3]) For any graph G, (i) $\gamma(G) \leq \gamma_g(G) \leq dcc(G)$. (ii) $(\mathcal{G}) \leq dcc(\mathcal{G})$.*

Result 1.5 ([3]).

(i) $dcc(P_n) = \gamma(P_n) = \lceil \frac{n}{3} \rceil$, $n \geq 4$.

(ii) Every $\mathcal{X}$- critical vertex is in every dcc-set.

Theorem 1.6 ([8]). If G is a graph without isolated vertices, then $\gamma(G) \leq 2$.

Theorem 1.7([2]) For a connected bipartite graph G of order $n \geq 3$, $dsg(G) = ds(G)$.

Result 1.8 ([3]).

(i) $dcc(K_n) = dcc(\overline{K_n}) = n$ and $dcc(K_{m,n}) = 2$.

(ii) Let G be a connected bipartite graph with vertex bipartition (X, Y). If there exists a γ-set S of G such that $S \cap X \neq \emptyset$, then $dcc(G) = \gamma(G)$, otherwise $dcc(G) = \gamma(G) + 1$. In particular $dcc(C_n) = \lceil \frac{n}{3} \rceil$, if n is even.

(iii) If every vertex v of a graph G forms a color class of some chromatic partition of G, then $dcc(G) = n$. In particular $dcc(C_n) = n$, if n is odd and

$$dcc\,(W_n) = \begin{cases} n, \text{ if } n \text{ is even;} \\ \\ 3, \text{ if } n \text{ is odd} \end{cases}, \text{ where } W_n \text{ is a whell with}$$

(iv) $dcc(P) = 5$, where P is the Petersen graph.

Theorem 1.9 ([3]). Let G be a connected graph with bipartition (X, Y) where $|X| \leq |Y|$ and $n \geq 3$. Then $dcc(G) = \gamma(G) + 1$ if and only if every vertex in X has at least two neighbours which are leaves.

2. Main Results

Analogous to $ds(G)$ and $dsg(G)$ in [2] we define a number $dccs(G)$ using which we can characterize graphs for which $dcc(G) = n - 1$.

Definition 2.1. *Let G be a graph. The dcc-saturation number of a vertex v, denoted by $dccs(v)$, equals the minimum integer k such that the vertex v is contained in a dcc-set of cardinality k and $dccs(G) = \max\{dccs(v) : v \in V\}$.*

Remark 2.2. *(i) If S is a dcc-set, then for any $u \in V - S$, $S \cup \{u\}$ is a dcc-set and hence $dcc(G) \leq dccs(G) \leq dcc(G) + 1$.*

(ii) For every $v \in V$, $dcc(G) \leq dccs(v) \leq dccs(G)$.

(iii) $ds(G) \leq dsg(G) \leq dccs(G)$.

Definition 2.3. *A graph G is said to be in class I or class II according as $dccs(G) = dcc(G)$ or $dccs(G) = dcc(G) + 1$.*

It can be verified that the following graphs belong to class I:-

(i) All $\mathcal{X}$-critical graphs, in particular K_n, for every n and the wheel W_n, n even.

(ii) The cycle C_n, for every n.

(iii) The Petersen graph in view of a theorem given in [3].

Proposition 2.4. *For $n \geq 4$, P_n is a class I graph if and only if $n \equiv 1 \pmod 3$.*

Proof. Clearly P_4 is a class I graph. Let $n \geq 5$. If $n \not\equiv 1(mod\ 3)$, then by a Theorem in [2], the domsaturation number $d_s(P_n) = \gamma(P_n) + 1$ and consequently $dccs(P_n) \geq d_s(P_n) = dcc(P_n) + 1$. Hence P_n is a class II graph.

Conversely, suppose $n \equiv 1 \pmod 3$. Let $n = 3k + 1; k \geq 2$. Let the vertices of P_n be $\{1, 2, ..., 3k + 1\}$. Then $D_1 = \{1, 3, 6, .., 3(k - 1), 3k\}$, $D_2 = \{2, 5, ..., 3k - 1, 3k\}$ and $D_3 = \{1, 4, 7, ..., 3k - 2, 3k + 2\}$ are dcc-sets and so $dccs(P_n) = dcc(G)$. □

Proposition 2.5. *Let H be a connected graph with $n \geq 3$. Then the corona $H \circ K_1$, is a class I graph if and only if H has no critical vertex.*

Proof. To prove this result, we prove that G is a Class II graph if and only if H has a critical vertex.

Let $dccs(G) = dcc(G) + 1$. Then $dccs(v) = dcc(G) = |V(H)|$ for every $v \in V(H)$. As $dccs(H \circ K_1) = dcc(H \circ K_1) + 1$, there exists a leaf v_0 for which $dccs(v_0) = dcc(H \circ K_1) + 1$. Let u_0 be the support of v_0. Then u_0 is a critical vertex; otherwise $(V(H) - \{u_0\}) \cup \{v_0\}$ becomes a dcc-set containing v_0 of cardinality $dcc(G)$.

Conversely, if u_0 is a critical vertex of H, then u_0 is also a critical vertex of G and so $dccs(v_0) = dcc(G)+1$ where v_0 is a leaf having u_0 as its support. So G is a class II graph. □

Lemma 2.6. *For a connected graph G, $dccs(G) = n$ if and only if G has at most one vertex that is not $\mathcal{X}$-critical.*

Proof. If $dccs(G) = n$, then $dcc(G) = n$ or $n - 1$. If $dcc(G) = n$, then by Theorem 1.2, G becomes a $\mathcal{X}$-critical graph with n vertices.

Let $dcc(G) = n - 1$. Let S be a dcc-set of G. Then, $S = V - \{u\}$ for some $u \in V$. Clearly u is not a critical vertex. As $dccs(G) = n$, we have $dccs(u) = n$. Suppose $w \neq u$ is not critical, then $V - \{w\}$ is a dcc-set containing u, contradicting the fact $dccs(u) = n$.

Conversely, if all the vertices of G are critical vertices, then by Theorem 1.3, $dcc(G) = n = dccs(G)$. If there exists a unique vertex u that is not critical, then $V - \{u\}$ is the only dcc-set in G and so $dccs(G) = n$. □

Lemma 2.7. *For a connected graph G, $dcc(G) = dccs(G) = n - 1$ if and only if G has exactly two vertices u and v that are not critical and they satisfy at least one of the following conditions:*

(a) *one is a support and the other is its only adjacent leaf.*

(b) $\chi(G - \{u, v\}) < (\mathcal{G})$.

Proof. Let G be a connected graph of order $n \geq 2$ with $dcc(G) = dccs(G) = n - 1$. Obviously $(\mathcal{G}) \geq \in$. As $dcc(G) = n - 1$, by Theorem 1.2, G is not a $\mathcal{X}$-critical graph. Let H be a $\mathcal{X}$-critical subgraph of G of maximum size. Let $|V(H)| = k$. As $(\mathcal{H}) = (\mathcal{G})$, $V(H)$ is a chromatic partition cover of G. So $n - 1 = dcc(G) \leq k + n - k2 \leq \frac{n+k}{2}$. This implies that $n \leq k + 2$. Now $n \neq k$, for otherwise G becomes a $\mathcal{X}$-critical graph. So $n = k + 2$ or $k + 1$.

Case 1: Let $n = k + 2$. Let $V(G) - V(H) = \{u, v\}$. Suppose u and v are non-adjacent or both are not leaves, then $V(H)$ becomes a dcc-set of G, a contradiction to $dcc(G) = n - 1$. Therefore u and v are adjacent and one of them, say v, is a leaf. Every vertex w distinct from u and v are critical vertex of G, for otherwise $dcc(G) = p - 2$. So (a) is true in this case.

Case 2: Let $n = k + 1$. When $(\mathcal{G}) = \in$, we have $G \simeq P_3$ satisfying (b). Assume that $(\mathcal{G}) \geq \ni$.

Then $k \geq 3$. Let $V(G) - V(H) = \{v\}$. By definition of H, $H = G - \{v\}$. As $dccs(G) = n - 1$, there exists a vertex u in H such that $V(G) - \{u\}$ is a dcc-set of G containing v. Obviously u and v are not critical vertices of G.

Claim:

item[(i)] No vertex in G is a leaf. item[(ii)] $G - u$ is a $\mathcal{X}$-critical subgraph of G. item[(iii)] u and v are not adjacent in G. item[(iv)] $N(u) \cup N(v) = V(G)$. item[(v)] u and v are the only vertices in G that are not critical.

v cannot be a leaf, for otherwise $dccs(v) = n$. Since H is a $\mathcal{X}$-critical subgraph, no vertex of H is a leaf in G. Hence (i) is true. Suppose (ii) is not true. Then there exists a vertex w in $G - \{u\}$ such that $(\mathcal{G} - \{\sqcap, \sqsupseteq\}) = (\mathcal{G} - \sqcap) = (\mathcal{G})$. Hence $V - \{u, w\}$ becomes a dcc-set in view of (i), a contradiction.

Suppose u and v are adjacent. Then as u and v are not critical vertices, the set $V - \{u, v\}$ is a dcc-set by (i), a contradiction and so (iii) is true. Next we prove that $N(u) \cup N(v) = V(G)$. Suppose $w \notin N(u) \cup N(v)$, then $(\mathcal{G} - \{\sqsubseteq, \sqsupseteq\}) = (\mathcal{G}) - \infty$ as $G - v$ is a $\mathcal{X}$-critical subgraph of G. Similarly $\chi(G - \{u, w\}) = (\mathcal{G}) - \infty$. This implies that u, v, w belong to every dcc-set of G, a contradiction to the fact that $V - \{u\}$ is a dcc-set containing v. This proves (iv).

Suppose $w \neq u, v$, such that w is not a critical vertex of G. By (iv) w is adjacent to either u or v, say u. As u, v, w are not critical vertices, none of the sets $\{u\}$, $\{v\}$, $\{w\}$ will form a color class for any chromatic partition of G. Since u and w are adjacent, they belong to different color classes of every chromatic partition of G. This implies that $V - \{u, v\}$ is a *dcc* set of G by (i), a contradiction. If $\{u, v\}$ is not a color class of any chromatic partition of G, then $V(G) - \{u, v\}$ is a *dcc*-set, a contradiction. So $\chi(G - \{u, v\}) < (\mathcal{G})$ satisfying (b).

The converse is easily proved. □

Theorem 2.8. *For a connected graph G, $dcc(G) = n - 1$ if and only if G has exactly one vertex that is not critical or two vertices u and v, both are not critical vertices satisfying any one of the following properties:*

(i) One is a support and the other is its adjacent leaf.

(ii) $\chi(G - \{u, v\}) < (\mathcal{G})$.

Proof. If $dcc(G) = n - 1$, then $dccs(G) = n - 1$ or n. So by Lemmas 2.6 and 2.7, we get the required result. □

Corollary 2.9. *If G is a connected bipartite graph, then $dcc(G) = n - 1$ if and only if $G \simeq P_3$.*

Proof. Let $dcc(G) = n - 1$. Obviously $n \geq 3$. When $n > 4$ by Theorem 2.8, G will have at least two critical vertices which is impossible. So $G \simeq P_3$. The converse in obvious. □

Theorem 2.10. *For a connected bipartite graph $G(\neq K_2)$, $ds(G) = dccs(G)$.*

Proof. If $G = K_{1,n-1}$, then $ds(G) = 2 = dcc(G)$. Therefore assume that $G \neq K_{1,n-1}$. Then every dominating set has vertices from each partition set. That means, every dominating set is also a chromatic partition cover and hence $dcc(G) \leq \gamma(G)$. Since $\gamma(G) \leq dcc(G)$ is evident, we have that $dcc(G) = \gamma(G)$. Hence $\gamma(G) \leq ds(G) \leq dcc(G) + 1 = \gamma(G) + 1$. Therefore, if $ds(G) \neq dccs(G)$, then $ds(G) = \gamma(G)$ and $dccs(G) = \gamma(G) + 1$. From the first equality follows that every vertex lies on a minimum dominating set. From the second that there is a vertex that does not lie on a minimum dcc-set or rather (since $dcc(G) = \gamma(G)$) on a minimum dominating set. Hence, a contradiction and $dc(G) = dccs(G)$ follows. □

The classification theorem for $dsg(G) = ds(G)$ given in [2] follows as a corollary of Theorem 2.10.

Corollary 2.11. *For a connected bipartite graph* $G(\neq K_2)$, $dsg(G) = ds(G) = dccs(G)$.

Proof. Since $ds(G) \leq dsg(G) \leq dccs(G)$, the result follows from Theorem 2.10. □

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