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A class of multivalent functions defined by generalized Ruscheweyh derivatives involving a general fractional derivative operator

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Abstract

The main aim of the present paper is to obtain a new class of multivalent functions which is defined by making use of the generalized Ruscheweyh derivatives involving a general fractional derivative operator. We study the region of starlikeness and convexity of the class $\Omega_p(\alpha, \beta, \gamma)$. Also we apply the Fractional calculus techniques to obtain the applications of the class $\Omega_p(\alpha, \beta, \gamma)$. Finally, the familiar concept of δ-neighborhoods of p-valent functions for above mentioned class are employed.

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1. Introduction

Let A denote the class of functions that are analytic in the open unit disk $U = \{z \in C : |z| < 1\}$ and let A_p be the subclass of A consisting of the functions f of the form

$$f(z) = z^p - \sum_{k=n+p}^{\infty} a_k z^k, \quad (n \in N) \tag{1.1}$$

where p is some positive integer and f is analytic and p-valent in U.

The generalized fractional derivative operator of order λ, introduced by Srivastava and Saxena [9], [10], is defined as

$$J_{0,z}^{\lambda,\mu,\nu} f(z) = \begin{cases} \frac{1}{\Gamma(1-\lambda)} \frac{d}{dz} \Big\{ z^{\lambda-\mu} \int_0^z (z-\zeta)^{-\lambda} \\ \quad \cdot\, {}_2F_1\left(\mu-\lambda, 1-\nu; 1-\lambda; 1-\frac{\zeta}{z}\right) f(\zeta) d\zeta \Big\}, \\ \qquad\qquad\qquad\qquad (0 \le \lambda < 1) \\ \frac{d^n}{dz^n} J_{0,z}^{\lambda-n,\mu,\nu} f(z), \quad (n \le \lambda < n+1, n \in N) \end{cases} \tag{1.2}$$

where f is an analytic function in a simply connected region of the z-plane containing the origin, and the multiplicity of $(z-\zeta)^{-\lambda}$ is removed by requiring $log(z-\zeta)$ to be real when $z-\zeta > 0$, provided further that,

$$f(z) = O(|z|^k), \qquad (z \to 0) \tag{1.3}$$

In terms of gamma function, we have

$$J_{0,z}^{\lambda,\mu,\nu} z^\rho = \frac{\Gamma(\rho+1)\Gamma(\rho-\mu+\nu+2)}{\Gamma(\rho-\mu+1)\Gamma(\rho-\lambda+\nu+2)} z^{\rho-\mu}, \tag{1.4}$$

$$(0 \le \lambda < 1, \rho > max\{0, \mu-\nu-1\} - 1)$$

It follows at once from the above definition that

$$J_{0,z}^{\lambda,\lambda,\nu} f(z) = D_z^\lambda f(z) = \frac{1}{\Gamma\lambda} \int_0^z \frac{f(t)}{(z-t)^{1-\delta}} dt, \quad (0 \le \lambda < 1). \tag{1.5}$$

where $D_z^\lambda f(z)$ is the fractional derivative operator of order λ. Furthermore, in terms of gamma function, we have

$$D_z^\lambda z^\rho = \frac{\Gamma(\rho+1)}{\Gamma(\rho-\delta+1)} z^{\rho-\lambda}, \quad (0 \leq \lambda) \tag{1.6}$$

Similary, the fractional integral opearator of order λ is

$$D_z^{-\lambda} f(z) = \frac{1}{\Gamma(1-\lambda)} \frac{d}{dz} \int_0^z \frac{f(t)}{(z-t)^\delta} dt, \tag{1.7}$$

where f is an analytic function in a simply connected region of the z-plane containing the origin, and the multiplicity of $(z-t)^{-\lambda}$ is removed by requiring $log(z-t)$ to be real when $z-t>0$. In terms of gamma function,

$$D_z^{-\lambda} z^\rho = \frac{\Gamma(\rho+1)}{\Gamma(\rho+\delta+1)} z^{\rho+\lambda}. \tag{1.8}$$

The generalized Ruscheweyh derivatives $\mathbf{J}_p^{\lambda,\mu} f$, $\mu > -1$ of $f \in A_p$ is defined by Goyal and Goyal [2] as follows:

$$\begin{aligned} \mathbf{J}_p^{\lambda,\mu} f(z) &= \tfrac{\Gamma(\mu-\lambda+\nu+2)}{\Gamma(\nu+2)\Gamma(\mu+1)} z^p J_{0,z}^{\lambda,\mu,\nu}(z^{\mu-p} f(z)) \\ &= z^p - \textstyle\sum_{k=n+p}^\infty a_k B_p^{\lambda,\mu}(k) z^k \end{aligned} \tag{1.9}$$

where

$$B_p^{\lambda,\mu}(k) = \frac{\Gamma(k-p+1+\mu)\Gamma(\nu+2+\mu-\lambda)\Gamma(k+\nu-p+2)}{\Gamma(k-p+1)\Gamma(k+\nu-p+2+\mu-\lambda)\Gamma(\nu+2)\Gamma(1+\mu)}$$

(1.10)

For $\lambda = \mu$, this generalized Ruscheweyh derivatives get reduced to Ruscheweyh derivatives of f(z) of order λ (see, e.g. [12]):

$$\begin{aligned} D^\lambda f(z) &= \tfrac{z^p}{\Gamma(\lambda+1)} \tfrac{d^\lambda}{dz^\lambda}(z^{\lambda-p} f(z)) \\ &= z^p + \textstyle\sum_{k=n+p}^\infty a_k B_k(\lambda) z^k \end{aligned}$$

(1.11)

where

$$B_k(\lambda) = \frac{\Gamma(\lambda+k)}{\Gamma(\lambda+p)\Gamma(k-p+1)} \tag{1.12}$$

For p=1, (1.11) reduces to ordinary Ruscheweyh derivatives for univalent functions [8].

The operation * is the convolution (Hadamard product) of two power series

$$f(z) = z^p - \sum_{k=n+p}^{\infty} a_k z^k \text{ and } g(z) = z^p - \sum_{k=n+p}^{\infty} b_k z^k$$

defined as

$$(f*g)(z) = z^p - \sum_{k=n+p}^{\infty} a_k b_k z^k \tag{1.13}$$

A function $f \in \Omega_p$ is said to be in the class $\Omega_p(\alpha, \beta, \lambda)$ if and only if

$$Re\left\{\frac{z(\mathbf{J}_p^{\lambda,\mu} f(z))'}{(1-\alpha)(\mathbf{J}_p^{\lambda,\mu} f(z)) + \alpha z^2 (\mathbf{J}_p^{\lambda,\mu} f(z))''}\right\} > \beta \tag{1.14}$$

for $z \in U$ and $0 \leq \alpha < 1$, $0 \leq \beta < p$ and $\lambda > -1$.

The class $\Omega_p(\alpha, \beta, \lambda)$ contains many well-known classes of analytic functions such as:

- For $\alpha = \lambda = 0$, $\Omega_p(\alpha, \beta, \lambda)$ reduces to the class $S^*(\beta)$ of starlike functions of order β.

- For $\alpha = \lambda = -1$, $\Omega_p(\alpha, \beta, \lambda)$ reduces to the class $K(\beta)$ of convex functions of order β.

2. Main Results

The coefficient bounds for the functions $f \in \Omega_p(\alpha, \beta, \lambda)$ are found in the following theorem:

Theorem 2.1. *Let $f \in A_p$, $z \in U$ be of the form (1.1). Then $f \in \Omega_p(\alpha, \beta, \lambda)$ iff*

$$\sum_{k=n+p}^{\infty} \frac{\alpha\beta(1+k-k^2) + k - \beta}{p - \beta + \alpha\beta(1+p-p^2)} B_p^{\lambda,\mu}(k) a_k < 1 \tag{2.1}$$

where $0 \leq \alpha < 1$, $0 \leq \beta < p$ and $\lambda > -1$.

Proof. Since $f \in \Omega_p(\alpha, \beta, \lambda)$

$$Re\left\{\frac{z(\mathbf{J}_p^{\lambda,\mu} f(z))'}{(1-\alpha)(\mathbf{J}_p^{\lambda,\mu} f(z)) + \alpha z^2(\mathbf{J}_p^{\lambda,\mu} f(z))''}\right\} > \beta \tag{2.2}$$

Making use of equation (1.9) in the above inequality, we obtain

$$Re\left\{\frac{pz^{p-1} - \sum_{k=n+p}^{\infty} k a_k B_p^{\lambda,\mu}(k) z^{k-1}}{(1-\alpha+\alpha p(p-1))z^p - \sum_{k=n+p}^{\infty}[(1-\alpha)+\alpha(k(k-1))] a_k B_p^{\lambda,\mu}(k) z^k}\right\} > \beta \tag{2.3}$$

Therefore, we obtain

$$\sum_{k=n+p}^{\infty} \frac{\alpha\beta(1+k-k^2)+k-\beta}{p-\beta+\alpha\beta(1+p-p^2)} B_p^{\lambda,\mu}(k) a_k < 1. \tag{2.4}$$

In this theorem, we will show that this class is closed under linear combination.

Theorem 2.2. *Let for* $j \in \{1, 2, 3, ..., m\}$

$$f_j(z) = z^p - \sum_{k=n+p}^{\infty} a_{k,j} z^k \in \Omega_p(\alpha, \beta, \lambda)$$

Then for $0 < P_j < 1$, $\sum_{j=1}^{m} P_j = 1$, *the function* $F(z)$ *defined by*

$$F(z) = \sum_{j=1}^{m} P_j f_j(z) \tag{2.5}$$

is also in $\Omega_p(\alpha, \beta, \lambda)$.

Proof. For every $j \in \{1, 2, 3, ..., m\}$, we obtain

$$\sum_{k=n+p}^{\infty} \frac{\alpha\beta(1+k-k^2)+k-\beta}{p-\beta+\alpha\beta(1+p-p^2)} B_p^{\lambda,\mu}(k) a_{k,j} < 1. \tag{2.6}$$

Since

$$F(z) = \sum_{j=1}^{m} P_j \left(z^p - \sum_{k=n+p}^{\infty} a_{k,j} z^k\right) = z^p - \sum_{k=n+p}^{\infty} \left(\sum_{j=1}^{m} P_j a_{k,j}\right) z^k \tag{2.7}$$

Therefore,

$$\sum_{k=n+p}^{\infty} \frac{\alpha\beta(1+k-k^2)+k-\beta}{p-\beta+\alpha\beta(1+p-p^2)} B_p^{\lambda,\mu}(k) \left(\sum_{j=1}^{m} P_j a_{k,j}\right) < \sum_{j=1}^{m} P_j = 1 \tag{2.8}$$

which proves the Theorem.

Theorem 2.3. *Let*

$$f(z) = z^p - \sum_{k=n+p}^{\infty} a_k z^k$$

and

$$g(z) = z^p - \sum_{k=n+p}^{\infty} b_k z^k$$

belong to $\Omega_p(\alpha, \beta, \lambda)$. *Then the function*

$$G(z) = z^p - \sum_{k=n+p}^{\infty} (a_k^2 + b_k^2) z^k$$

is in $\in \Omega_p(\alpha, \beta, \lambda_1)$, *where*

$$\lambda_1 \le \inf_k \left[\frac{(k-p)[\alpha\beta(1+k-k^2)+k-\beta]}{2[p-\beta+\alpha\beta(1+p-p^2)]} (B_p^{\lambda,\mu}(k))^2 - 1\right]. \tag{2.9}$$

Proof.

Since f, $g \in \Omega_p(\alpha, \beta, \lambda)$,

$$\sum_{k=n+p}^{\infty} \left[\frac{\alpha\beta(1+k-k^2)+k-\beta}{p-\beta+\alpha\beta(1+p-p^2)} B_p^{\lambda,\mu}(k)\right]^2 a_k^2$$

$$\le \left[\sum_{k=n+p}^{\infty} \frac{\alpha\beta(1+k-k^2)+k-\beta}{p-\beta+\alpha\beta(1+p-p^2)} B_p^{\lambda,\mu}(k) a_k\right]^2 < 1. \tag{2.10}$$

Similarly,

$$\sum_{k=n+p}^{\infty} \left[\frac{\alpha\beta(1+k-k^2)+k-\beta}{p-\beta+\alpha\beta(1+p-p^2)} B_p^{\lambda,\mu}(k)\right]^2 b_k^2$$

$$\leq \left[\sum_{k=n+p}^{\infty} \tfrac{\alpha\beta(1+k-k^2)+k-\beta}{p-\beta+\alpha\beta(1+p-p^2)} B_p^{\lambda,\mu}(k) b_k\right]^2 < 1. \tag{2.11}$$

Therefore

$$\sum_{k=n+p}^{\infty} \frac{1}{2}\left[\frac{\alpha\beta(1+k-k^2)+k-\beta}{p-\beta+\alpha\beta(1+p-p^2)} B_p^{\lambda,\mu}(k)\right]^2 (a_k^2+b_k^2) < 1 \tag{2.12}$$

Now, we must show that

$$\sum_{k=n+p}^{\infty} \frac{\alpha\beta(1+k-k^2)+k-\beta}{p-\beta+\alpha\beta(1+p-p^2)} B_p^{\lambda_1,\mu}(k)(a_k^2+b_k^2) < 1 \tag{2.13}$$

This inequality holds if

$$\frac{\alpha\beta(1+k-k^2)+k-\beta}{p-\beta+\alpha\beta(1+p-p^2)} B_p^{\lambda_1,\mu}(k) \leq \frac{1}{2}\left[\frac{\alpha\beta(1+k-k^2)+k-\beta}{p-\beta+\alpha\beta(1+p-p^2)} B_p^{\lambda,\mu}(k)\right]^2 \tag{2.14}$$

which is equivalent to

$$B_p^{\lambda_1,\mu}(k) = \frac{1}{2}\frac{\alpha\beta(1+k-k^2)+k-\beta}{p-\beta+\alpha\beta(1+p-p^2)}\left[B_p^{\lambda,\mu}(k)\right]^2 \tag{2.15}$$

Since $\frac{\lambda_1+1}{k-p} \leq B_p^{\lambda_1,\mu}(k)$, we obtain

$$\frac{\lambda_1+1}{k-p} \leq \frac{1}{2}\frac{\alpha\beta(1+k-k^2)+k-\beta}{p-\beta+\alpha\beta(1+p-p^2)}\left[B_p^{\lambda,\mu}(k)\right]^2 \tag{2.16}$$

and this gives the required result.

A modified Komatu operator $K_c^{\gamma} : A \to A$ is defined for $\gamma \geq 0$ and $c > -p$ as

$$K_{c,p}^{\gamma} f(z) = \frac{(c+p)^{\gamma}}{\Gamma(\gamma) z^c} \int_0^1 t^c (log\frac{1}{t})^{\gamma-1} f(tz)dt \tag{2.17}$$

It can be easily verified that for $f \in A_p$

$$K_{c,p}^{\gamma} f(z) = z^p - \sum_{k=p+1}^{\infty} \left(\frac{c+p}{c+k}\right)^{\gamma} a_k z^k \tag{2.18}$$

Theorem 2.4. *If $f \in \Omega_p(\alpha, \beta, \lambda)$, then $K^{\gamma}_{c,p} f \in \Omega_p(\alpha, \beta, \lambda)$.*

Proof. Since $f \in \Omega_p(\alpha, \beta, \lambda)$ and $\left\{\frac{c+p}{c+k}\right\}^{\gamma} < 1$, we have

$$\sum_{k=n+p}^{\infty} \frac{\alpha\beta(1+k-k^2)+k-\beta}{p-\beta+\alpha\beta(1+p-p^2)} \left(\frac{c+p}{c+k}\right)^{\gamma} B_p^{\lambda,\mu}(k) a_k < 1 \tag{2.19}$$

This completes the proof.

3. Radius of starlikeness and convexity

Now we obtain the radii of starlikeness and convexity for the functions $K^{\gamma}_{c,p} f$.

Theorem 3.1. *The function $K^{\gamma}_{c,p} f$ is starlike of order η in*

$|z| < r_1(\alpha, \beta, \lambda, c, \gamma, \eta)$, where

$$r_1(\alpha, \beta, \lambda, c, \gamma, \eta)$$

$$= \inf_k \left[\frac{1-\eta}{k-\eta-p+1} \frac{\alpha\beta(1+k-k^2)+k-\beta}{p-\beta+\alpha\beta(1+p-p^2)} \left(\frac{c+p}{c+k}\right)^{\gamma} B_p^{\lambda,\mu}(k) a_k\right]^{\frac{1}{k-p}} \tag{3.1}$$

Proof. We must show that

$$\left|\frac{z(K^{\gamma}_{c,p} f(z))'}{K^{\gamma}_{c,p} f(z)} - p\right| < 1 - \eta. \tag{3.2}$$

i.e.

$$\left|\frac{z(K^{\gamma}_{c,p} f(z))' - pK^{\gamma}_{c,p} f(z)}{K^{\gamma}_{c,p} f(z)}\right| \leq \frac{\sum_{k=n+p}^{\infty}(\frac{c+p}{c+k})^{\gamma}(k-p)a_k|z|^{k-p}}{1-\sum_{k=n+p}^{\infty}(\frac{c+p}{c+k})^{\gamma} a_k|z|^{k-p}} < 1 - \eta \tag{3.3}$$

or to show that

$$\sum_{k=n+p}^{\infty} (\frac{c+p}{c+k})^{\gamma}(k-p)a_k|z|^{k-p} + (1-\eta) \sum_{k=n+p}^{\infty} \left(\frac{c+p}{c+k}\right)^{\gamma} a_k|z|^{k-p} < 1 - \eta$$

(3.4)

Now, in view of (2.1), the theorem holds if

$$(3.5)\ |z|^{k-p} < \frac{1-\eta}{k-p+1-\eta}\frac{\alpha\beta(1+k-k^2)+k-\beta}{p-\beta+\alpha\beta(1+p-p^2)}\left(\frac{c+p}{c+k}\right)^{\gamma} B_p^{\lambda,\mu}(k)$$

This proves the result.

Theorem 3.2. *The function $K_{c,p}^{\gamma}f$ is convex of order η in*

$|z| < r_2(\alpha,\beta,\lambda,c,\gamma,\eta)$, *where*

$$r_2(\alpha,\beta,\lambda,c,\gamma,\eta)$$

$$=\inf_k \left[\frac{p(p-\eta)}{k(k-\eta)}\frac{\alpha\beta(1+k-k^2)+k-\beta}{p-\beta+\alpha\beta(1+p-p^2)}\left(\frac{c+p}{c+k}\right)^{\gamma} B_p^{\lambda,\mu}(k)a_k\right]^{\frac{1}{k-p}}$$

(3.6)

Proof. Noting the fact that $K_{c,p}^{\gamma}f$ is convex iff $z(K_{c,p}^{\gamma}f)'$ is starlike. Therefore, we must show that,

$$\left|\frac{z(K_{c,p}^{\gamma}f(z))''}{(K_{c,p}^{\gamma}f(z))'}\right| < 1-\eta. \tag{3.7}$$

i.e.

$$\frac{p(p-1)z^{p-1}-\sum_{k=n+p}^{\infty}(\frac{c+p}{c+k})^{\gamma}(k-p)a_k|z|^{k-1}}{pz^{p-1}-\sum_{k=n+p}^{\infty}(\frac{c+p}{c+k})^{\gamma}a_k|z|^{k-1}} < 1-\eta \tag{3.8}$$

or to show that

$$\sum_{k=n+p}^{\infty}(\frac{c+p}{c+k})^{\gamma}[k(k-\eta)]a_k|z|^{k-p} < p(p-\eta) \tag{3.9}$$

Now, by (2.1), the last inequality holds if

$$(3.10)\quad |z|^{k-p} < \frac{p(p-\eta)}{k(k-\eta)}\frac{\alpha\beta(1+k-k^2)+k-\beta}{p-\beta+\alpha\beta(1+p-p^2)}\left(\frac{c+p}{c+k}\right)^{\gamma} B_p^{\lambda,\mu}(k)$$

This complete the proof.

Theorem 3.3. *If $f \in \Omega_p(\alpha, \beta, \lambda)$, then the function $F_p^\mu(z)$, $z \in U$ defined by*

$$F_p^\mu(z) = (1-\mu)z^p + p\mu \int_0^z \frac{f(t)}{t}dt, \ \ 0 \le \mu < \frac{2}{p} \tag{3.11}$$

is in $\Omega_p(\alpha, \beta, \lambda)$.

Proof. We have

$$F_p^\mu(z) = (1-\mu)z^p + p\mu\left[\int_0^z t^{p-1}dt - \sum_{k=n+p}^{\infty}\int_0^z a_k t^{k-1}dt\right] = z^p - \sum_{k=n+p}^{\infty} a_k \frac{p\mu}{k} z^k \tag{3.12}$$

Now, by (2.1), we obtain

$$\sum_{k=n+p}^{\infty} \frac{\alpha\beta(1+k-k^2)+k-\beta}{p-\beta+\alpha\beta(1+p-p^2)}\left(\frac{p\mu}{k}\right) B_p^{\lambda,\mu}(k)a_k$$

$$\le \textstyle\sum_{k=n+p}^{\infty} \frac{\alpha\beta(1+k-k^2)+k-\beta}{p-\beta+\alpha\beta(1+p-p^2)}\left(\frac{2}{k}\right) B_p^{\lambda,\mu}(k)a_k < 1 \tag{3.13}$$

and this proves the theorem.

Remark 3.4. *$F_p^\mu(z)$ is starlike of order η in $|z| < r_1^p(\alpha, \beta, \lambda, \eta, \mu)$, where*

$$r_1^p(\alpha, \beta, \lambda, \eta, \mu)$$

$$= \inf_k \left[\tfrac{k(1-\eta)}{p\mu(p-k+1-\eta)} \tfrac{\alpha\beta(1+k-k^2)+k-\beta}{p-\beta+\alpha\beta(1+p-p^2)} B_p^{\lambda,\mu}(k)a_k\right]^{\frac{1}{k-p}}. \tag{3.14}$$

Also, $F_p^\mu(z)$ is convex of order η in $|z| < r_1^p(\alpha, \beta, \lambda, \eta, \mu)$, where

$$r_2^p(\alpha, \beta, \lambda, \eta, \mu) = \inf_k \left[\frac{(p-\eta)}{p\mu(k-\eta)} \frac{\alpha\beta(1+k-k^2)+k-\beta}{p-\beta+\alpha\beta(1+p-p^2)} B_p^{\lambda,\mu}(k)a_k\right]^{\frac{1}{k-p}}. \tag{3.15}$$

The proof of the above remark is made by similar arguments of the Theorems 3.1 and 3.2.

4. Fractional Calculus on $\Omega_p(\alpha, \beta, \lambda)$

In this section, we apply the fractional calulus techniques and discuss the properties of the family $\Omega_p(\alpha, \beta, \lambda)$ (see [10]). In this theorem, we find the distortion bounds for $f(z)$.

Theorem 4.1. *Let $f \in \Omega_p(\alpha, \beta, \lambda)$, $\lambda \geq 0$. Then*

$$|z|^{p+\delta}\frac{\Gamma(p+1)}{\Gamma(p+\delta+1)}\left[1-M|z|^n\right] \leq |D_z^{-\delta}f(z)| \leq |z|^{p+\delta}\frac{\Gamma(p+1)}{\Gamma(p+\delta+1)}\left[1+M|z|^n\right]$$
(4.1)

where

$$M = \frac{(p+1)_n[p-\beta+\alpha\beta(1+p-p^2)](\nu+\mu-\lambda+2)_n\Gamma(n+1)}{(p+\delta+1)_n[p+n+\beta\{\alpha(1+p+n-(p+n)^2)-1\}](\mu+1)_n(\nu+2)_n}$$

and $f(z)$ is analytic function

Proof.

By equation (1.8), we have

$$\frac{\Gamma(\delta+p+1)}{\Gamma(p+1)}z^{-\delta}D_z^{-\delta}f(z) = z^p - \sum_{k=n+p}^{\infty} a_k H_p(k,\delta)z^k \tag{4.2}$$

where

$$H_p(k,\delta) = \frac{\Gamma(\delta+p+1)\Gamma(k+1)}{\Gamma(k+\delta+1)\Gamma(p+1)} \tag{4.3}$$

But $H_p(k,\delta)$ is a decreasing function for $k \geq n+p$ and also $B_p^{\lambda,\mu}(k)$ is increasing function of k, thus, we have

$$H_p(k,\delta) \leq \frac{\Gamma(\delta+p+1)\Gamma(n+p+1)}{\Gamma(n+p+\delta+1)\Gamma(p+1)} = \frac{(p+1)_n}{(\delta+p+1)_n} \tag{4.4}$$

and

$$B_p^{\lambda,\mu}(k) \geq \frac{(\mu+1)_n(\nu+2)_n}{(\mu-\lambda+\nu+2)_n\Gamma(n+1)} \tag{4.5}$$

So, we conclude that

$$\left|\frac{\Gamma(\delta+p+1)}{\Gamma(p+1)}z^{-\delta}D_z^{-\delta}f(z)\right|$$

$\leq |z|^p + \frac{(p+1)_n}{(\delta+p+1)_n}|z|^{n+p}\sum_{k=n+p}^{\infty} a_k$

$\leq |z|^p \frac{\Gamma(p+1)}{\Gamma(p+\delta+1)}[1+M|z|^{n+p}]$
(4.6)

where M is defined in the theorem statement. Thus, we get

$$\left|D_z^{-\delta} f(z)\right| \leq |z|^{p+\delta} \frac{\Gamma(p+1)}{\Gamma(\delta+p+1)}[1+M|z|^n] \tag{4.7}$$

Also, we have

$$\left|\frac{\Gamma(\delta+p+1)}{\Gamma(p+1)} z^{-\delta} D_z^{-\delta} f(z)\right|$$

$\geq |z|^p - \frac{(p+1)_n}{(\delta+p+1)_n}|z|^{n+p} \sum_{k=n+p}^{\infty} a_k$

$\geq |z|^p \frac{\Gamma(p+1)}{\Gamma(p+\delta+1)}[1-M|z|^n]$
(4.8)

Then

$$\left|D_z^{-\delta} f(z)\right| \geq |z|^{p+\delta} \frac{\Gamma(p+1)}{\Gamma(\delta+p+1)}[1-M|z|^n] \tag{4.9}$$

This completes the proof of the theorem.

Theorem 4.2. *Let $f \in \Omega_p(\alpha, \beta, \lambda)$, $\lambda \geq 0$. Then*

$$|z|^{p-\delta} \frac{\Gamma(p+1)}{\Gamma(p-\delta+1)}\left[1-N|z|^n\right] \leq |D_z^{\delta} f(z)| \leq |z|^{p-\delta} \frac{\Gamma(p+1)}{\Gamma(p-\delta+1)}\left[1+N|z|^n\right]$$
(4.10)
where

$$N = \frac{(p+1)_n[p-\beta+\alpha\beta(1+p-p^2)](\nu+\mu-\lambda+2)_n\Gamma(n+1)}{(p-\delta+1)_n[p+n+\beta\{\alpha(1+p+n-(p+n)^2)-1\}](\mu+1)_n(\nu+2)_n}$$

and $f(z)$ is analytic function

Proof. By equation (1.6), we have

$$\frac{\Gamma(p-\delta+1)}{\Gamma(p+1)} z^{\delta} D_z^{\delta} f(z) = z^p - \sum_{k=n+p}^{\infty} a_k R_p(k,\delta) z^k \tag{4.11}$$

where

$$R_p(k,\delta)=\frac{\Gamma(p-\delta+1)\Gamma(k+1)}{\Gamma(k-\delta+1)\Gamma(p+1)} \tag{4.12}$$

But $R_p(k,\delta)$ is a decfreasing function for $k \geq n+p$ and also $B_p^{\lambda,\mu}(k)$ is increasing function of k, thus, we have

$$R_p(k,\delta) \leq \frac{\Gamma(p-\delta+1)\Gamma(n+p+1)}{\Gamma(n+p-\delta+1)\Gamma(p+1)}=\frac{(p+1)_n}{(p-\delta+1)_n} \tag{4.13}$$

and

$$B_p^{\lambda,\mu}(k) \geq \frac{(\mu+1)_n(\nu+2)_n}{(\mu-\lambda+\nu+2)_n\Gamma(n+1)} \tag{4.14}$$

So, we conclude that

$$\left|\frac{\Gamma(p-\delta+1)}{\Gamma(p+1)}z^{\delta}D_z^{\delta}f(z)\right|$$

$\leq |z|^p+\frac{(p+1)_n}{(p-\delta+1)_n}|z|^{n+p}\sum_{k=n+p}^{\infty}a_k$

$\leq |z|^p\frac{\Gamma(p+1)}{\Gamma(p-\delta+1)}[1+N|z|^{n+p}]$

(4.15)

where N is defined in the theorem statement. Then, we get

$$\left|D_z^{\delta}f(z)\right| \leq |z|^{p-\delta}\frac{\Gamma(p+1)}{\Gamma(p-\delta+1)}[1+N|z|^n] \tag{4.16}$$

Also, we have

$$\left|\frac{\Gamma(p-\delta+1)}{\Gamma(p+1)}z^{\delta}D_z^{\delta}f(z)\right|$$

$\geq |z|^p-\frac{(p+1)_n}{(p-\delta+1)_n}|z|^{n+p}\sum_{k=n+p}^{\infty}a_k$

$\geq |z|^p\frac{\Gamma(p+1)}{\Gamma(p+\delta+1)}[1-N|z|^n]$

(4.17)

Then

$$\left|D_z^{\delta}f(z)\right| \geq |z|^{p-\delta}\frac{\Gamma(p+1)}{\Gamma(p-\delta+1)}[1-N|z|^n] \tag{4.18}$$

This completes the proof.

Letting $\delta=1$ in Theorem 4.1, we obtain

Corollary 4.3. *Let $f \in \Omega_p(\alpha, \beta, \lambda)$, $\lambda \geq 0$. Then*

$$\frac{|z|^{p+1}}{(p+1)}\left[1 - M|z|^n\right] \leq \left|\int_0^z f(t)dt\right| \leq \frac{|z|^{p+1}}{(p+1)}\left[1 + M|z|^n\right] \tag{4.19}$$

where

$$M = \frac{(p+1)_n[p-\beta+\alpha\beta(1+p-p^2)](\nu+\mu-\lambda+2)_n\Gamma(n+1)}{(p+n+1)_n[p+n+\beta\{\alpha(1+p+n-(p+n)^2)-1\}](\mu+1)_n(\nu+2)_n}$$

and $f(z)$ is analytic function

Letting $\delta = 0$ in Theorem 4.2, we obtain

Corollary 4.4. *Let $f \in \Omega_p(\alpha, \beta, \lambda)$, $\lambda \geq 0$. Then*

$$|z|^p\left[1 - N|z|^n\right] \leq |f(z)| \leq |z|^p\left[1 + N|z|^n\right] \tag{4.20}$$

where

$$N = \frac{[p-\beta+\alpha\beta(1+p-p^2)](\nu+\mu-\lambda+2)_n\Gamma(n+1)}{[p+n+\beta\{\alpha(1+p+n-(p+n)^2)-1\}](\mu+1)_n(\nu+2)_n}$$

and $f(z)$ is analytic function

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