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On triple difference sequences of real numbers in probabilistic normed spaces

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Abstract

In this paper we define concept of triple Δ-statistical convergent sequences in probabilistic normed space and give some results. Also we introduce the notions of Δ-statistical limit point and Δ-statistical cluster point and investigate their different properties.

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1. Introduction

The notion of probabilistic normed space (PN-space) is a generalization of normed linear space. In an ordinary normed linear space norm of vectors are represented by a positive number. But in a PN-space, the norm of vectors are represented by probability distribution functions rather than a positive number. The notion of PN-spaces was first introduced by Serstnev [25] in 1963. For detailed history, development and applications in different subjects of the notion of probabilistic normed spaces, one may refer to Alotaibi [1], Alsina etal ([2], [3]), Constantin etal [5], Esi [6], Karakus [11], Menger [18], Lafuerza etal ([14], [15]), Lafuerza etal [16], Schweizer and Sklar ([23], [24]), Tripathy etal [34].

As a generalization of ordinary convergence for sequences of real numbers, the notion of statistical convergence was first introduced by Fast [9]. After then it was studied by many researchers like Connor [4], Fridy [10], Karakus [11], Karakus and Demirci [12], Salat [20], Tripathy ([26], [27]), Tripathy and Baruah [28], Tripathy etal [29], Tripathy and Dutta [32], Tripathy and Sarma [33]. Statistical convergence has been studied in abstract spaces such as the fuzzy number space by Esi ([6], [8]), Fast [9]), locally convex spaces by Maddox [17]. Karakus [11] introduced the notion of statistical convergence in PN-spaces and followed by Ideal convergence by Tripathy etal [34], in normed linear spaces by Kolk [13]. Karakus and Demirci [12], studied statistical convergence of double sequences in PN-spaces. In recent times Esi and Özdemir [7] introduced generalized Δ^m-statistical convergence in PN-spaces for single generalized difference sequences. Also sequences of fuzzy numbers has been studied recently by Tripathy and Borgohain [30] and Tripathy and Debnath [31].

The notion of double sequence was initiated by Priengsheim [19]. In this paper we introduce the concept of statistical convergence of triple difference sequence in probabilistic normed spaces and establish some basic properties in PN-spaces.

2. Definitions and Preliminaries

Definition 2.1. A function $f : R \to R_0^+$ is called a *distribution function* if it is non-decreasing and left continuous with $\inf_{t\in R} f(t) = 0$ and

$\sup_{t\in R} f(t) = 1$.

Throughout D denotes the set of all distribution functions.

Definition 2.2. A *triangular norm* or t-norm is a binary operation on $[0,1]$ which is continuous, commutative, associative, non-decreasing and has 1 as neutral element, i.e., it is the continuous mapping $*:[0,1]\times[0,1]\to[0,1]$ such that for all $a,b,c\in[0,1]$
1) $a*1=a$.
2) $a*b=b*a$.
3) $c*d\geq a*b$ if $c\geq a$ and $d\geq b$.
4) $(a*b)*c=a*(b*c)$.

Example 2.1. Consider the $*$ operation defined by $a*b=\max\{a+b-1,0\}$. Then $*$ is a t-norm. Similarly one may consider $a*b=ab$, $a*b=\min\{a,b\}$ on [0,1] and verify that these are also t-norms.

Definition 2.3. A triplet $(X,N,*)$ is called a probabilistic normed space or a PN-space if X is a real vector space, N is a mapping from X into D (for $x\in X$ the distribution function $N(x)$ is denoted by N_x and $N_x(t)$ is the value of N_x at $t\in R$) and satisfies the following conditions:
(PN-1)$N_x(0)=0$,
(PN-2) $N_x(t)=1$ for all $t>0$ if and only if $x=0$
(PN-3) $N_{\alpha x}(t)=N_x(\frac{t}{|\alpha|})$ for all $\alpha\in R-0$
(PN-4) $N_{x+y}(s+t)\geq N_x(s)*N_y(t)$ for all $x,y\in X$ and $s,t\in R_0^+$.

Example 2.2. Let $(X,||.||)$ be a normed linear space and $\mu\in D$ with $\mu(0)=0$ and $\mu\neq h$, where

$$h(t)=\begin{cases}0, & \text{for all } t\leq 0;\\ 1, & \text{for all } t>0.\end{cases}$$

Define

$$N_x(t)=\begin{cases}h(t), & \text{for } x=0;\\ \mu(\frac{t}{||x||}), & \text{for } x\neq 0,\end{cases}$$

where $x\in X, t\in R$. Then $(X,N,*)$ is a PN-space. We define the functions μ and μ' on R by

$$\mu(x) = \begin{cases} 0, & \text{for } x \leq 0; \\ \frac{x}{1+x}, & \text{for } x > 0. \end{cases}$$

and

$$\mu'(x) = \begin{cases} 0, & \text{for } x \leq 0; \\ \exp(\frac{-1}{x}), & \text{for } x > 0. \end{cases}$$

Then we obtain the following well known $*$ norms

$$N_x(t) = \begin{cases} h(t), & \text{for } x = 0; \\ \frac{t}{t+||x||}, & \text{for } x \neq 0. \end{cases}$$

and

$$N'_x(t) = \begin{cases} h(t), & \text{for } x = 0; \\ \exp(\frac{-||x||}{t}), & \text{for } x \neq 0. \end{cases}$$

We recall the concepts of convergence and Cauchy sequences for single sequences in a probabilistic normed space.

Definition 2.4. Let $(X, N, *)$ be a PN-space. Then a sequence $x =< x_k >$ is said to be convergent to $L \in X$ with respect to the probabilistic norm N if for every $\varepsilon > 0$ and $\lambda \in (0, 1)$, there exists a positive integer k_0 such that $N_{x_k - L}(\varepsilon) > 1 - \lambda$, whenever $k > k_0$. It is denoted by $N - \lim x_k = L$ or $x_k \xrightarrow{N} L$ as $k \to \infty$.

Definition 2.5. Let $(X, N, *)$ be a PN-space. Then a sequences $x =< x_k >$ is said to be Cauchy sequence with respect to the probabilistic norm N if for every $\varepsilon > 0$ and $\lambda \in (0, 1)$, there exists a positive integer k_0 such that $N_{x_k - x_l}(\varepsilon) > 1 - \lambda$, whenever $k, l > k_0$.

Definition 2.6. Let $(X, N, *)$ be a PN-space. Then a sequences $x =< x_k >$ is said to be bounded in X if there is $r \in R$ such that $N_{x_k}(r) > 1 - \lambda$ where $\lambda \in (0, 1)$, we denote by ℓ_∞^N the spaces of all bounded sequences in PN-space.

Definition 2.7. A triple sequence $x =< x_{nlk} >$ has Pringsheim limit L (denoted by $P - \lim x = L$) provided that given $\varepsilon > 0$ there exists $n_0 \in N$

such that $|x_{nlk} - L| < \varepsilon$ whenever $n, l, k > n_0$.

Now we introduce the following notions.

Definition 2.8. A subset $K \in N \times N \times N$ is said to have triple asymptotic density $\delta_3(K)$ if $\delta_3(K) = \lim_{n,l,k\to\infty} \frac{1}{nlk} \sum_{i_1=1}^{n} \sum_{i_2=1}^{l} \sum_{i_3=1}^{k} \chi_K(i_1, i_2, i_3)$ exists,

where χ_K is the characteristic function of K.

Definition 2.9. A real triple sequence $x =< x_{nlk} >$ is said to be Δ-statistically convergent to L, provided that for each $\varepsilon > 0$. There exists $m = m(\varepsilon), p = p(\varepsilon) andq = q(\varepsilon) suchthat \delta_3(\{(n, l, k) \in N^3 : n \leq m, l \leq p, k \leq q, |\Delta x_{nlk} - \Delta x_m pq| \geq \varepsilon\}) = 0$.

Definition 2.10. A real triple sequence $x =< x_{nlk} >$ is said to be Δ-statistically Cauchy, provided that for each $\varepsilon > 0$. There exists $m = m(\varepsilon), p = p(\varepsilon) and$q=q($\varepsilon$) such that $\delta_3(\{(n, l, k) \in N^3 : n \leq m, l \leq p, k \leq q, |\Delta x_{nlk} - \Delta x_{mpq}| \geq \varepsilon\}) = 0$.

Definition 2.11 Let $(X, N, *)$ be a PN-space. Then a triple sequences $x =< x_{nlk} >$ is said to be Δ-convergent to $L \in X$ with respect to the probabilistic norm N provided that for every $\varepsilon > 0$ and $\lambda \in (0, 1)$, there is a positive integer k_0 such that $N_{\Delta x_{nlk} - L}(\varepsilon) > 1 - \lambda$ whenever $n \geq k_0, l \geq k_0, k \geq k_0$. In this case we write $N_\Delta - lim x_{nlk} = L$, where $\Delta x_{nlk} = x_{nlk} - x_{n,l+1,k} - x_{n,l,k+1} + x_{n,l+1,k+1} - x_{n+1,lk} + x_{n+1,l+1,k} + x_{n+1,l,k+1} - x_{n+1,l+1,k+1}$ and $\Delta^0 x_{nlk} =< x_{nlk} >$.

Definition 2.12. Let $(X, N, *)$ be a probabilistic normed space. A triple sequence $x =< x_{nlk} >$ is said to be Δ-Cauchy in X with respect to the probabilistic norm N if for every $\varepsilon > 0$ and $\lambda \in (0, 1)$, there exists positive integer k_0, k_1, k_2 such that $N_{\Delta x_{nlk} - \Delta x_{pqr}}(\varepsilon) > 1 - \lambda$, whenever $n, p \geq k_0$, $l, q \geq k_1$, $k, r \geq k_2$.

Definition 2.13. Let $(X, N, *)$ be a probabilistic normed space. A triple sequence $x =< x_{nlk} >$ is said to be Δ-statistically convergent to L in X with respect to the probabilistic norm N if for every $\varepsilon > 0$ and $\lambda \in (0, 1)$, $\delta_3(\{(n, l, k) \in N^3 : N_{\Delta x_{nlk} - L}(\varepsilon) \leq 1 - \lambda\}) = 0$. In this case we write $st_{N\Delta} - \lim x_{nlk} = L$.

Definition 2.14. Let $(X, N, *)$ be a probabilistic normed space. A triple sequence $x =< x_{nlk} >$ is said to be Δ-statistically Cauchy in X with respect to the probabilistic norm N if for every $\varepsilon > 0$ and $\lambda \in (0,1)$, there exist positive integers N, M and P such that $\delta_3(\{(n,l,k) \in N^3 : N_{\Delta x_{nlk} - \Delta x_{pqr}}(\varepsilon) \leq 1 - \lambda\}) = 0$ for all $n, p \geq N$, $l, q \geq M$, $k, r \geq N$.

Definition 2.15 Let $(X, N, *)$ be a probabilistic normed space. For $x \in X$, $t > 0$ and $0 < r < 1$, the ball centered at x with radius r is defined by $B(x, r, t) = \{y \in X : N_{x-y}(t) > 1 - r\}$.

3. Main results

Theorem 3.1. Let $(X, N, *)$ be a PN-space, then for every $\varepsilon > 0$ and $\lambda \in (0,1)$, the following statements are equivalent.

$(i) st_{N\Delta} - \lim x_{nlk} = L.$

(ii) $\delta_3(\{(n,l,k) \in N^3 : N_{\Delta x_{nlk} - L}(\varepsilon) \leq 1 - \lambda\}) = 0.$

$(iii) \delta_3(\{(n,l,k) \in N^3 : N_{\Delta x_{nlk} - L}(\varepsilon) > 1 - \lambda\}) = 1.$

(iv) $s^3_\Delta - \lim N_{\Delta x_{nlk} - L}(\varepsilon) = 1.$

Proof. $(i) \Rightarrow (ii)$ Suppose $st_{N\Delta} - \lim x_{nlk} = L$. Then by definition we have, for every $\varepsilon > 0$ and $\lambda \in (0,1)$, we have $\delta_3(\{(n,l,k) \in N^3 : N_{\Delta x_{nlk} - L}(\varepsilon) \leq 1 - \lambda\}) = 0$.

$(ii) \Rightarrow (iii)$ Let $\varepsilon > 0$ and $\lambda \in (0,1)$, then we have

$\delta_3(\{(n,l,k) \in N^3 : N_{\Delta x_{nlk} - L}(\varepsilon) > 1 - \lambda\})$
$= 1 - \delta_3(\{(n,l,k) \in N^3 : N_{\Delta x_{nlk} - L}(\varepsilon) \leq 1 - \lambda\})$
$= 1$ by (ii).

$(iii) \Rightarrow (iv)$ Let $\varepsilon > 0$ and $\lambda \in (0,1)$, then we have

$\{(n,l,k) \in N^3 : |N_{\Delta x_{nlk} - L}(\varepsilon) - 1| \geq \lambda\}$ $\{(n,l,k) \in N^3 : N_{\Delta x_{nlk} - L}(\varepsilon) \leq 1 - \lambda\} \cup \{(n,l,k) \in N^3 : N_{\Delta x_{nlk} - L}(\varepsilon) \geq 1 + \lambda\}$.

Therefore we have from the finite additivity property of density $\delta_3(\{(n,l,k) \in N^3 : |N_{\Delta x_{nlk}-L}(\varepsilon) - 1| \geq \lambda\})$
$= \delta_3(\{(n,l,k) \in N^3 : N_{\Delta x_{nlk}-L}(\varepsilon) \leq 1-\lambda\}) + \delta_3(\{(n,l,k) \in N^3 : N_{\Delta x_{nlk}-L}(\varepsilon) \geq 1+\lambda\})$.

Since, $\delta_3(\{(n,l,k) \in N^3 : N_{\Delta x_{nlk}-L}(\varepsilon) \leq 1-\lambda\}) = 0$ and $\delta_3(\{(n,l,k) \in N^3 : N_{\Delta x_{nlk}-L}(\varepsilon) \geq 1+\lambda\}) = 0$.

Hence $\delta_3(\{(n,l,k) \in N^3 : |N_{\Delta x_{nlk}-L}(\varepsilon) - 1| \geq \lambda\}) = 0 \Rightarrow s^3_\Delta - \lim N_{\Delta x_{nlk}-L}(\varepsilon) = 1$.

$(iv) \Rightarrow (i)$ By hypothesis for a given $\varepsilon > 0$ and $\lambda \in (0,1)$, we have $\delta_3(\{(n,l,k) \in N^3 : |N_{\Delta x_{nlk}-L}(\varepsilon) - 1| \geq \lambda\}) = 0$ i.e., $\delta_3(\{(n,l,k) \in N^3 : N_{\Delta x_{nlk}-L}(\varepsilon) \leq 1-\lambda\}) + \delta_3(\{(n,l,k) \in N^3 : N^{(\varepsilon)-1}_{\Delta x_{nlk}-L} + \lambda\}) = 0$.
$\Rightarrow \delta_3(\{(n,l,k) \in N^3 : N_{\Delta x_{nlk}-L}(\varepsilon) \leq 1-\lambda\}) = 0$, since $\delta_3(\{(n,l,k) \in N^3 : N_{\Delta x_{nlk}-L}(\varepsilon) \geq 1+\lambda\}) = 0$.

The following result is on the uniqueness of the limit, if it exists.

Theorem 3.2. Let $(X, N, *)$ be a PN-space. If a sequence $x =< x_{nlk} >$ is Δ-statistically convergent with respect to the probabilistic norm, then $st_{N\Delta} - \lim x_{nlk}$ is unique.

Proof. Let $st_{N\Delta} - \lim x_{nlk} = L_1$ and $st_{N\Delta} - \lim x_{nlk} = L_2$, where $x =< x_{nlk} >$ is a triple sequence. For a given $\lambda > 0$ we choose $\gamma \in (0,1)$ such that $(1-\gamma) \star (1-\gamma) > 1-\lambda$. Then for any $\varepsilon > 0$, we define the following sets.

$$K_{N,1}(\gamma,\varepsilon) = \{(n,l,k) \in N^3 : N_{\Delta x_{nlk}-L_1}(\varepsilon) \leq 1-\gamma\},$$

$$K_{N,2}(\gamma,\varepsilon) = \{(n,l,k) \in N^3 : N_{\Delta x_{nlk}-L_2}(\varepsilon) \leq 1-\gamma\}.$$

Since $st_{N\Delta} - \lim x_{nlk} = L_1$, $\delta_3(\{K_{N,1}(\gamma,\varepsilon)\}) = 0$, for all $\varepsilon > 0$.

Furthermore using $st_{N\Delta} - \lim x_{nlk} = L_2$ we get $\delta_3(\{K_{N,2}(\gamma,\varepsilon)\}) = 0$, for all $\varepsilon > 0$. Now let $K_N(\gamma,\varepsilon) = K_{N,1}(\gamma,\varepsilon) \cap K_{N,2}(\gamma,\varepsilon)$. Then $\delta_3(\{K_N(\gamma,\varepsilon)\}) = 0$, which implies that $\delta_3(\{N^3 - K_N(\gamma,\varepsilon)\}) = 1$. If $(n,l,k) \in \{N^3 - K_N(\gamma,\varepsilon)\}$, then $N_{L_1-L_2}(\varepsilon) \geq N_{\Delta x_{nlk}-L_1}(\frac{\varepsilon}{2}) \star N_{\Delta x_{nlk}-L_2}(\frac{\varepsilon}{2}) >$

$(1-\gamma)\star(1-\gamma) > 1-\lambda$. Since $\lambda > 0$ is arbitrary we get $N_{L_1-L_2}(\varepsilon) = 1$ for all $\varepsilon > 0$, which yields $L_1 = L_2$. Therefore we conclude that $st_{N\Delta} - limit$ of triple sequence is unique.

Theorem 3.3. Let $(X, N, *)$ be a PN-space. If $N_\Delta - \lim x_{nlk} = L$, then $st_{N\Delta} - \lim x_{nlk} = L$, but not necessarily conversely.

proof. By hypothesis $x =< x_{nlk} >$, Δ-converges to L with respect to the probabilistic norm N. Therefore for every $\lambda > 0$ and $\varepsilon > 0$ there exists a positive integer k_0 such that $N_{\Delta x_{nlk}-L}(\varepsilon) > 1-\lambda$ for all $n \geq k_0$, $l \geq k_0$, $k \geq k_0$. Thus the set $\{(n,l,k) \in N^3 : N_{\Delta x_{nlk}-L}(\varepsilon) \leq 1-\lambda\}$ has finitely many terms. Since every finite subset of N^3 has density zero, we see that $\delta_3(\{(n,l,k) \in N^3 : N_{\Delta x_{nlk}-L}(\varepsilon) \leq 1-\lambda\}) = 0$.

Theorem 3.4. Let $(X, N, *)$ be a PN-space and $x =< x_{nlk} >$ be a triple sequence. Then $st_{N\Delta} - \lim x_{nlk} = L$ if and only if there exists a subset $K = \{(n,l,k) : n,l,k = 1,2,3,4,...\} \subset N^3$ such that $\delta_3(K) = 1$ and $N_\Delta - \lim_{\substack{(n,l,k)\in K \\ n,l,k\to\infty}} x_{nlk} = L$.

Proof. Suppose $st_{N\Delta} - \lim x_{nlk} = L$. Now for every $\varepsilon > 0$ and $r \in N$, let

$$K(r,\varepsilon) = \{(n,l,k) \in N^3 : N_{\Delta x_{nlk}-L}(\varepsilon) \leq 1 - \frac{1}{r}\}. \tag{3.1}$$

$M(r,\varepsilon) = \{(n,l,k) \in N^3 : N_{\Delta x_{nlk}-L}(\varepsilon) > 1 - \frac{1}{r}\}$

Then $\delta_3\{K(r,\varepsilon)\} = 0$ and

$$M(1,\varepsilon) \supset M(2,\varepsilon) \supset M(3,\varepsilon) \supset .... \supset M(i,\varepsilon) \supset M(i+1,\varepsilon) \supset .... \tag{3.2}$$

$$\delta_3\{M(r,\varepsilon)\} = 1 \text{ for } r = 1,2,3,.... \tag{3.3}$$

Now we have to show that for $(n,l,k) \in M(r,\varepsilon)$ the sequence $x = x_{nlk}$ is N_Δ-convergent to L.

Suppose $x =< x_{nlk} >$ be not N_Δ-convergent to L. Therefore there exists $\gamma > 0$ such that the set $\{(n,l,k) \in N_3 : N_{\Delta x_{nlk}-L}(\varepsilon) \leq 1-\gamma\}$ has infinitely many terms.

Let $M(\gamma,\varepsilon) = \{(n,l,k) \in N^3 : N_{\Delta x_{nlk}-L}(\varepsilon) > 1-\gamma\}$, $\gamma > \frac{1}{r}, (r = 1,2,3,...)$

Then $\delta_3\{M(\gamma,\varepsilon)\} = 0$ and by (3.2) we have $M(r,\varepsilon) \subset M(\gamma,\varepsilon)$. Hence $M(r,\varepsilon) = 0$ which contradicts (3.3).

Therefore $x =< x_{nlk} >$ is N_Δ-convergent to L.

Conversely suppose that there exists a subset $K = \{(n,l,k) : n,l,k = 1,2,3,4,...\} \subset N^3$ such that $\delta_3(K) = 1$ and $N_\Delta - \lim_{\substack{(n,l,k)\in K \\ n,l,k\to\infty}} x_{nlk} = L$.

Then there exists $k_0 \in N$, such that for every $\gamma \in (0,1)$ and $\varepsilon > 0$,

$N_{\Delta x_{nlk}-L}(\varepsilon) > 1-\gamma$ for $n \geq k_0$, $l \geq k_0$, $k \geq k_0$.

Now, $M(\gamma,\varepsilon) = \{(n,l,k) \in N^3 : N_{\Delta x_{nlk}-L}(\varepsilon) \leq 1-\gamma\}$
$\subset N^3 - \{(n_{k_0+1}, l_{k_0+1}, k_{k_0+1}), (n_{k_0+2}, l_{k_0+2}, k_{k_0+2}), (n_{k_0+3}, l_{k_0+3}, k_{k_0+3}), ...\}$.

Therefore $\delta_3(M(\gamma,\varepsilon)) \geq 1-1 = 0$.

Hence $st_{N\Delta} - \lim x_{nlk} = L$. This completes the proof.

Theorem 3.5. Let $(X,N,*)$ be a PN-space and $x =< x_{nlk} >$ be a sequence whose terms are in the vector space X . Then the following conditions are equivalent.

(a) x is Δ-statistically Cauchy sequence with respect to the probabilistic norm N.

(b) There exists an increasing index sequence $K = \{(k_1,k_2,k_3)\}$ of N^3 such that $\delta_3(K) = 1$ and the subsequence $\{(x_{k_1,k_2,k_3}\}_{(k_1,k_2,k_3)\in K}$ is a Δ-Cauchy sequence with respect to the probabilistic norm N.

Theorem 3.6. Let $(X,N,*)$ be a PN-space. Then

(i) If $st_{N\Delta} - \lim x_{nlk} = \xi$ and $st_{N\Delta} - \lim y_{nlk} = \eta$, then $st_{N\Delta} - \lim(x_{nlk} + y_{nlk}) = \xi + \eta$.

(*ii*) If $st_{N\Delta} - \lim x_{nlk} = \xi$ and $\alpha \in R$, then $st_{N\Delta} - \lim \alpha x_{nlk} = \alpha\xi$.

(*iii*) If $st_{N\Delta} - \lim x_{nlk} = \xi$ and $st_{N\Delta} - \lim y_{nlk} = \eta$, then $st_{N\Delta} - \lim(x_{nlk} - y_{nlk}) = \xi - \eta$.

Proof. (*i*) Let $st_{N\Delta} - \lim x_{nlk} = \xi$ and $st_{N\Delta} - \lim y_{nlk} = \eta$. For a given $\varepsilon > 0$ and $\lambda \in (0,1)$ we choose $\gamma \in (0,1)$ such that $(1-\gamma) \star (1-\gamma) > 1-\lambda$. Then we define the following sets. $K_{N,1}(\gamma,\varepsilon) = \{(n,l,k) \in N^3 : N_{\Delta x_{nlk}-\xi}(\varepsilon) \leq 1-\gamma\}$, $K_{N,2}(\gamma,\varepsilon) = \{(n,l,k) \in N^3 : N_{\Delta x_{nlk}-\eta}(\varepsilon) \leq 1-\gamma\}$. Since $st_{N\Delta} - \lim x_{nlk} = \xi$, $\delta_3\{K_{N,1}(\gamma,\varepsilon)\} = 0$, for all $\varepsilon > 0$.

Further using $st_{N\Delta} - \lim x_{nlk} = \xi$ we get $\delta_3\{K_{N,2}(\gamma,\varepsilon)\} = 0$, for all $\varepsilon > 0$.

Let $K_N(\gamma,\varepsilon) = K_{N,1}(\gamma,\varepsilon) \cap K_{N,2}(\gamma,\varepsilon)$.

Then we observe that $\delta_3\{K_N(\gamma,\varepsilon)\} = 0$, which implies that $\delta_3\{N^3 - K_N(\gamma,\varepsilon)\} = 1$. If $(n,l,k) \in \{N^3 - K_N(\gamma,\varepsilon)\}$, then we have
$N_{(\Delta x_{nlk}-\xi)+(\Delta y_{nlk}-\eta)}(\varepsilon) \geq N_{\Delta x_{nlk}-\xi}(\frac{\varepsilon}{2}) \star N_{\Delta y_{nlk}-\eta}(\frac{\varepsilon}{2})$
$> (1-\gamma) \star (1-\gamma) > 1-\lambda$.

This shows that $\delta_3\{(n,l,k) \in N^3 : N_{\Delta x_{nlk}-\xi)+(\Delta y_{nlk}-\eta)}(\varepsilon) \leq 1-\lambda = 0$.

Hence $st_{N\Delta} - \lim(x_{nlk} + y_{nlk}) = \xi + \eta$.

(*ii*) Let $st_{N\Delta} - \lim x_{nlk} = \eta$, $\lambda \in (0,1)$ and $\varepsilon > 0$. First we consider the case of $\alpha = 0$. In this case, $N_{0\Delta x_{nlk}-0\xi}(\varepsilon) = N_0(\varepsilon) = 1 > 1-\lambda$.

So we have $N_\Delta - \lim 0x_{nlk} = 0$. Then from Theorem 3.2 we have $st_{N\Delta} - \lim 0x_{nlk} = 0$.

Let $\alpha \in R(\alpha \neq 0)$. Since $st_{N\Delta} - \lim x_{nlk} = \xi$, we define the following set

$K_N(\gamma,\varepsilon) = \{(n,l,k) \in N^3 : N_{\Delta x_{nlk}-\xi}(\varepsilon) \leq 1-\gamma\}$, then we can say $\delta_3\{K_N(\gamma,\varepsilon)\} = 0$ for all $\varepsilon > 0$. In this case $\delta_3\{N^3 - K_N(\gamma,\varepsilon)\} = 1$. If $(n,l,k) \in N^3 - K_N(\gamma,\varepsilon)$, then

$N_{\alpha\Delta x_{nlk}-\alpha\xi}(\varepsilon) = N_{\Delta x_{nlk}-\xi}(\frac{\varepsilon}{|\alpha|})$
$\geq N_{\Delta x_{nlk}-\xi}(\varepsilon) \star N_0(\frac{\varepsilon}{|\alpha|} - \varepsilon)$

$= N_{\Delta x_{nlk}-\xi}(\varepsilon) \star 1$
$= N_{\Delta x_{nlk}-\xi}(\varepsilon) > 1-\lambda,\ \alpha \in R(\alpha \neq 0)$
This shows that $\delta_3\{(n,l,k) \in N^3 : N_{\alpha\Delta x_{nlk}-\alpha\xi}(\varepsilon) \leq 1-\lambda\} = 0$

Hence $st_{N\Delta} - \lim \alpha x_{nlk} = \alpha\xi$.

(iii) From (i) and (ii) by putting $\alpha = -1$, one can get (iii).

4. Statistical limit point and statistical cluster point of the class of difference triple sequences with respect to the probabilistic norm

Definition 4.1. Let $(X, N, *)$ be a PN-space. A subset Y of X is said to be bounded if for every $r \in (0,1)$, there exists $t_0 > 0$ such that $N_x(t_0) > 1-r$ for all $x \in Y$.

Definition 4.2. Let $(X, N, *)$ be a PN-space, then $L \in X$ is called a Δ-limit point of the triple sequence $x =< x_{nlk} >$ with respect to the probabilistic norm N provided that there is a subsequence of x that Δ-converges to L with respect to the probabilistic norm N. Let $\Omega_{N\Delta}(x)$ denote the set of all limit points of the sequence x. Let $\{(x_{n(i_1)l(i_2)k(i_3)})\}$ be a subsequence of $x =< x_{nlk} >$ and $K = \{(n(i_1), l(i_2), k(i_3)) \in N^3, i_1, i_2, i_3 \in N\}$, then we abbreviate $\{(x_{n(i_1)l(i_2)k(i_3)})\}$ by $\{x\}_K$, which in case $\delta_3(K) = 0$, $\{x\}_K$ is called a subsequence of density zero or thin subsequence. On the other hand $\{x\}_K$ is a non-thin subsequence of x if K does not have density zero.

Definition 4.3. Let $(X, N, *)$ be a PN-space.Then $\xi \in X$ is called a Δ-statistical limit point of the triple sequence $x =< x_{nlk} >$ with respect to the probabilistic norm N provided that there is a non-thin subsequence of x that Δ-converges to $\xi \in X$ with respect to the probabilistic norm. In this case we say ξ is an $st_{N\Delta}$-limit point of sequence $x =< x_{nlk} >$. Throughout $\Lambda_{N\Delta}(x)$ denotes the set of all $st_{N\Delta}$-limit points of the sequence x.

Definition 4.4. Let $(X, N, *)$ be a PN-space. Then $\gamma \in X$ is called a Δ-statistical cluster point of the sequence $x =< x_{nlk} >$ with respect to the probabilistic norm N provided that for $\varepsilon > 0$ and $\lambda \in (0,1)$, $\lim - \sup \delta_3(\{(n,l,k) \in N^3 : N_{\Delta x_{nlk}-\gamma}(\varepsilon) > 1-\lambda\}) > 0$. In this case we say that $\gamma \in X$ is an $st_{N\Delta}$-cluster point of the sequence $x =< x_{nlk} >$.

Throughout $\Gamma_{N\Delta}(x)$ denote the set of all $st_{N\Delta}$-cluster points of the sequence x.

Definition 4.5. A probabilistic normed space $(X, N, *)$ is said to be Δ-complete if every Δ-Cauchy sequence is Δ-convergent in X with respect to the probabilistic norm N.

Theorem 4.1. Let $(X, N, *)$ be a PN-space, then for any sequence $x =< x_{nlk} >\in X$, $\Lambda_{N\Delta}(x) \subset \Gamma_{N\Delta}(x)$.

Proof. Let $\xi \in \Lambda_{N\Delta}(x)$, then there is a non-thin subsequence $(x_{n(i_1)l(i_2)k(i_3)})$ of $x =< x_{nlk} >$ that Δ-converges to ξ with respect to the probabilistic norm N, i.e.
$\delta_3\{(n(i_1), l(i_2), k(i_3)) \in N^3 : N_{\Delta x_{n(i_1)l(i_2)k(i_3)}-\xi}(\varepsilon) > 1-\lambda\} = d > 0$.

Since

$\{(n, l, k) \in N^3 : N_{\Delta x_{nlk}-\xi}(\varepsilon) > 1-\lambda\} \supset \{(n(i_1), l(i_2), k(i_3)) \in N^3 : N_{\Delta x_{n(i_1)l(i_2)k(i_3)}-\xi}(\varepsilon) > 1-\lambda\}$.

For every $\varepsilon > 0$, we have $\{(n, l, k) \in N^3 : N_{\Delta x_{nlk}-\xi}(\varepsilon) > 1-\lambda\} \supseteq \{(n(i_1), l(i_2), k(i_3)) \in N^3 : i_1, i_2, i_3 \in N\} - \{(n(i_1), l(i_2), k(i_3)) \in N^3 : N_{\Delta x_{n(i_1)l(i_2)k(i_3)}-\xi}(\varepsilon) \leq 1-\lambda\}$.

Since $(x_{n(i_1)l(i_2)k(i_3)})$ is Δ-convergent to ξ with respect to the probabilistic norm N, the set $\{(n(i_1), l(i_2), k(i_3)) \in N^3 : N_{\Delta x_{n(i_1)l(i_2)k(i_3)}-\xi}(\varepsilon) \leq 1-\lambda\}$ is finite, for any $\varepsilon > 0$, therefore

$\limsup \delta_3(\{(n, l, k) \in N^3 : N_{\Delta x_{nlk}-\xi}(\varepsilon) > 1-\lambda\})$

$\geq \limsup \delta_3\{(n(i_1), l(i_2), k(i_3)) \in N^3 : i_1, i_2, i_3 \in N\}$

- $\limsup \delta_3\{(n(i_1), l(i_2), k(i_3)) \in N^3 : N_{\Delta x_{n(i_1)l(i_2)k(i_3)}-\xi}(\varepsilon) \leq 1-\lambda\}$.

Hence $\limsup \delta_3(\{(n, l, k) \in N^3 : N_{\Delta x_{nlk}-\xi}(\varepsilon) > 1-\lambda\}) > 0$, which implies $\xi \in \Gamma_{N\Delta}(x)$.

Thus $\Lambda_{N\Delta}(x) \subset \Gamma_{N\Delta}(x)$.

Theorem 4.2. Let $(X, N, *)$ be a PN-space. Then for any sequence $x =< x_{nlk} >\in X$, $\Gamma_{N\Delta}(x) \subset \Omega_{N\Delta}(x)$.

Proof. Let $\gamma \in \Gamma_{N\Delta}(x)$, then $\delta_3(\{(n,l,k) \in N^3 : N_{\Delta x_{nlk}-\gamma}(\varepsilon) > 1-\lambda\}) > 0$ for every $\varepsilon > 0$ and $\lambda \in (0,1)$. Let $\{x\}_K$ be a non-thin subsequence of x such that $K = \{(n(i_1), l(i_2), k(i_3)) \in N^3 : N_{\Delta x_{n(i_1)l(i_2)k(i_3)}-\gamma}(\varepsilon) > 1-\lambda\}$ for every $\varepsilon > 0$ and $\delta_3(K) \neq 0$. Since there are infinitely many elements in K,$\gamma \in \Omega_{N\Delta}(x)$.

Thus $\Gamma_{N\Delta}(x) \subset \Omega_{N\Delta}(x)$.

Theorem 4.3. Let $(X, N, *)$ be a PN-space, then for any sequence $x =< x_{nlk} >\in X$, $st_{N\Delta} - \lim x_{nlk} = L$, implies $\Lambda_{N\Delta}(x) = \Gamma_{N\Delta}(x) = \{L\}$.

Proof. First we prove that $\Lambda_{N\Delta}(x) = \{L\}$. Suppose that $\Lambda_{N\Delta}(x) = \{L, M\}$ be such that $L \neq M$. In this case, there exist non-thin subsequences $\{x_{n(i_1)l(i_2)k(i_3)}\}$ and $\{x_{p(i_1)q(i_2)r(i_3)}\}$ of $x =< x_{nlk} >$ those Δ-converge to L and M respectively with respect to the probabilistic norm N. Since $\{x_{p(i_1)q(i_2)r(i_3)}\}$ is Δ-convergent to M with respect to the probabilistic norm N, so for every $\varepsilon > 0$ and $\lambda \in (0,1)$, $K = \{(p(i_1), q(i_2), r(i_3)) \in N^3 : N_{\Delta x_{p(i_1)q(i_2)r(i_3)}-M}(\varepsilon) \leq 1-\lambda\}$ is a finite set and so $\delta_3(K) = 0$.

Then $\{(p(i_1), q(i_2), r(i_3)) \in N^3 : i_1, i_2, i_3 \in N\}$

$$= \{(p(i_1), q(i_2), r(i_3)) \in N^3 : N_{\Delta x_{p(i_1)q(i_2)r(i_3)}-M}(\varepsilon) > 1-\lambda\}$$

$$\cup \{(p(i_1), q(i_2), r(i_3)) \in N^3 : N_{\Delta x_{p(i_1)q(i_2)r(i_3)}-M}(\varepsilon) \leq 1-\lambda\}.$$

Which implies

$$\delta_3\{(p(i_1), q(i_2), r(i_3)) \in N^3 : N_{\Delta x_{p(i_1)q(i_2)r(i_3)}-M}(\varepsilon) > 1-\lambda\} \neq 0. \tag{4.1}$$

Since $st_{N\Delta} - \lim x_{nlk} = L$.

$$\delta_3\{(n,l,k) \in N^3 : N_{\Delta x_{nlk}-L}(\varepsilon) \leq 1-\lambda\} = 0, \text{ for every } \varepsilon > 0. \tag{4.2}$$

Therefore we can write $\delta_3\{(n,l,k) \in N^3 : N_{\Delta x_{nlk}-L}(\varepsilon) > 1-\lambda\} \neq 0$.

For every $L \neq M$, we have

$$\{(p(i_1), q(i_2), r(i_3)) \in N^3 : N_{\Delta x_{p(i_1)q(i_2)r(i_3)}-M}(\varepsilon) > 1-\lambda\}$$
$$\cap\{(n,l,k) \in N^3 : N_{\Delta x_{nlk}-L}(\varepsilon) > 1-\lambda\} = \emptyset.$$

Hence $\{(p(i_1), q(i_2), r(i_3)) \in N^3 : N_{\Delta x_{p(i_1)q(i_2)r(i_3)}-M}(\varepsilon) > 1-\lambda\} \subseteq \{(n,l,k) \in N^3 : N_{\Delta x_{nlk}-L}(\varepsilon) \leq 1-\lambda\}$.

Therefore

$$\limsup \delta_3\{(p(i_1), q(i_2), r(i_3)) \in N^3 : N_{\Delta x_{p(i_1)q(i_2)r(i_3)}-M}(\varepsilon) > 1-\lambda\}$$
$$\leq \limsup \delta_3\{(n,l,k) \in N^3 : N_{\Delta x_{nlk}-L}(\varepsilon) \leq 1-\lambda\} = 0.$$

This contradicts (4.1).

Hence $\Lambda_{N\Delta}(x) = \{L\}$.

Next we show that $\Gamma_{N\Delta}(x) = \{L\}$. Suppose that $\Gamma_{N\Delta}(x) = \{L, Q\}$ such that $L \neq Q$. Then

$$\limsup \delta_3\{(n,l,k) \in N^3 : N_{\Delta x_{nlk}-Q}(\varepsilon) > 1-\lambda\} \neq 0. \tag{4.3}$$

Since $\{(n,l,k) \in N^3 : N_{\Delta x_{nlk}-L}(\varepsilon) > 1-\lambda\} \cap \{(n,l,k) \in N^3 : N_{\Delta x_{nlk}-Q}(\varepsilon) > 1-\lambda\} = \emptyset$ for every $L \neq Q$, so $\{(n,l,k) \in N^3 : N_{\Delta x_{nlk}-L}(\varepsilon) \leq 1-\lambda\} \supseteq \{(n,l,k) \in N^3 : N_{\Delta x_{nlk}-Q}(\varepsilon) > 1-\lambda\}$.

Therefore

$$\limsup \delta_3(\{(n,l,k) \in N^3 : N_{\Delta x_{nlk}-L}(\varepsilon) \leq 1-\lambda\}) \tag{4.4}$$
$$\geq \limsup \delta_3(\{(n,l,k) \in N^3 : N_{\Delta x_{nlk}-Q}(\varepsilon) > 1-\lambda\}).$$

From (4.3), the right hand side of (4.4) is greater than zero and from (4.2) the left hand side of (4.4) equals zero. This leads to a contradiction.

Hence $\Gamma_{N\Delta}(x) = \{L\}$.

Theorem 4.4. Let $(X, N, *)$ be a PN-space. Then the set $\Gamma_{N\Delta}$ is closed in X for each $x =< x_{nlk} >$ of elements of X.

Proof. Let $y \in \overline{\Gamma_{N\Delta}(x)}$. Let $0 < r < 1$ and $t > 0$, there exists $\gamma \in \Gamma_{N\Delta}(x) \cap B(y, r, t)$ such that $B(y, r, t) = \{x \in X : N_{y-x}(t) > 1 - r\}$.

Choose $\eta > 0$ such that $B(\gamma, \eta, t) \subset B(y, r, t)$, then we have

$\{(n, l, k) \in N^3 : N_{y-\Delta x_{nlk}} > 1 - r\} \supset \{(n, l, k) \in N^3 : N_{\gamma-\Delta x_{nlk}}(t) > 1 - \eta\}$.

Since $\gamma \in \Gamma_{N\Delta}(x)$ so $\limsup \delta_3\{(n, l, k) \in N^3 : N_{\gamma-\Delta x_{nlk}}(t) > 1 - \eta\} > 0$.

Hence $\limsup \delta_3\{(n, l, k) \in N^3 : N_{y-\Delta x_{nlk}}(t) > 1 - r\} > 0$.

Thus $y \in \Gamma_{N\Delta}(x)$.This completes the proof.

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