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The forcing connected detour number of a graph

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Abstract

For two vertices u and v in a graph $G = (V, E)$, the detour distance $D(u, v)$ is the length of a longest u–v path in G. A u–v path of length $D(u, v)$ is called a u–v detour. A set $S \subseteq V$ is called a detour set of G if every vertex in G lies on a detour joining a pair of vertices of S. The detour number $dn(G)$ of G is the minimum order of its detour sets and any detour set of order $dn(G)$ is a detour basis of G. A set $S \subseteq V$ is called a connected detour set of G if S is detour set of G and the subgraph $G[S]$ induced by S is connected. The connected detour number $cdn(G)$ of G is the minimum order of its connected detour sets and any connected detour set of order $cdn(G)$ is called a connected detour basis of G. A subset T of a connected detour basis S is called a forcing subset for S if S is the unique connected detour basis containing T. A forcing subset for S of minimum cardinality is a minimum forcing subset of S. The forcing connected detour number of S, denoted by $fcdn(S)$, is the cardinality of a minimum forcing subset for S. The forcing connected detour number of G, denoted by $fcdn(G)$, is $fcdn(G) = min\{fcdn(S)\}$, where the minimum is taken over all connected detour bases S in G. The forcing connected detour numbers of certain standard graphs are obtained. It is shown that for each pair a, b of integers with $0 \leq a < b$ and $b \geq 3$, there is a connected graph G with $fcdn(G) = a$ and $cdn(G) = b$.

Key Words : *Detour, connected detour set, connected detour basis, connected detour number, forcing connected detour number.* **AMS Subject Classification :** *05C12.*

1. Introduction

By a graph $G = (V, E)$ we mean a finite undirected graph without loops or multiple edges. The order and size of G are denoted by p and q respectively. We consider connected graphs with at least two vertices. For basic definitions and terminologies we refer to [1, 5]. For vertices u and v in a connected graph G, the detour distance $D(u, v)$ is the length of a longest u–v path in G. A u–v path of length $D(u, v)$ is called a u–v detour. It is known that the detour distance is a metric on the vertex set V. Detour distance and detour center of a graph were studied in [2, 4].

A vertex x is said to lie on a u–v detour P if x is a vertex of P including the vertices u and v. A set $S \subseteq V$ is called a detour set if every vertex v in G lies on a detour joining a pair of vertices of S. The detour number $dn(G)$ of G is the minimum order of a detour set and any detour set of order $dn(G)$ is called a detour basis of G. A vertex v that belongs to every detour basis of G is a detour vertex in G. If G has a unique detour basis S, then every vertex in S is a detour vertex in G. These concepts were studied in [3]. A set $S \subseteq V$ is called a connected detour set of G if S is a detour set of G and the subgraph $G[S]$ induced by S is connected. The connected detour number $cdn(G)$ of G is the minimum order of its connected detour sets and any connected detour set of order $cdn(G)$ is called a connected detour basis of G. A vertex v in G is a *connected detour vertex* if v belongs to every connected detour basis of G. If G has a unique connected detour basis S, then every vertex in S a connected detour vertex of G. The connected detour number of a graph was introduced and studied in [6].

For the graph G given in Figure 1.1, the sets $S_1 = \{v_1, v_3\}$, $S_2 = \{v_1, v_5\}$ and $S_3 = \{v_1, v_4\}$ are the three detour bases of G so that $dn(G) = 2$. It is clear that no two element subset of V is a connected detour set of G. However the set $S_4 = \{v_1, v_2, v_3\}$ is a connected detour basis of G so that $cdn(G) = 3$. Also the set $S_5 = \{v_1, v_2, v_5\}$ is another connected detour basis of G. Thus there can be more than one connected detour basis for a graph G.

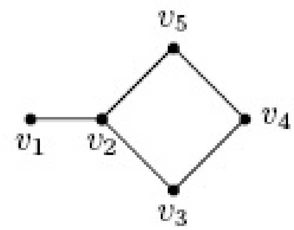

Figure 1.1: G

Graphs are often used to model network of real life problems and some definite part is always present in a minimum possible spanning set in a particular problem. For each connected detour basis S in a connected graph G, there is always some subset T of S that uniquely determines S as the connected detour basis containing T. Such subsets are called forcing subsets for S, and in this paper we briefly describe the properties satisfied by these sets in a graph.

The following theorem is used in the sequel.

Theorem 1.1. [6] All the end vertices and all the cut vertices of a connected graph G belong to every connected detour set of G.

Throughout this paper G denotes a connected graph with at least two vertices.

2. The Forcing Connected Detour Number

Definition 2.1. *Let G be a connected graph and S a connected detour basis of G. A subset $T \subseteq S$ is called a forcing subset for S if S is the unique connected detour basis containing T. A forcing subset for S of minimum cardinality is a minimum forcing subset of S. The forcing connected detour number of S, denoted by $fcdn(S)$, is the cardinality of a minimum forcing subset for S. The forcing connected detour number of G, denoted by $fcdn(G)$, is $fcdn(G) = \min\{fcdn(S)\}$, where the minimum is taken over all connected detour bases S in G.*

Example 2.2. *For the graph G given in Figure 2.1, $S_1 = \{u, s, w, t, v\}$ is the unique connected detour basis of G so that $fcdn(G) = 0$ and for the graph G given in Figure 1.1, $S_2 = \{v_1, v_2, v_3\}$ and $S_3 = \{v_1, v_2, v_5\}$ are the only connected detour bases of G so that $fcdn(G) = 1$.*

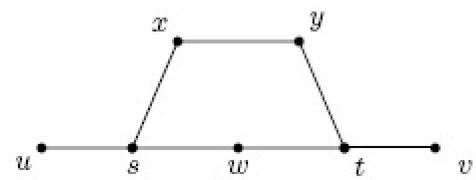

Figure 2.1: G

The next theorem follows immediately from the definitions of connected detour number and forcing connected detour number of a connected graph G.

Theorem 2.3. *For every connected graph G, $0 \leq fcdn(G) \leq cdn(G)$.*

Remark 2.4. *The lower bound in Theorem 2.3 is sharp. For the graph G given in Figure 2.2, $fcdn(G) = 0$. Also, all the inequalities in Theorem 2.3 can be strict. For the graph G given in Figure 1.1, $cdn(G) = 3$ and $fcdn(G) = 1$. Thus $0 < fcdn(G) < cdn(G)$.*

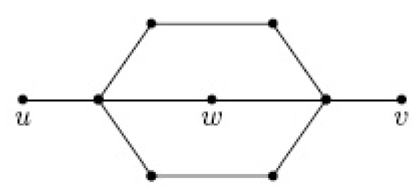

Figure 2.2: G

The following theorem is an easy consequence of the definition of forcing connected detour number of a graph.

Theorem 2.5. *Let G be a connected graph. Then*
a) $fcdn(G) = 0$ if and only if G has a unique connected detour basis,
b) $fcdn(G) = 1$ if and only if G has at least two connected detour bases, one of which is a unique connected detour basis containing one of its elements, and
c) $fcdn(G) = cdn(G)$ if and only if no connected detour basis of G is the unique connected detour basis containing any of its proper subsets.

Theorem 2.6. *Let G be a connected graph and W be the set of all connected detour vertices of G. Then $fcdn(G) \leq cdn(G) - |W|$.*

Proof. *Let S be a connected detour basis S of G. Then $cdn(G) = |S|$, $W \subseteq S$ and S is the unique connected detour basis containing $S - W$. Thus $fcdn(S) \leq |S - W| = |S| - |W| = cdn(G) - |W|$ and the result follows.* □

Corollary 2.7. *If G is a connected graph with k end-vertices and l cut-vertices, then $fcdn(G) \leq cdn(G) - k - l$.*

Proof. *This follows from Theorems 1.1 and 2.6.* □

Remark 2.8. *The bound in Theorem 2.6 is sharp. For the graph G given in Figure 1.1, $cdn(G) = 3$, $|W| = 2$ and $fcdn(G) = 1$ as in Remark 2.4. Also, the inequality in Theorem 2.6 can be strict. For the graph G of Figure 2.3, the sets $S_1 = \{v_1, v_2, v_3, v_4, v_7, v_8\}$, and $S_2 = \{v_1, v_2, v_5, v_6, v_7, v_8\}$ are the two connected detour bases of G and $W = \{v_1, v_2, v_7, v_8\}$ so that $cdn(G) = 6$, $|W| = 4$ and $fcdn(G) = 1$. Thus $fcdn(G) < cdn(G) - |W|$. Moreover, the bound in Corollary 2.7 is also sharp. For the graph G given in Figure 1.1, $cdn(G) = 3$, $k = 1$, $l = 1$ and $fcdn(G) = 1$. Also, the inequality in Corollary 2.7 can be strict. For the graph G of Figure 2.3, $cdn(G) = 6$, $k = 2$, $l = 2$ and $fcdn(G) = 1$. Thus $fcdn(G) < cdn(G) - k - l$.*

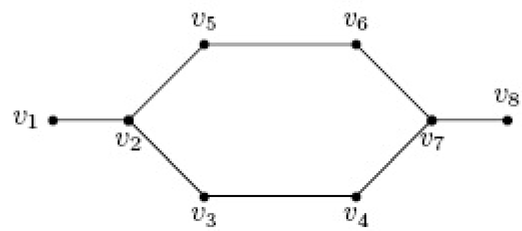

Figure 2.3: G

In the following theorems we proceed to find the forcing numbers of certain graphs G.

Theorem 2.9. *Let G be the complete graph K_p $(p \geq 2)$ or the cycle C_p or the complete bipartite graph $K_{m,n}$ $(2 \leq m \leq n)$. Then a set S of vertices is a connected detour basis if and only if S consists of any two adjacent vertices of G. Furthermore, $cdn(G) = 2$ for each of these graphs.*

Proof. *If G is the complete graph K_p $(p \geq 2)$ or the cycle C_p, then it is clear that any set of two adjacent vertices is a connected detour basis of G. Let G be the complete bipartite graph $K_{m,n}$ $(2 \leq m \leq n)$. Let X and Y be the bipartite sets of $K_{m,n}$ $(2 \leq m \leq n)$ with $X = \{x_1, x_2, \ldots, x_m\}$. Let $u \in X$ and $v \in Y$. It is clear that $D(u, v) = 2m - 1$. Let $y \in Y - \{v\}$. Then the vertex y lies on a u–v detour $P : u = x_1, y, x_2, y_1, x_3, y_2, \ldots, x_{m-1}, y_{m-2}, x_m, v$, where $y_1, y_2, \ldots, y_{m-2} \in Y - \{v, y\}$. Thus the set $\{u, v\}$ is a connected detour basis of $K_{m,n}$.*

Now, let S be a connected detour basis of G. Let S' be any set consisting of two adjacent vertices of G. Then as in the first part of this theorem S' is a connected detour basis of G. Hence $|S| = |S'| = 2$ and it follows that the two vertices of S are adjacent. The converse is obvious. □

Theorem 2.10. *a) If G is the complete graph $K_p(p \geq 3)$ or the the cycle C_p or the complete bipartite graph $K_{m,n}(m, n \geq 2)$, then $cdn(G) = fcdn(G) = 2$.*
b) If G is a tree of order $p \geq 2$, then $cdn(G) = p$ and $fcdn(G) = 0$.

Proof. *a) By Theorem 2.9, a set S of vertices is a connected detour basis of G if and only if S consists of two adjacent vertices of G. For each vertex v in G there are at least two vertices adjacent with v. Thus the vertex v belongs to more than one connected detour basis of G. Hence it follows that no set consisting of a single vertex is a forcing subset for any connected detour basis of G. Thus $fcdn(G) = 2$. Also, by Theorem 2.9, $cdn(G) = 2$ and the result follows.*
*b) By Theorem 1.1, $cdn(G) = p$. The set of all vertices of a tree is the unique connected detour basis so that $fcdn(G) = 0$ by Theorem 2.5(*a*).* □

Theorem 2.11. *Let G be a connected graph with cut-vertices and S a connected detour set of G. Then for any cut-vertex v of G, every component of $G - v$ contains an element of S.*

Proof. *Let v be a cut-vertex of G such that one of the components, say C of $G - v$ contains no vertex of S. Let $u \in V(C)$. Since S is a connected detour set of G, there exist vertices $x, y \in S$ such that the vertex u lies on some x–y detour $P : x = u_0, u_1, \ldots, u, \ldots, u_t = y$ in G. Let P_1 be the x–u subpath of P and P_2 be the u–y subpath of P. Since v is a cut-vertex of G both P_1 and P_2 contain v so that P is not a detour, which is a contradiction. Thus every component of $G - v$ contains an element of S.* □

Theorem 2.12. *Let $G = (K_{n_1} \cup K_{n_2} \cup ... \cup K_{n_r} \cup kK_1) + v$ be a block graph of order $p \geq 4$ such that $r \geq 1$, each $n_i \geq 2$ and $n_1 + n_2 + \ldots + n_r + k = p - 1$. Then $cdn(G) = r + k + 1$.*

Proof. *Let u_1, u_2, $\ldots$, u_k be the end-vertices of G. Let S be any connected detour set of G. Then by Theorem 1.1, $v \in S$ and $u_i \in S\,(1 \leq i \leq k)$. Also by Theorem 2.11, S contains a vertex from each component $K_{n_i}\,(1 \leq i \leq r)$. Now, choose exactly one vertex v_i from each K_{n_i} such that $v_i \in S$. Then $|S| \geq r + k + 1$. Let $T = \{v, v_1, v_2, \ldots, v_r, u_1, u_2, \ldots, u_k\}$. Since every vertex of G lies on a detour joining a pair of vertices of T, it follows that T is a detour basis of G. Also, since $G[T]$ is connected, $cdn(G) = r + k + 1$.* □

Now, in view of Theorem 2.3, we have the following realization result.

Theorem 2.13. *For each pair a, b of integers with $0 \leq a < b$ and $b \geq 3$, there is a connected graph G with $fcdn(G) = a$ and $cdn(G) = b$.*

Proof. **Case 1:** $a = 0$. *For each* $b \geq 3$, *let* G *be a tree with* b *vertices. Then* $fcdn(G) = 0$ *and* $cdn(G) = b$ *by Theorem 2.10(b).*
Case 2: $a \geq 1$. *For each integer* i *with* $1 \leq i \leq a$, *let* F_i *be a copy of the complete graph* K_2, *where* $V(F_i) = \{u_i, v_i\}$ *and let* $H = K_{1,b-a-1}$ *be the star whose vertex set is* $W = \{z_1, z_2, \ldots, z_{b-a-1}, v\}$. *Then the graph* G *is obtained by joining the central vertex* v *of* H *to the vertices of* $F_1, F_2, \ldots, F_a$. *The graph* G *is connected and is shown in Figure 2.4.*

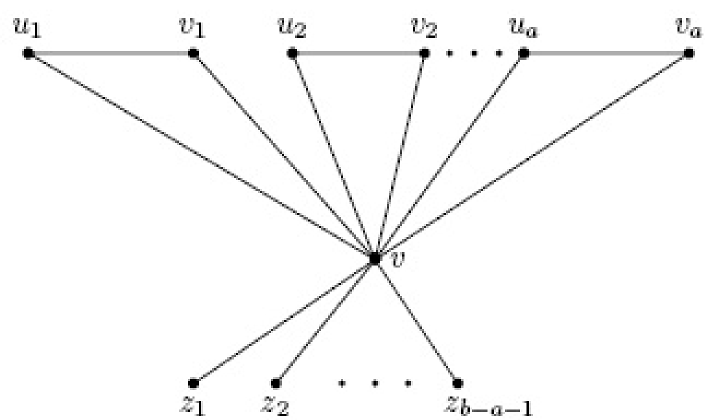

Figure 2.4: G

By Theorem 2.12, $cdn(G) = b$. *Now, we show that* $fdn(G) = a$. *It is clear that* W *is the set all connected detour vertices of* G. *Hence it follows from Theorem 2.6 that* $fcdn(G) \leq cdn(G) - |W| = b - (b - a) = a$. *Now, since* $cdn(G) = b$, *it follows from Theorem 2.11 that any connected detour basis of* G *is of the form* $S = W \cup \{x_1, x_2, \ldots, x_a\}$, *where* $x_i \in \{u_i, v_i\}$ $(1 \leq i \leq a)$. *Let* T *be a subset of* S *with* $|T| < a$. *Then there is a vertex* x_j $(1 \leq j \leq a)$ *such that* $x_j \notin T$. *Let* y_j *be a vertex of* F_j *distinct from* x_j. *Then* $S' = (S - \{x_j\}) \cup \{y_j\}$ *is also a connected detour basis such that it contains* T. *Thus* S *is not the unique connected detour basis containing*

T and so T is not a forcing set of S. Since this is true for all connected detour bases of G, it follows that $fcdn(G) \geq a$ and so $fcdn(G) = a$. $\square$

References

[1] F. Buckley and F. Harary, Distance in Graphs, Addison-Wesley, Reading MA, (1990).

[2] G. Chartrand, H. Escuadro and P. Zhang, Detour distance in graphs, J. Combin. Math. Combin. Comput., 53, pp. 75-94, (2005).

[3] G. Chartrand, L. Johns, and P. Zhang, Detour Number of a Graph, Util. Math. **64***, pp. 97–113, (2003).*

[4] G. Chartrand and P. Zhang, Distance in Graphs–Taking the Long View, AKCE J. Graphs.1. No.1, pp. 1–13, (2004).

[5] G. Chartrand and P. Zang, Introduction to Graph Theory, Tata McGraw-Hill, (2006).

[6] A. P. Santhakumaran and S. Athisayanathan, The connected detour number of a graph, J. Combin. Math. Combin. Comput., 69, pp. 205–218, (2009).

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