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On generating functions of biorthogonal polynomials suggested by the Laguerre polynomials

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Abstract

In this note, we have obtained some novel bilateral generating functions involving Konhauser biorthogonal polynomials, $Y_n^{\alpha}(x;k)$ which is converted into trilateral generating functions with Tchebycheff polynomials by group theoretic method. As special cases, we have obtained the corresponding results on generalised Laguerre polynomials. Some applications are also given here.

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1. Introduction

In 1965[1], Konhauser extended the notion of a particular pair of biorthogonal polynomial sets as introduced by Spencer and Fano [3] and established general properties of biorthogonal sets. In [2], Konhauser also introduced two sets of polynomials $\{Y_n^{\alpha}(x;k)\}$ and $\{Z_n^{\alpha}(x;k)\}$, which are biorthogonal with respect to the weight function $x^{\alpha}e^{-x}$ over the interval $(0,\infty)$, $\alpha > -1$, k is a positive integer. For previous works on these polynomials one can see the works[[10]-[15]]. For $k = 1$, these polynomials reduce to the generalized Laguerre polynomials, $L_n^{\alpha}(x)$. In the present paper we are interested only on $Y_n^{\alpha}(x;k)$. In [5], Carlitz gave an explicit representation for the polynomials $Y_n^{\alpha}(x;k)$ in the following form:

$\mathrm{Y}_n^{\alpha}(x;k) = \frac{1}{n!}\sum_{i=0}^{n}\frac{x^i}{i!}\sum_{j=0}^{i}(-1)^j \begin{pmatrix} i \\ j \end{pmatrix} \left(\frac{j+\alpha+1}{k}\right)_n$, where $(a)_n$ is the pochhammer symbol [9].

The aim at presenting this paper is to obtain the trilateral generating functions for the Konhauser biorthogonal polynomials, $Y_n^{\alpha}(x;k)$ with Tchebycheff polynomials by the group-theoretic method. At first we shall obtain the following theorem on bilateral generating functions.

Theorem 1.1. *If there exists a unilateral generating relation of the form*

$$G(x,\ w) = \sum_{n=0}^{\infty} a_n\ Y_n^{\alpha}(x;k)\ w^n \tag{1.1}$$

then

$$(1+kw)^{\frac{(1+\alpha-k)}{k}} exp[x\{1-(1+kw)^{\frac{1}{k}}\}]G(x(1+kw)^{\frac{1}{k}}, wv) = \sum_{n=0}^{\infty} w^n \sigma_n(x,v), \tag{1.2}$$

where $\sigma_n(x,\ v) = \sum_{p=0}^{n} a_p\ k^{n-p} \begin{pmatrix} n \\ p \end{pmatrix} Y_n^{\alpha+pk-nk}(x;k)v^p.$

Furthermore, we would like to point it out that we have given some applications of our theorem in this paper.

2. Operator and extended form of the group

At first, we seek a linear partial differential operator R of the form: $R = A_1(x,y,z)\,\frac{\partial}{\partial x} + A_2(x,y,z)\,\frac{\partial}{\partial y} + A_3(x,y,z)\,\frac{\partial}{\partial z} + A_0(x,y,z)$, where each $A_i (i = 0,1,2,3)$ is a function of x, y and z which is independent of n, α such that

$$R[Y_n^{\alpha}(x;k)\ y^{\alpha} z^n] = c(n,\alpha) Y_{n+1}^{\alpha-k}(x;k) y^{\alpha-k} z^{n+1}, \tag{2.1}$$

where $c(n,\alpha)$ is a function of n, α and is independent of x, y and z.

Using (2.1) and with the help of the differential recurrence relation:

$$x\frac{d}{dx}[Y_n^{\alpha}(x;k)] = k(n+1)Y_{n+1}^{\alpha-k}(x;k) + (x+k-\alpha-1)Y_n^{\alpha}(x;k) \tag{2.2}$$

we easily obtain the following linear partial differential operator: $R = xy^{-k} z\,\frac{\partial}{\partial x} + y^{-(k-1)} z\,\frac{\partial}{\partial y} - (x+k-1)y^{-k} z$ such that

$$R[Y_n^{\alpha}(x;k)\ y^{\alpha} z^n] = k(n+1)\ Y_{n+1}^{\alpha-k}(x;k) y^{\alpha-k} z^{n+1}. \tag{2.3}$$

The extended form of the group generated by R is given by $e^{wR} f(x,\ y,\ z) = (1+kwy^{-k}z)^{\frac{1-k}{k}}\ exp[\ x\{1-(1+kwy^{-k}z)^{\frac{1}{k}}\}\ ]$

$$\times\ f\left(x(1+kwy^{-k}z)^{\frac{1}{k}},\ y(1+kwy^{-k}z)^{\frac{1}{k}}, z\right), \tag{2.4}$$

where $f(x,y,z)$ is an arbitrary function and w is an arbitrary constant.

3. Derivation of generating function

Now writing $f(x,y,z) = Y_n^{\alpha}(x;k)\, y^{\alpha} z^n$ in (2.4), we get $e^{wR}\left(Y_n^{\alpha}(x;k)\, y^{\alpha} z^n\right) = (1+kwy^{-k}z)^{\frac{1+\alpha-k}{k}}\ exp[\ x\{1-(1+kwy^{-k}z)^{\frac{1}{k}}\}\ ]y^{\alpha} z^n$

$$\times\ Y_n^{\alpha}\left(x(1+kwy^{-k}z)^{\frac{1}{k}}; k\right). \tag{3.1}$$

Again, on the other hand, with the help of (2.3) we have

$$e^{wR}\left(Y_n^{\alpha}(x;k)\ y^{\alpha} z^n\right) = \sum_{m=0}^{\infty} \frac{w^m}{m!}(n+1)_m\ k^m Y_{n+m}^{\alpha-mk}(x;k) y^{\alpha-mk} z^{n+m}. \tag{3.2}$$

Equating (3.1) and (3.2) and then substituting $wy^{-k}z = t$, we get

$$(1+\text{kt})^{\frac{1+\alpha-k}{k}}\ exp[\ x\{1-(1+kt)^{\frac{1}{k}}\}\ ]Y_n^{\alpha}\left(x(1+kt)^{\frac{1}{k}};k\right)$$

$$(3.3)\qquad = \sum_{m=0}^{\infty} k^m \begin{pmatrix} n+m \\ m \end{pmatrix} Y_{n+m}^{\alpha-mk}(x;k)t^m,$$

which is also found derived in [8] by the classical method.

Corollary 3.1. *Putting n=0 in (3.3), we get the following generating relation:*

$$(3.4)\ (1+kt)^{\frac{1+\alpha-k}{k}}\ exp[\ x\{1-(1+kt)^{\frac{1}{k}}\}\ ] = \sum_{m=0}^{\infty} k^m Y_m^{\alpha-mk}(x;k)t^m,$$

which is found derived in [8, 12, 14] by different methods.

Special case 1 If we put $k=1$, then $Y_n^{\alpha}(x;k)$ reduces to the generalized Laguerre polynomials, $L_n^{\alpha}(x)$. Thus putting $k=1$ in (3.3), we get the following generating relation on Laguerre polynomials:

$$(3.5)\ (1+t)^{\alpha}\ exp(-xt)L_n^{\alpha}\left(x(1+t)\right) = \sum_{m=0}^{\infty} \begin{pmatrix} n+m \\ m \end{pmatrix} L_{n+m}^{\alpha-m}(x)t^m,$$

which is found derived in [6, 7, 19,20,21,24].

Sub case 1 Putting n=0 in the above relation, we get the following generating relation:

$$(3.6)\qquad (1+t)^{\alpha}\ exp(-xt) = \sum_{m=0}^{\infty} L_m^{\alpha-m}(x)t^m,$$

which are found derived in [26].

Sub case 2 Using the relation $L_n^{\alpha-n}(x) = \frac{(-x)^n}{n!}C_n(\alpha;x)$, $[4, p.227]$ and from the above generating relations we get the following generating relations on Charlier polynomials [4, p. 226]:

$$(3.7)(1-\frac{y}{x})^{\alpha+n}\ exp(y)C_n\left(\alpha+n;x-y\right) = \sum_{m=0}^{\infty} \frac{y^m}{m!} C_{n+m}\left(\alpha+n;x\right).$$

Replacing $\alpha - n$ by α, we get

$$(3.8) \qquad (1-\frac{y}{x})^{\alpha}\ exp(y)C_n\Bigg(\alpha; x-y\Bigg) = \sum_{m=0}^{\infty} \frac{y^m}{m!} C_{n+m}\Bigg(\alpha; x\Bigg)$$

and putting n=0 in the above relation, we get

$$(3.9) \qquad (1-\frac{y}{x})^{\alpha}\ exp(y) = \sum_{m=0}^{\infty} \frac{y^m}{m!} C_m\Bigg(\alpha; x\Bigg),$$

which are found derived in [16;p.84 , 17; p. 36].

Now we proceed to prove the Theorem 1.

4. Proof of the theorem 1

Let us now consider the generating relation of the form:

$$(4.1) \qquad G(x,w) = \sum_{n=0}^{\infty} a_n\ Y_n^{\alpha}(x;k)\ w^n.$$

Replacing w by wvz and multiplying both sides of (4.1) by y^{α} and finally operating e^{wR} on both sides, we get

$$(4.2) \qquad e^{wR}\Bigg[y^{\alpha}\ G(x,wvz)\Bigg] = e^{wR}\Bigg[\sum_{n=0}^{\infty} a_n\ Y_n^{\alpha}(x;k)\ y^{\alpha} z^n (wv)^n\Bigg].$$

Now the left number of (4.2), with the help of (2.4), reduces to

$$(1+\text{kwy}^{-k}z)^{\frac{1+\alpha-k}{k}}\ exp[\ x\{1-(1+kwy^{-k}z)^{\frac{1}{k}}\}\ ]y^{\alpha}$$

$$(4.3) \qquad \times\ G\Bigg(x(1+kwy^{-k}z)^{\frac{1}{k}}, wvz\Bigg).$$

The right number of (4.2), with the help of(2.3), becomes

$$(4.4) \qquad = \sum_{n=0}^{\infty}\sum_{p=0}^{n} a_{n-p} w^n k^p \begin{pmatrix} n \\ p \end{pmatrix} Y_n^{\alpha-pk}(x;k) y^{\alpha-pk} z^n v^{n-p}.$$

Now equating (4.3) and (4.4) and then substituting $y = z = 1$, we get

$$(1+kw)^{\frac{(1+\alpha-k)}{k}} exp[x\{1-(1+kw)^{\frac{1}{k}}\}]G(x(1+kw)^{\frac{1}{k}}, wv) = \sum_{n=0}^{\infty} w^n \sigma_n(x,v),$$

(4.5)

where $\sigma_n(x,\ v) = \sum_{p=0}^{n}\ a_p\ k^{n-p}\ \left(\begin{array}{c} n \\ p \end{array}\right)\ Y_n^{\alpha+pk-nk}(x;k)v^p.$

This completes the proof of the theorem and does not seem to have appeared in the earlier works.

Special case 2 Now putting k=1 in our Theorem 1.1 we get the following result on generalised Laguerre polynomials:

Theorem 4.1. *If there exists a unilateral generating relation of the form*

$$G(x,\ w) = \sum_{n=0}^{\infty}\ a_n\ L_n^{(\alpha)}(x)\ w^n \tag{4.6}$$

then

$$(1+w)^{\alpha} exp(-xw) G(x(1+w), wv) = \sum_{n=0}^{\infty} w^n \sigma_n(x,v), \tag{4.7}$$

where $\sigma_n(x,\ v) = \sum_{p=0}^{n}\ a_p\ \left(\begin{array}{c} n \\ p \end{array}\right)\ L_n^{(\alpha+p-n)}(x)v^p,$

which is found derived in [20, 23, 24].

5. Applications

A1. As an application of our Theorem 1.1, we consider the following generating relation[5,11]:

$$(1-w)^{\frac{-(1+\alpha)}{k}} exp\left\{ -x[(1-w)^{\frac{-1}{k}} - 1]\right\} = \sum_{p=0}^{\infty}\ Y_n^{\alpha}(x;k)w^n. \tag{5.1}$$

Taking $a_n = 1$, we get G(x,w)=(1-w)$^{\frac{-(1+\alpha)}{k}} exp\left\{ -x[(1-w)^{\frac{-1}{k}} - 1]\right\}$.

By applying our Theorem 1.1, we get

$$(1+kw)^{\frac{(1+\alpha-k)}{k}} (1-wv)^{\frac{-(1+\alpha)}{k}} exp\left\{ x[1-(1+kw)^{\frac{1}{k}}\ (1-wv)^{\frac{-1}{k}}]\right\} = \sum_{n=0}^{\infty} w^n \sigma_n(x,v), \tag{5.2}$$

where $\sigma_n(x,\ v) = \sum_{p=0}^{n}\ k^{n-p} \begin{pmatrix} n \\ p \end{pmatrix}\ Y_n^{\alpha+pk-nk}(x;k)v^p.$

It is of interest to mention that the result (5.2) for$k = 1$ is also obtained by applying Theorem 4.1, on (5.1) for $k = 1$.

A2. As an application of our Theorem 4.1, we consider the following generating relation[25]:

$$(5.3)\qquad (1-w)^{-c}\ {}_1F_1\left[c; 1+\alpha; \frac{-xw}{1-w}\right] = \sum_{p=0}^{\infty} \frac{(c)_n}{(1+\alpha)_n} L_n^{(\alpha)}(x) w^n.$$

Taking $a_n = \frac{(c)_n}{(1+\alpha)_n}$, we get G(x,w)=(1-w)$^{-c}$ ${}_1F_1\left[c; 1+\alpha; \frac{-xw}{1-w}\right]$.

By applying our Theorem 4.1, we get

$$(1+w)^{\alpha} exp(-xw)(1-wv)^{-c}\ {}_1F_1\left[c; 1+\alpha; \frac{-xwv(1+w)}{1-wv}\right] = \sum_{n=0}^{\infty} w^n \sigma_n(x,v),$$
(5.4)

where $\sigma_n(x,\ v) = \sum_{p=0}^{n}\ \frac{(c)_p}{(1+\alpha)_p} \begin{pmatrix} n \\ p \end{pmatrix}\ L_n^{(\alpha+p-n)}(x)v^p.$

6. Trilateral generating functions of biorthogonal polynomials

In this Section the above bilateral generating function has been converted to trilateral generating relation with Tchebycheff polynomial by means of the relation $\mathrm{T}_n(x) = \frac{1}{2}\left[\left(x+\sqrt{x^2-1}\right)^n + \left(x-\sqrt{x^2-1}\right)^n\right]$, utilizing the method of Chongdar and Chatterjea [27].

Now to convert the above bilateral generating relation into a trilateral generating relation with Tchebycheff polynomial as done in [27], we notice that

$$\sum_{n=0}^{\infty} w^n\sigma_n(x,v)T_n(u) = \frac{1}{2}\Big[(1+k\rho_1)^{\frac{(1+\alpha-k)}{k}} exp[x\{1-(1+k\rho_1)^{\frac{1}{k}}\}]G(x(1+k\rho_1)^{\frac{1}{k}}, \rho_1 v)\ + (1+k\rho_2)^{\frac{(1+\alpha-k)}{k}} exp[x\{1-(1+k\rho_2)^{\frac{1}{k}}\}]G(x(1+k\rho_2)^{\frac{1}{k}}, \rho_2 v)\Big],$$

where $\rho_1 = w(u+\sqrt{u^2-1}\ )$ *and* $\rho_2 = w(u-\sqrt{u^2-1}\ )$. Thus we have the following general theorem

Theorem 6.1. *If there exists a unilateral generating relation of the form*

$$G(x,\ w) = \sum_{n=0}^{\infty} a_n\ Y_n^{\alpha}(x;k)\ w^n \tag{6.1}$$

then

$$\frac{1}{2}\Bigg[(1+k\rho_1)^{\frac{(1+\alpha-k)}{k}} exp[x\{1-(1+k\rho_1)^{\frac{1}{k}}\}]G(x(1+k\rho_1)^{\frac{1}{k}}, \rho_1 v)$$

$$+ (1+k\rho_2)^{\frac{(1+\alpha-k)}{k}} exp[x\{1-(1+k\rho_2)^{\frac{1}{k}}\}]G(x(1+k\rho_2)^{\frac{1}{k}}, \rho_2 v)\Bigg]$$

$$=\textstyle\sum_{n=0}^{\infty} w^n \sigma_n(x,v) T_n(u), \tag{6.2}$$

where $\sigma_n(x,\ v) = \sum_{p=0}^{n}\ a_p\ k^{n-p} \begin{pmatrix} n \\ p \end{pmatrix} Y_n^{\alpha+pk-nk}(x;k)\, v^p$

which is believed to be new.

Special Case 4 By putting k=1 in our Theorem 6.1, we get the general theorem on Laguerre polynomials, $L_n^{\alpha}(x)$ found derived in [27, 28, 29].

Again using (3.3) we get,

$$\textstyle\sum_{m=0}^{\infty} k^m \begin{pmatrix} n+m \\ m \end{pmatrix} Y_{n+m}^{\alpha-mk}(x;k) t^m T_m(y)$$

$$= \frac{1}{2}\Bigg[\left\{1+kt(y+\sqrt{y^2-1})\right\}^{\frac{1+\alpha-k}{k}} exp\left(x - x\{1+kt(y+\sqrt{y^2-1})\}^{\frac{1}{k}}\right)$$

$$Y_n^{\alpha}\left(x\{1+kt(y+\sqrt{y^2-1})\}^{\frac{1}{k}}; k\right) + \left\{1+kt(y-\sqrt{y^2-1})\right\}^{\frac{1+\alpha-k}{k}}$$

$$exp\left(x - x\{1+kt(y-\sqrt{y^2-1})\}^{\frac{1}{k}}\right) Y_n^{\alpha}\left(x\{1+kt(y-\sqrt{y^2-1})\}^{\frac{1}{k}}; k\right)\Bigg] \tag{6.3}$$

Corollary 6.2. *Putting n=0 in 6.3, we get the following generating relation:*

$$\sum_{m=0}^{\infty} k^m Y_m^{\alpha-mk}(x;k)t^m T_m(y)$$

$$= \frac{1}{2}\Bigg[\Bigg\{1+kt(y+\sqrt{y^2-1})\Bigg\}^{\frac{1+\alpha-k}{k}} exp\Bigg(x - x\{1+kt(y+\sqrt{y^2-1})\}^{\frac{1}{k}}\Bigg)$$

$$+\Bigg\{1+kt(y-\sqrt{y^2-1})\Bigg\}^{\frac{1+\alpha-k}{k}} exp\Bigg(x - x\{1+kt(y-\sqrt{y^2-1})\}^{\frac{1}{k}}\Bigg)\Bigg]$$

(6.4)

Special Case 5 For k=1, we get the following generating relations on Laguerre polynomials, $L_n^{\alpha}(x)$:

$$\sum_{m=0}^{\infty} \binom{n+m}{m} L_{n+m}^{\alpha-m}(x)t^m T_m(y)$$

$$= \frac{1}{2}\Bigg[\Bigg\{1+t(y+\sqrt{y^2-1})\Bigg\}^{\alpha} exp\Bigg(-xt(y+\sqrt{y^2-1})\Bigg) L_n^{\alpha}\Bigg(x\{1+t(y+\sqrt{y^2-1})\}\Bigg)$$

$$+\Bigg\{1+t(y-\sqrt{y^2-1})\Bigg\}^{\alpha} exp\Bigg(-xt(y-\sqrt{y^2-1})\Bigg) L_n^{\alpha}\Bigg(x\{1+t(y-\sqrt{y^2-1})\}\Bigg)\Bigg]$$

(6.5)

and

$$\sum_{m=0}^{\infty} L_m^{\alpha-m}(x)t^m T_m(y)$$

$$= \frac{1}{2}\Bigg[\Bigg\{1+t(y+\sqrt{y^2-1})\Bigg\}^{\alpha} exp\Bigg(-xt(y+\sqrt{y^2-1})\Bigg)$$

$$+\Bigg\{1+t(y-\sqrt{y^2-1})\Bigg\}^{\alpha} exp\Bigg(-xt(y-\sqrt{y^2-1})\Bigg)\Bigg] \tag{6.6}$$

Therefore the generating relations (6.3-6.6) are believed to be new.

Finally using (5.2) and (5.4) we get,
$\sum_{n=0}^{\infty} w^n \sigma_n(x,v) T_n(y)$

$$= \frac{1}{2}\Bigg[\Big\{1 + kw(y + \sqrt{y^2-1})\Big\}^{\frac{(1+\alpha-k)}{k}} \Big\{1 - wv(y + \sqrt{y^2-1})\Big\}^{\frac{-(1+\alpha)}{k}}$$

$$\times\ exp\Big\{ - x\Big[1 - \Big\{1 + kw(y + \sqrt{y^2-1})\Big\}^{\frac{1}{k}} \Big\{1 - wv(y + \sqrt{y^2-1})\Big\}^{\frac{-1}{k}}\Big]\Big\}$$

$$+ \Big\{1 + kw(y - \sqrt{y^2-1})\Big\}^{\frac{(1+\alpha-k)}{k}} \Big\{1 - wv(y - \sqrt{y^2-1})\Big\}^{\frac{-(1+\alpha)}{k}}$$

$$\times\ exp\Big\{ - x\Big[1 - \Big\{1 + kw(y - \sqrt{y^2-1})\Big\}^{\frac{1}{k}} \Big\{1 - wv(y - \sqrt{y^2-1})\Big\}^{\frac{-1}{k}}\Big]\Big\}\Bigg]$$

(6.7)

where

$\sigma_n(x,\ v) = \sum_{p=0}^{n}\ k^{n-p} \begin{pmatrix} n \\ p \end{pmatrix} Y_n^{\alpha+pk-nk}(x;k)v^p.$

and

$$\sum_{n=0}^{\infty} w^n \sigma_n(x,v) T_n(y)$$

$$= \frac{1}{2}\Bigg[\Big\{1+w(y+\sqrt{y^2-1})\Big\}^{\alpha} \Big\{1-wv(y+\sqrt{y^2-1})\Big\}^{c} exp\Big\{-xw(y+\sqrt{y^2-1})\Big\}$$

(6.8)

$$\times\ {}_1F_1\left[c; 1+\alpha; \frac{-xwv(y+\sqrt{y^2-1})\left\{1+w(y+\sqrt{y^2-1})\right\}}{1-wv(y+\sqrt{y^2-1})}\right]$$

$$+\left\{1+w(y-\sqrt{y^2-1})\right\}^{\alpha}\left\{1-wv(y-\sqrt{y^2-1})\right\}^{c} exp\left\{-xw(y-\sqrt{y^2-1})\right\}$$

$$\times\ {}_1F_1\left[c; 1+\alpha; \frac{-xwv(y-\sqrt{y^2-1})\left\{1+w(y-\sqrt{y^2-1})\right\}}{1-wv(y-\sqrt{y^2-1})}\right]\Bigg]$$

where

$$\sigma_n(x,\ v) = \sum_{p=0}^{n} \frac{(c)_p}{(1+\alpha)_p} \begin{pmatrix} n \\ p \end{pmatrix} L_n^{(\alpha+p-n)}(x)v^p,$$

which are believed to be new.

7. Conclusions

From the above discussion, it is clear that whenever one knows a generating relation of the form (1.1, 4.6) then the corresponding bilateral generating function can at once be written down from (1.2, 4.7). So one can get a large number of bilateral generating functions by attributing different suitable values to a_n in (1.1, 4.6).

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