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On Zweier I-convergent sequence spaces

Vakeel A. Khan
Khalid Ebadullah
and
Yasmeen
Aligarh Muslim University, India
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Abstract

In this article we introduce the Zweier I-convergent sequence spaces $\mathcal{Z}^I, \mathcal{Z}_0^I$ and $\mathcal{Z}_\infty^I$. We prove the decomposition theorem and study topological, algebraic properties and have established some inclusion relations of these spaces.

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1. Introduction

Let $\mathbf{N}$,$\mathbf{R}$ and $\mathbf{C}$ be the sets of all natural, real and complex numbers respectively. We write

$$\omega = \{x = (x_k) : x_k \in \mathbf{R} \ or \ \mathbf{C} \ \},$$

the space of all real or complex sequences.

Let ℓ_∞, c and c_0 denote the Banach spaces of bounded,convergent and null sequences respectively normed by

$$||x||_\infty = \sup_k |x_k|.$$

A sequence space λ with linear topology is called a K-space provided each of maps $p_i \longrightarrow \mathbf{C}$ defined by $p_i(x) = x_i$ is continuous for all $i \in \mathbf{N}$.

A K-space λ is called an FK-space provided λ is a complete linear metric space.

An FK-space whose topology is normable is called a BK-space.

Let λ and μ be two sequence spaces and $A = (a_{nk})$ be an infinite matrix of real or complex numbers (a_{nk}), where $n, k \in \mathbf{N}$. Then we say that A defines a matrix mapping from λ to μ, and we denote it by writting $A : \lambda \longrightarrow \mu$.

If for every sequence $x = (x_k) \in \lambda$ the sequence $Ax = \{(Ax)_n\}$, the A transform of x is in μ, where

$$(Ax)_n = \sum_k a_{nk} x_k, \ (n \in \mathbf{N}). \tag{1.1}$$

By $(\lambda : \mu)$, we denote the class of matrices A such that $A : \lambda \longrightarrow \mu$. Thus, $A \in (\lambda : \mu)$ if and only if series on the right side of (1) converges for each $n \in \mathbf{N}$ and every $x \in \lambda$.

The approach of constructing new sequence spaces by means of the matrix domain of a particular limitation method have recently been employed by Altay,Başar and Mursaleen [1], Başar and Altay [2], Malkowsky [13], Ng and Lee [14], and Wang [21].

Şengönül[18] defined the sequence $y = (y_i)$ which is frequently used as the Z^p transform of the sequence $x = (x_i)$ i. e.,

$$y_i = px_i + (1-p)x_{i-1}$$

where $x_{-1} = 0$, $1 < p < \infty$ and Z^p denotes the matrix $Z^p = (z_{ik})$ defined by

$$z_{ik} = \begin{cases} p, (i = k), \\ 1-p, (i-1 = k); (i, k \in \mathbf{N}), \\ 0, \text{otherwise.} \end{cases}$$

Following Başar and Altay[2], Şengönül[18] introduced the Zweier sequence spaces $\mathcal{Z}$ and $\mathcal{Z}_0$ as follows :

$$\mathcal{Z} = \{x = (x_k) \in \omega : Z^p x \in c\}$$

$$\mathcal{Z}_0 = \{x = (x_k) \in \omega : Z^p x \in c_0\}.$$

Here we list below some of the results of Şengönül [18] which we will need as a reference in order to establish analogously some of the results of this article.

Theorem 1.1. The sets $\mathcal{Z}$ and $\mathcal{Z}_0$ are linear spaces with the co-ordinate wise addition and scalar multiplication which are the BK-spaces with the norm

$$||x||_{\mathcal{Z}} = ||x||_{\mathcal{Z}_0} = ||Z^p x||_c \text{[See (Theorem 2.1. [18])].}$$

Theorem 1.2. The sequence spaces $\mathcal{Z}$ and $\mathcal{Z}_0$ are linearly isomorphic to the spaces c and c_0 respectively, i.e $\mathcal{Z} \cong c$ and $\mathcal{Z}_0 \cong c_0$ [See (Theorem 2.2.[18])]

Theorem 1.3. The inclusions $\mathcal{Z}_0 \subset \mathcal{Z}$ strictly hold for $p \neq 1$. [See (Theorem 2.3. [18])].

Theorem 1.4. $\mathcal{Z}_0$ is solid.[See (Theorem 2.6.[18])].

Theorem 1.5. $\mathcal{Z}$ is not a solid sequence space.[See (Theorem 3.6. [18])].

The concept of statistical convergence was first introduced by Fast [7] and also independently by Buck [3] and Schoenberg [17] for real and complex sequences.Further this concept was studied by Connor [4, 5], Connor, Fridy and Kline [6] and many others. Statistical convergence is a generalization of the usual notion of convergence that parallels the usual theory of convergence. A sequence $x = (x_k)$ is said to be statistically convergent to L if for a given $\varepsilon > 0$

$$\lim_k \frac{1}{k}|\{i : |x_i - L| \geq \varepsilon, i \leq k\}| = 0.$$

The notion of I-convergence generalizes and unifies different notions of convergence including the notion of statistical convergence. At the initial stage it was studied by Kostyrko, Šalát, Wilczyński [12]. Later on it was studied by Šalát, Tripathy, Ziman [15, 16]. Recentlly further it was studied by Tripathy [19, 20, 21, 22, 23, 24, 25, 26, 27], and V. A.Khan and Khalid Ebadullah [9-11].

Here we give some preliminaries about the notion of I-convergence.

Let X be a non empty set. Then a family of sets I$\subseteq 2^X$(2^X denoting the power set of X) is said to be an ideal if I is additive i.e A,B$\in$I $\Rightarrow$A$\cup$ B$\in$I and hereditary i.e A$\in$I, B$\subseteq$A$\Rightarrow$B$\in$I.

A non-empty family of sets $\pounds(I) \subseteq 2^X$ is said to be filter on X if and only if $\emptyset \notin \pounds(I)$, for A, B$\in \pounds(I)$ we have A$\cap$B$\in \pounds(I)$ and for each A$\in \pounds(I)$ and A$\subseteq$B implies B$\in \pounds(I)$.

An Ideal I$\subseteq 2^X$ is called non-trivial if I$\neq 2^X$.
A non-trivial ideal I$\subseteq 2^X$ is called admissible if $\{\{x\} : x \in X\}$ $\subseteq$I. A non-trivial ideal I is maximal if there cannot exist any non-trivial ideal J$\neq$I containing I as a subset.

For each ideal I, there is a filter $\pounds$(I) corresponding to I. i.e

$$\pounds(I) = \{K \subseteq N : K^c \in I\}, \quad \text{where} \quad K^c = N - K.$$

Definition 1.6. A sequence $(x_k) \in \omega$ is said to be I-convergent to a number L if for every $\varepsilon > 0$.

$$\{k \in N : |x_k - L| \geq \varepsilon\} \in I.$$

In this case we write $I - \lim x_k = L$. The space c^I of all I-convergent sequences to L is given by

$$c^I = \{(x_k) \in \omega : \{k \in \mathbf{N} : |x_k - L| \geq \varepsilon\} \in I, \text{for some L} \in \mathbf{C}\ \}.$$

Definition 1.7. A sequence $(x_k) \in \omega$ is said to be I-null if $L = 0$. In this case we write $I - \lim x_k = 0$.

Definition 1.8. A sequence $(x_k) \in \omega$ is said to be I-Cauchy if for every $\varepsilon > 0$ there exists a number m $=$ m(ε) such that

$$\{k \in N : |x_k - x_m| \geq \varepsilon\} \in I.$$

Definition 1.9. A sequence $(x_k) \in \omega$ is said to be I-bounded if there exists $M > 0$ such that

$$\{k \in N : |x_k| > M\} \in I.$$

Example 1.10. Take for I the class I_f of all finite subsets of $\mathbf{N}$. Then I_f is a non-trivial admissible ideal and I_f convergence coincides with the usual convergence with respect to the metric in X. (see [12]).

Definition 1.11. For $I = I_\delta$ and $A \subset \mathbf{N}$ with $\delta(A) = 0$ respectively. I_δ is a non-trivial admissible ideal, I_δ-convergence is said to be logarithmic statistical convergence(see[12]).

Definition 1.12. A map $\hbar$ defined on a domain $D \subset X$ i.e $\hbar : D \subset X \to \mathbf{R}$ is said to satisfy Lipschitz condition if

$$|\hbar(x) - \hbar(y)| \leq K|x - y|,$$

where K is known as the Lipschitz constant.The class of K-Lipschitz functions defined on D is denoted by $\hbar \in (D, K)$(see[15,16]).

Definition 1.13. A convergence field of I-covergence is a set

$$F(I) = \{x = (x_k) \in l_\infty : \text{there exists } I - \lim x \in \mathbf{R}\}.$$

The convergence field $F(I)$ is a closed linear subspace of l_∞ with respect to the supremum norm, $F(I) = l_\infty \cap c^I$(See [15,16]).

Define a function $\hbar : F(I) \to \mathbf{R}$ such that $\hbar(x) = I - \lim x$, for all $x \in F(I)$, then the function $\hbar : F(I) \to \mathbf{R}$ is a Lipschitz function. (see [15, 16]).

Definition 1.14. Let $(x_k), (y_k)$ be two sequences. We say that $(x_k) = (y_k)$ for almost all k relative to I (a.a.k.r.I), if

$$\{k \in \mathbf{N} : x_k \neq y_k\} \in I(see[19, 20]).$$

The following Lemmas will be used for establishing some results of this article :

Lemma 1.15. Let E be a sequence space. If E is solid then E is monotone.(see [8],page 53).

Lemma 1.16. If I $\subset 2^N$ and M$\subseteq$N. If M $\notin$I, then M$\cap$N $\notin$I. (see [19,20]).

2. Main Results

In this section we introduce the following classes of sequence spaces :

$$\mathcal{Z}^I = \{x = (x_k) \in \omega : \{k \in \mathbf{N} : I - \lim Z^p x = L, \quad \text{for some L} \in \mathbf{C}\ \}\};$$

$$\mathcal{Z}_0^I = \{x = (x_k) \in \omega : \{k \in \mathbf{N} : I - \lim Z^p x = 0\}\};$$

$$\mathcal{Z}_\infty^I = \{x = (x_k) \in \omega : \{k \in \mathbf{N} : \sup_k |Z^p x| < \infty\}\}.$$

We also denote by

$$m_{\mathcal{Z}}^I = \mathcal{Z}_\infty \cap \mathcal{Z}^I$$

and

$$m_{\mathcal{Z}_0}^I = \mathcal{Z}_\infty \cap \mathcal{Z}_0^I.$$

Throughout the article, for the sake of conveinence now we will denote by
$Z^p(x_k) = x^/, Z^p(y_k) = y^/, Z^p(z_k) = z^/$ for $x, y, z \in \omega$.

Theorem 2.1. The classes of sequences $\mathcal{Z}^I, \mathcal{Z}_0^I, m_{\mathcal{Z}}^I$ and $m_{\mathcal{Z}_0}^I$ are linear spaces.

Proof. We shall prove the result for the space $\mathcal{Z}^I$.

The proof for the other spaces will follow similarly.

Let $(x_k), (y_k) \in \mathcal{Z}^I$ and let α, β be scalars. Then

$$I - \lim |x_k^/ - L_1| = 0, \quad \text{for some L}_1 \in \mathbf{C};$$

$$I - \lim |y_k^{/} - L_2| = 0, \text{ for some } \quad \mathrm{L}_2 \in \mathbf{C};$$

That is for a given $\varepsilon > 0$, we have

$$A_1 = \{k \in N : |x_k^{/} - L_1| > \frac{\varepsilon}{2}\} \in I,$$

$$A_2 = \{k \in N : |y_k^{/} - L_2| > \frac{\varepsilon}{2}\} \in I. \tag{2.1}$$

we have

$$|(\alpha x_k^{/} + \beta y_k^{/}) - (\alpha L_1 + \beta L_2)| \leq |\alpha|(|x_k^{/} - L_1|) + |\beta|(|y_k^{/} - L_2|)$$

$$\leq |x_k^{/} - L_1| + |y_k^{/} - L_2|$$

Now, by (1) and (2), $\{k \in \mathrm{N}: |(\alpha x_k^{/} + \beta y_k^{/}) - (\alpha L_1 + \beta L_2)| > \epsilon\} \subset A_1 \cup A_2$.

Therefore $(\alpha x_k + \beta y_k) \in \mathcal{Z}^I$

Hence $\mathcal{Z}^I$ is a linear space.

Theorem 2.2. The spaces $m_{\mathcal{Z}}^I$ and $m_{\mathcal{Z}_0}^I$ are normed linear spaces,normed by

$$||x_k^{/}||_* = \sup_k |Z^p(x)|, \tag{2.2}$$

where $x_k^{/} = Z^p(x)$.

Proof: It is clear from Theorem 2.1 that $m_{\mathcal{Z}}^I$ and $m_{\mathcal{Z}_0}^I$ are linear spaces.

It is easy to verify that (3) defines a norm on the spaces $m_{\mathcal{Z}}^I$ and $m_{\mathcal{Z}_0}^I$.

Theorem 2.3. A sequence $x = (x_k) \in m_{\mathcal{Z}}^I$ I-converges if and only if for every $\epsilon > 0$ there exists $N_\epsilon \in \mathbf{N}$ such that

$$\{k \in \mathbf{N} : |x_k^{/} - x_{N_\epsilon}^{/}| < \varepsilon\} \in m_{\mathcal{Z}}^I \tag{2.3}$$

Proof. Suppose that $L = I - \lim x^{/}$. Then

$$B_\varepsilon = \{k \in \mathbf{N} : |x_k^{/} - L| < \frac{\varepsilon}{2}\} \in m_{\mathcal{Z}}^I \quad \text{for all } \varepsilon > 0$$

. Fix an $N_\varepsilon \in B_\varepsilon$. Then we have

$$|x^{/}_{N_\varepsilon} - x^{/}_k| \leq |x^{/}_{N_\varepsilon} - L| + |L - x^{/}_k| < \frac{\epsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

which holds for all $k \in B_\varepsilon$.

Hence $\{k \in \mathbf{N} : |x^{/}_k - x^{/}_{N_\varepsilon}| < \varepsilon\} \in m^I_{\mathcal{Z}}$.

Conversely, suppose that $\{k \in \mathbf{N} : |x^{/}_k - x^{/}_{N_\varepsilon}| < \varepsilon\} \in m^I_{\mathcal{Z}}$.
That is $\{k \in \mathbf{N} : |x^{/}_k - x^{/}_{N_\varepsilon}| < \varepsilon\} \in m^I_{\mathcal{Z}}$ for all $\varepsilon > 0$. Then the set

$$C_\varepsilon = \{k \in \mathbf{N} : x^{/}_k \in [x^{/}_{N_\varepsilon} - \varepsilon, x^{/}_{N_\varepsilon} + \varepsilon]\} \in m^I_{\mathcal{Z}} \text{ for all } \varepsilon > 0.$$

Let $J_\varepsilon = [x^{/}_{N_\varepsilon} - \varepsilon, x^{/}_{N_\varepsilon} + \varepsilon]$.If we fix an $\epsilon > 0$ then we have $C_\epsilon \in m^I_{\mathcal{Z}}$ as well as $C_{\frac{\varepsilon}{2}} \in m^I_{\mathcal{Z}}$. Hence $C_\varepsilon \cap C_{\frac{\varepsilon}{2}} \in m^I_{\mathcal{Z}}$.This implies that

$$J = J_\varepsilon \cap J_{\frac{\varepsilon}{2}} \neq \phi$$

that is

$$\{k \in \mathbf{N} : x^{/}_k \in J\} \in m^I_{\mathcal{Z}}$$

that is

$$diam J \leq diam J_\varepsilon$$

where the diam of J denotes the length of interval J. In this way, by induction we get the sequence of closed intervals

$$J_\varepsilon = I_0 \supseteq I_1 \supseteq ..... \supseteq I_k \supseteq ....$$

with the property that $diam I_k \leq \frac{1}{2} diam I_{k-1}$ for (k=2,3,4,.....) and

$\{k \in \mathbf{N} : x^{/}_k \in I_k\} \in m^I_{\mathcal{Z}}$ for (k=1,2,3,4,......).

Then there exists a $\xi \in \cap I_k$ where $k \in \mathbf{N}$ such that $\xi^{/} = I - \lim x^{/}$, that is $L = I - \lim x^{/}$.

Theorem 2.4. Let I be an admissible ideal.Then the following are equivalent.
(a) $(x_k) \in \mathcal{Z}^I$;

(b) there exists $(y_k) \in \mathcal{Z}$ such that $x_k = y_k$, for a.a.k.r.I;

(c) there exists $(y_k) \in \mathcal{Z}$ and $(z_k) \in \mathcal{Z}_0^I$ such that $x_k = y_k + z_k$ for all $k \in \mathbf{N}$ and $\{k \in \mathbf{N} : |y_k - L| \geq \epsilon\} \in I$;

(d) there exists a subset $K = \{k_1 < k_2....\}$ of $\mathbf{N}$ such that $K \in \pounds(I)$ and $\lim\limits_{n\to\infty} |x_{k_n} - L| = 0$.

Proof.(a) implies (b). Let $(x_k) \in \mathcal{Z}^I$. Then there exists $L \in \mathbf{C}$ such that

$$\{k \in \mathbf{N} : |x_k^{/} - L| \geq \varepsilon\} \in I$$

.Let (m_t) be an increasing sequence with $m_t \in \mathbf{N}$ such that

$$\{k \leq m_t : |x_k^{/} - L| \geq \frac{1}{t}\} \in I.$$

Define a sequence (y_k) by

$$y_k = x_k, \text{ for all } \; k \leq m_1.$$

For $m_t < k \leq m_{t+1}, t \in \mathbf{N}$.

$$y_k = \begin{cases} x_k, & \text{if } |x_k^{/} - L| < t^{-1}, \\ & L, \text{otherwise.} \end{cases}$$

Then $(y_k) \in \mathcal{Z}$ and form the following inclusion

$$\{k \leq m_t : x_k \neq y_k\} \subseteq \{k \leq m_t : |x_k^{/} - L| \geq \epsilon\} \in I.$$

We get $x_k = y_k$, for a.a.k.r.I.
(b) implies (c).For $(x_k) \in \mathcal{Z}^I$.

Then there exists $(y_k) \in \mathcal{Z}$ such that $x_k = y_k$, for a.a.k.r.I.

Let $K = \{k \in \mathbf{N} : x_k \neq y_k\}$, then $K \in I$.

Define a sequence (z_k) by

$$z_k = \begin{cases} x_k - y_k, & \text{if } k \in K, \\ & 0, \text{otherwise.} \end{cases}$$

Then $z_k \in \mathcal{Z}_0^I$ and $y_k \in \mathcal{Z}$.

(c) implies (d).Let $P_1 = \{k \in \mathbf{N} : |z_k| \geq \varepsilon\} \in I$ and

$$K = P_1^c = \{k_1 < k_2 < k_3 < ...\} \in \pounds(I)$$

. Then we have $\lim_{n\to\infty} |x_{k_n} - L| = 0$.

(d) implies (a). Let $K = \{k_1 < k_2 < k_3 < ...\} \in \pounds(I)$ and $\lim_{n\to\infty} |x_{k_n} - L| = 0$.

Then for any $\epsilon > 0$, and by Lemma , we have

$$\{k \in \mathbf{N} : |x_k^{/} - L| \geq \epsilon\} \subseteq K^c \cup \{k \in K : |x_k^{/} - L| \geq \epsilon\}.$$

Thus $(x_k) \in \mathcal{Z}^I$.

Theorem 2.5. The inclusions $\mathcal{Z}_0^I \subset \mathcal{Z}^I \subset \mathcal{Z}_\infty^I$ are proper.

Proof: Let $(x_k) \in \mathcal{Z}^I$. Then there exists L $\in$ C such that

$$I - \lim |x_k^{/} - L| = 0$$

We have $|x_k^{/}| \leq \frac{1}{2}|x_k^{/} - L| + \frac{1}{2}|L|$.
Taking the supremum over k on both sides we get $(x_k) \in \mathcal{Z}_\infty^I$.
The inclusion $\mathcal{Z}_0^I \subset \mathcal{Z}^I$ is obvious.

Theorem 2.6. The function $\hbar : m_{\mathcal{Z}}^I \to \mathbf{R}$ is the Lipschitz function,where $m_{\mathcal{Z}}^I = \mathcal{Z}^I \cap \mathcal{Z}_\infty$, and hence uniformly continuous.

Proof:Let $x, y \in m_{\mathcal{Z}}^I$, $x \neq y$.Then the sets

$$A_x = \{k \in \mathbf{N} : |x_k^{/} - \hbar(x^{/})| \geq ||x^{/} - y^{/}||_*\} \in I,$$

$$A_y = \{k \in \mathbf{N} : |y_k^{/} - \hbar(y^{/})| \geq ||x^{/} - y^{/}||_*\} \in I.$$

Thus the sets,

$$B_x = \{k \in \mathbf{N} : |x_k^{/} - \hbar(x^{/})| < ||x^{/} - y^{/}||_*\} \in m_{\mathcal{Z}}^I,$$

$$B_y = \{k \in \mathbf{N} : |y_k^{/} - \hbar(y^{/})| < ||x^{/} - y^{/}||_*\} \in m_{\mathcal{Z}}^I.$$

Hence also $B = B_x \cap B_y \in m_{\mathcal{Z}}^I$, so that $B \neq \phi$.

Now taking k in B,

$$|\hbar(x^/) - \hbar(y^/)| \leq |\hbar(x^/) - x_k^/| + |x_k^/ - y_k^/| + |y^/ - \hbar(y^/)| \leq 3||x^/ - y^/||_*.$$

Thus $\hbar$ is a Lipschitz function.

For $m_{\mathcal{Z}_0}^I$ the result can be proved similarly.

Theorem 2.7. If $x, y \in m_{\mathcal{Z}}^I$, then $(x.y) \in m_{\mathcal{Z}}^I$ and $\hbar(xy) = \hbar(x)\hbar(y)$.

Proof: For $\epsilon > 0$

$$B_x = \{k \in \mathbf{N} : |x^/ - \hbar(x^/)| < \varepsilon\} \in m_{\mathcal{Z}}^I,$$

$$B_y = \{k \in \mathbf{N} : |y^/ - \hbar(y^/)| < \varepsilon\} \in m_{\mathcal{Z}}^I.$$

Now,

$$|x^/.y^/ - \hbar(x^/)\hbar(y^/)| = |x^/.y^/ - x^/\hbar(y^/) + x^/\hbar(y^/) - \hbar(x^/)\hbar(y^/)|$$

$$\leq |x^/||y^/ - \hbar(y^/)| + |\hbar(y^/)||x^/ - \hbar(x^/)| \tag{2.4}$$

As $m_{\mathcal{Z}}^I \subseteq \mathcal{Z}_\infty$, there exists an $M \in \mathbf{R}$ such that $|x^/| < M$ and $|\hbar(y^/)| < M$.

Using eqn(5) we get

$$|x^/.y^/ - \hbar(x^/)\hbar(y^/)| \leq M\varepsilon + Mvar\varepsilon = 2M\varepsilon$$

For all $k \in B_x \cap B_y \in m_{\mathcal{Z}}^I$.

Hence $(x.y) \in m_{\mathcal{Z}}^I$ and $\hbar(xy) = \hbar(x)\hbar(y)$.

For $m_{\mathcal{Z}_0}^I$ the result can be proved similarly.

Theorem 2.8. The spaces $\mathcal{Z}_0^I$ and $m_{\mathcal{Z}_0}^I$ are solid and monotone .

Proof: We prove the result for the case $\mathcal{Z}_0^I$.

Let $(x_k) \in \mathcal{Z}_0^I$.Then

$$I - \lim_k |x_k^/| = 0 \tag{2.5}$$

Let (α_k) be a sequence of scalars with $|\alpha_k| \leq 1$ for all $k \in \mathbf{N}$. Then the result follows from (6) and the following inequality

$|\alpha_k x_k^{/}| \leq |\alpha_k||x_k^{/}| \leq |x_k^{/}|$ for all $k \in \mathbf{N}$.

That the space $\mathcal{Z}_0^I$ is monotone follows from the Lemma 1.15.

For $m_{\mathcal{Z}_0}^I$ the result can be proved similarly.

Theorem 2.9. The spaces $\mathcal{Z}^I$ and $m_{\mathcal{Z}}^I$ are neither monotone nor solid, if I is neither maximal nor $I = I_f$ in general .

Proof: Here we give a counter example.

Let I = I$_\delta$. Consider the K-step space X_K of X defined as follows,

Let $(x_k) \in X$ and let $(y_k) \in X_K$ be such that

$$(y_k^{/}) = \begin{cases} (x_k^{/}), \text{if k is odd,} \\ 1, otherwise. \end{cases}$$

Consider the sequence $(x_k^{/})$ defined by $(x_k^{/}) = k^{-1}$ for all $k \in \mathbf{N}$.

Then $(x_k) \in \mathcal{Z}^I$ but its K-stepspace preimage does not belong to $\mathcal{Z}^I$. Thus $\mathcal{Z}^I$ is not monotone. Hence $\mathcal{Z}^I$ is not solid.

Theorem 2.10. The spaces $\mathcal{Z}^I$ and $\mathcal{Z}_0^I$ are sequence algebras.

Proof: We prove that $\mathcal{Z}_0^I$ is a sequence algebra.

Let $(x_k), (y_k) \in \mathcal{Z}_0^I$. Then

$$I - \lim |x_k^{/}| = 0$$

and

$$I - \lim |y_k^{/}| = 0$$

Then we have

$$I - \lim |(x_k^{/}.y_k^{/})| = 0$$

Thus $(x_k.y_k) \in \mathcal{Z}_0^I$

Hence $\mathcal{Z}_0^I$ is a sequence algebra.

For the space $\mathcal{Z}^I$, the result can be proved similarly.

Theorem 2.11. The spaces $\mathcal{Z}^I$ and $\mathcal{Z}_0^I$ are not convergence free in general.

Proof: Here we give a counter example.

Let $I = I_f$. Consider the sequence $(x_k^/)$ and $(y_k^/)$ defined by

$$x_k^/ = \frac{1}{k} \ \text{ and } \ y_k^/ = k \ \text{ for all } k \in \mathbf{N}$$

Then $(x_k) \in \mathcal{Z}^I$ and $\mathcal{Z}_0^I$, but $(y_k) \notin \mathcal{Z}^I$ and $\mathcal{Z}_0^I$.

Hence the spaces $\mathcal{Z}^I$ and $\mathcal{Z}_0^I$ are not convergence free.

Theorem 2.12. If I is not maximal and $I \neq I_f$, then the spaces $\mathcal{Z}^I$ and $\mathcal{Z}_0^I$ are not symmetric.

Proof: Let $A \in I$ be infinite.

If

$$x_k^/ = \begin{cases} 1, \text{for } k \in A, \\ 0, otherwise. \end{cases}$$

Then by lemma 1.16. $x_k \in \mathcal{Z}_0^I \subset \mathcal{Z}^I$. Let $K \subset \mathbf{N}$ be such that $K \notin I$ and $\mathbf{N} - K \notin I$.Let $\phi : K \to A$ and $\psi : \mathbf{N} - K \to \mathbf{N} - A$ be bijections, then the map $\pi : \mathbf{N} \to \mathbf{N}$ defined by

$$\pi(k) = \begin{cases} \phi(k), \text{for } k \in K, \\ \psi(k), otherwise. \end{cases}$$

is a permutation on $\mathbf{N}$, but $x_{\pi(k)} \notin \mathcal{Z}^I$ and $x_{\pi(k)} \notin \mathcal{Z}_0^I$.

Hence $\mathcal{Z}^I$ and $\mathcal{Z}_0^I$ are not symmetric.

Theorem 2.13. The sequence spaces $\mathcal{Z}^I$ and $\mathcal{Z}_0^I$ are linearly isomorphic to the spaces c^I and c_0^I respectively, i.e $\mathcal{Z}^I \cong c^I$ and $\mathcal{Z}_0^I \cong c_0^I$.

Proof. We shall prove the result for the space $\mathcal{Z}^I$ and c^I.

The proof for the other spaces will follow similarly.

We need to show that there exists a linear bijection between the spaces $\mathcal{Z}^I$ and c^I. Define a map $T : \mathcal{Z}^I \longrightarrow c^I$ such that $x \to x^/ = Tx$

$$T(x_k) = px_k + (1-p)x_{k-1} = x_k^/$$

where $x_{-1} = 0, p \neq 1$, $1 < p < \infty$.

Clearly T is linear.

Further,it is trivial that $x = 0 = (0, 0, 0, ...)$, whenever $Tx = 0$ and hence injective.

Let $x_k^/ \in c^I$ and define the sequence $x = x_k$ by

$$x_k = M \sum_{i=0}^{k} (-1)^{k-i} N^{k-i} x_i^/ \qquad (i \in \mathbf{N}),$$

where $M = \frac{1}{p}$ and $N = \frac{1-p}{p}$.

Then we have

$$\lim_{k\to\infty} px_k + (1-p)x_{k-1}$$

$$= p \lim_{k\to\infty} M \sum_{i=0}^{k} (-1)^{k-i} N^{k-i} x_i^/ + (1-p) \lim_{k\to\infty} M \sum_{i=0}^{k-1} (-1)^{k-i} N^{k-i} x_i^/$$

$$= \lim_{k\to\infty} x_k^/$$

which shows that $x \in \mathcal{Z}^I$.

Hence T is a linear bijection.

Also we have $||x||_* = ||Z^p x||_c$.

Therefore,

$$||x||_* = \sup_{k\in\mathbf{N}} |px_k + (1-p)x_{k-1}|,$$

$$= \sup_{k\in\mathbf{N}} |pM \sum_{i=0}^{k} (-1)^{k-i} N^{k-i} x_i^{/} + (1-p)M \sum_{i=0}^{k-1} (-1)^{k-i} N^{k-i} x_i^{/}|$$

$$= \sup_{k\in\mathbf{N}} |x_k^{/}| = ||x^{/}||_{c^I}.$$

Hence $\mathcal{Z}^I \cong c^I$.

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Vakeel A. Khan
Department of Mathematics,
Aligarh Muslim University
Aligarh-202002,
India
e-mail: vakhanmaths@gmail.com

Khalid Ebadullah
Department of Applied Mathematics,
Z. H. College of Engineering and Technology,
Aligarh Muslim University ,
Aligarh-202002,
India
e-mail: khalidebadullah@gmail.com

and

Yasmeen
Department of Mathematics,
Aligarh Muslim University ,
Aligarh-202002,
India
e-mail : khany9828@gmail.com