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Difference sequence spaces in cone metric space

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Abstract

In this article we introduce the notion of difference bounded, convergent and null sequences in cone metric space. We investigate their different algebraic and topological properties.

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1. Introduction

The notion of difference sequence spaces (for single sequences) was introduced by Kizmaz [4] as follows:

$$Z(\Delta) = \{(x_k) \in w : (\Delta x_k) \in Z\},$$

for $Z = c, c_0$ and ℓ_∞, where $\Delta x_k = x_k - x_{k+1}$ for all $k \in N$.

The above spaces are Banach spaces normed by

$$||(x_k)|| = |x_1| + \sup_{k \geq 1} |\Delta x_k|.$$

Later on the notion of difference sequences was investigated from different aspects by Tripathy et al [5], Tripathy and Baruah [6], Tripathy and Borgogain [8], Tripathy and Chandra [9], Tripathy and Debnath [10], Tripathy and Dutta [12], Tripathy and Goswami [12] and many others.

The notion of cone metric space has been applied by various authors in different fields of research in these days. It has been applied for introducing and investigating different new classes of sequence spaces and studying their different algebraic and topological properties by Abdeljawad [1], Beg, Abbas and Nazir [2], Dhanorkar and Salunke [3] and many others. In this article we have investigated different properties of the notion of difference bounded, convergent and null sequences in cone metric spaces.

2. Definitions and Preliminaries

Definition 2.1. A subset P of a real Banach space E is called a cone if and only if

(i) P is closed, non-empty and $P \neq \{0\}$.

(ii) If $a, b \in R$, $a \geq 0, b \geq 0$ and $x, y \in P$, then $ax + by \in P$.

(iii) If both $x \in P$ and $-x \in P$ then $x = 0$.

For a given cone $P \subseteq E$, we can define a partial ordering $\leq$ with respect to P by $x \leq y$ if and only if $y - x \in P$, $x < y$ will stand for $x \leq y$ and

$x \neq y$, while $x << y$ will stand for $y - x \in intP$, where $intP$ denotes the interior of P.

Definition 2.2. A cone metric space is an ordered pair (X, d), where X is any set and $d : X \times X \to E$is a mapping satisfying:

(i) $0 < d(x, y)$ for all $x, y \in X$.

(ii) $d(x, y) = 0$ if and only if $x = y$.

(iii) $d(x, y) = d(y, x)$ for all $x, y \in X$.

(iv) $d(x, y) \leq d(x, z) + d(z, y)$ for all $x, y, z \in X$.

Example 2.1. Let $E = R^2, P = \{(x, y) \in E : x, y \geq 0\},\ X = R$ and $d : X \times X \to E$ defined by $d(x, y) = (|x - y|, \alpha|x - y|)$, where $\alpha \geq 0$, is a constant and $x, y \in X$. Then it is well known that (X, d) is a cone metric space.

Example 2.2. Let E be a Banach space, $P = \{(x, y, z) \in E : x, y, z \geq 0\}$ and (X, η) be a metric space with θ the zero element. Let $d : X \times X \to E$ be defined by $d(x, y) = (\eta(x, y), \alpha\eta(x, y), \beta\eta(x, y)), \alpha, \beta \geq 0$.

Then it can be easily verified that (X, d) is a cone metric space.

Definition 2.3. A sequence space E is said to be *solid* or *normal* if $\{\alpha_k x_k\} \in E$ whenever $\{x_k\} \in E$ and for all sequences (α_k) of scalars with $|\alpha_k| \leq 1$ for all $k \in N$.

Let $K = \{k_1 < k_2 < k_3, ...\} \subseteq N$ and E be a class of sequences. A *K-step set* of E is a set of sequences $\lambda_K^E = \{(x_{k_n}) \in w : (x_k) \in w\}$.

A *canonical pre-image of a sequence* $(x_{k_n}) \in \lambda_K^E$ is a sequence $(y_n) \in w$, defined as follows:

$$y_n = \begin{cases} x_n & \text{if } n \in K; \\ \overline{0}, & \text{otherwise,} \end{cases}$$

where $\overline{0}$ is the zero element.

Definition 2.4. *A canonical pre-image of a step set* λ_K^E is a set of canonical pre-images of all elements in λ_K^E i.e. Y is in canonical pre-image λ_K^E if and only if Y is canonical pre-image of some $X \in \lambda_K^E$.

Definition 2.5. A class of sequences E is said to be *monotone* if E contains the canonical pre-images of all its step sets.

The following remark is well known.

Remark 2.1. A class of sequences E is solid $\Rightarrow E$ is monotone.

Definition 2.6. A sequence space E is said to be *convergence free* if $\{x_n\} \in E$ implies $\{y_n\} \in E$ such that $y_n = \overline{0}$, whenever $x_n = \overline{0}$.

Definition 2.7. A class of sequences E is said to be *symmetric* if $(x_{\pi(n)}) \in E$, whenever $(x_k) \in E$, where π is a permutation of N.

Definition 2.8. A sequence space E is said to be a *sequence algebra* if $(x_k y_k) \in E$, whenever $(x_k), (y_k) \in E$.

In this article we introduced the following definitions.

Definition 2.9. Let (X, d) be a cone metric space. A sequence (x_k) in $c(\Delta)$ is said to be convergent to x if for every $\bar{c} \in E$ with $\theta << \bar{c}$, there exists n_0 such that for all $k \geq n_0$, $d(\Delta x_k, x) << \bar{c}$.

Definition 2.10. Let (X, d) be a cone metric space. A sequence (x_k) in $c(\Delta)$ is said to be Cauchy sequence if for every $\bar{c} \in E$ with $\theta << \bar{c}$ there exists n_0 such that for all $k, p \geq n_0$, $d(\Delta x_k, \Delta x_p) << \bar{c}$, where $\bar{c} = (c, \alpha c, \beta c)$, α, $\beta \geq 0$.

We procure the following results those will be used in establishing results of this article.

Lemma 2.1. Let (X, d) be a cone metric space and $(x_k), (y_k)$ be in $c(\Delta)$. If $\lim\limits_{k\to\infty} \Delta x_k = x$ and $\lim\limits_{k\to\infty} \Delta y_k = y$, then $\lim\limits_{k\to\infty} d(\Delta x_k, \Delta y_k) = d(x, y)$.

Lemma 2.2. Let (X, d) be a cone metric space and $(x_k) \in c(\Delta)$. If

$\lim\limits_{k\to\infty} \Delta x_k = x$ and $\lim\limits_{k\to\infty} \Delta x_k = y$, then $x = y$.

Lemma 2.3. Let (X, d) be a cone metric space and $(x_k) \in c(\Delta)$. If $\lim\limits_{k\to\infty} \Delta x_k = x$, then $d(\Delta x_k, x) = 0$.

3. Main results

We state the following result without proof.

Theorem 3.1.$(Z(\Delta), d)$, for $Z = c, c_0, \ell_\infty$ are cone metric spaces.

Theorem 3.2. Let x be a complete cone metric space. Then the classes of sequences $c(\Delta)$, $c_0(\Delta)$, $\ell_\infty(\Delta)$ are complete cone metric spaces w.r.t. the cone metric $\rho(x, y) = d(x_1, y_1) + \sup\limits_{k\in N} d(\Delta x_k, \Delta y_k)$.

Proof. Let $(x^{(i)})$ be a Cauchy sequence in $\ell_\infty(\Delta)$.

Then for a given $\bar{c} \in E$ with $\bar{c} = (c, \alpha c, \beta c)$ there exists n_0 such that

$\rho(x^i, x^j) << \bar{c}$ for all $i, j \geq n_0$.

$\Rightarrow d(x_1^i, x_1^j) + \underset{k\in N}{Sup}\, d(\Delta x_k^i, \Delta x_k^j) << \bar{c}$.

$\Rightarrow d(x_1^i, x_1^j) << \bar{c}$ and $d(\Delta x_k^i, \Delta x_k^j) << \bar{c}$, for all $k \in N$.

$\Rightarrow (\eta(x_k^i, x_k^j), \alpha\eta(x_k^i, x_k^j), \beta\eta(x_k^i, x_k^j)) << (c, \alpha c, \beta c)$.

and $(\eta(\Delta x_k^i, \Delta x_k^j), \alpha\eta(\Delta x_k^i, \Delta x_k^j), \beta\eta(\Delta x_k^i, \Delta x_k^j)) << (c, \alpha c, \beta c)$.

$\Rightarrow \eta(x_k^i, x_k^j) << c$ and $\eta(\Delta x_k^i, \Delta x_k^j) << c$.

Now, $\eta(x_k^i, x_k^j) << c$.

$\Rightarrow (x_1^i)$ is a Cauchy sequence in X.

Since X is a complete cone metric space, so (x_1^i) converges to L in X.

$$\Rightarrow d(x_1^i, L_1) << \frac{\bar{c}}{2} \tag{3.1}$$

and $\eta(\Delta x_k^i, \Delta x_k^j) << c$.

$\Rightarrow \eta(x_k^i - x_{k+1}^i, x_k^j - x_{k+1}^j) << c$.

$\Rightarrow \eta(x_t^i, x_t^j) << c$, where $x_t^i = x_k^i - x_{k+1}^i$ and $x_t^j = x_k^j - x_{k+1}^j$.

$\Rightarrow (x_t^i)$ is a Cauchy sequence in X.

Since X is complete, so (x_k^i) converges to L_k, for each $k \in N$.

Therefore $d(x_k^i, L_k) << c$, for all $k \in N$.

$\Rightarrow d(\Delta x_k^i, L_k) << \bar{c}$, for all $k \in N$.

$$\Rightarrow \sup_{k \in N} d(\Delta x_k^i, L_k) << \frac{\bar{c}}{2}. \tag{3.2}$$

Adding (3.1) and (3.2) we get,

$d(x_1^i, L_1) + \sup_{k \in N} d(\Delta x^i{}_k, L_k) << \bar{c}$.
$\Rightarrow d(x^i, L) << \bar{c}$.

$\Rightarrow (x^i)$ converges to L.

Hence $\ell_\infty(\Delta)$ is a complete metric space.

Theorem 3.3. The class of all sequences $c(\Delta),\ c_0(\Delta),\ \ell_\infty(\Delta)$ are neither solid nor normal.

Proof. The result can be verified by the following example.

Example 3.1. Let us consider the cases $c(\Delta)$ and $c_0(\Delta)$, similar example can be constructed for the case $\ell_\infty(\Delta)$. Let $E = R^3, P = (x, y, z) \in E : x, y, z \geq 0, X = R, d : X \times X \rightarrow E$ be defined by

$$d(x, y) = (|x - y|, \alpha|x - y|, \beta|x - y|),$$

where $\alpha, \beta \geq 0$ are constants. Consider the sequence (x_k) defined by $x_k = 1$, for all $k \in N$.

Then $\Delta x_k = 0$, for all $k \in N$.

Then clearly $(x_k) \in c(\Delta)$ is convergent to 0 with respect to the cone metric space considered.

We have $(\Delta x_k) = (0)$ is convergent.

Consider the canonical pre-image of (x_k) defined by

$(y_k) = (x_1, 0, x_3, 0, x_5, 0, ...) = (1, 0, 1, 0, 1, 0, ...)$.

Now consider the sequence of scalars (α_k) defined by $\alpha_k = (-1)^k$, for all $k \in N$.

Then it can be easily verified that the sequence $\Delta(\alpha_k x_k) = \Delta((-1)^k x_k)$ is not convergent with respect to the cone metric consider above.

Then $(\Delta x_k) = (1, -1, 1, -1, 1, -1, ...) \notin c(\Delta)$.

Hence the class of all convergent sequences is not normal and hence is not solid.

Theorem 3.4. The class of all sequences $c(\Delta)$, $c_0(\Delta)$, $\ell_\infty(\Delta)$ are not symmetric.

Proof. The result can be verified by the following example.

Example 3.2. Consider the cone metric space considered in Example 3.1. Consider the sequence (x_k) defined by

$$(x_k) = (1, 2, 3, 4, ..., k, ...).$$

Then $(\Delta x_k) = (-1, -1, -1, ...)$.

Hence (Δx_k) converges with respect to the cone metric space considered.

Considering the rearrangement (y_k) of (x_k) defined by $(y_k) = (1, 2, 4, 3, 9, 5, 16, 6, 25, 7, ...)$.

$(\Delta y_k) = (-1, -2, 1, -6, 4, -11, 10, ...)$.

Then it can be easily examined that the sequence (Δy_k) is not convergent with respect to the cone metric consider above.

Hence the class of all sequences $c(\Delta),\ c_0(\Delta),\ \ell_\infty(\Delta)$ are not symmetric.

Theorem 3.5. The class of all sequences $c(\Delta),\ c_0(\Delta),\ \ell_\infty(\Delta)$ are not sequence algebra.

Proof. The result can be verified by the following example.

Example 3.3. Consider the cone metric space considered in Example 3.1. Consider the sequence (x_k) and (y_k) defined by

$x_k = y_k = k$, for all $k \in N$.

Then $\Delta x_k = \Delta y_k = -1$, for all $k \in N$.

Therefore $(x_k), (y_k) \in c(\Delta)$.

Next we have $\Delta(x_k y_k) = k^2 - (k+1)^2 = -2k - 1$, for all $k \in N$.

Hence $d(\Delta(x_k y_k), \bar{0}) \to (\infty, \infty, \infty)$.

So the class of all sequences $c(\Delta),\ c_0(\Delta),\ \ell_\infty(\Delta)$ are not sequence algebra.

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