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The upper open monophonic number of a graph

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Abstract

For a connected graph G of order n, a subset S of vertices of G is a monophonic set of G if each vertex v in G lies on a x-y monophonic path for some elements x and y in S. The minimum cardinality of a monophonic set of G is defined as the monophonic number of G, denoted by $m(G)$. A monophonic set of cardinality $m(G)$ is called a m-set of G. A set S of vertices of a connected graph G is an open monophonic set of G if for each vertex v in G, either v is an extreme vertex of G and $v \in S$, or v is an internal vertex of a x-y monophonic path for some $x, y \in S$. An open monophonic set of minimum cardinality is a minimum open monophonic set and this cardinality is the open monophonic number, $om(G)$. An open monophonic set S of vertices in a connected graph G is a minimal open monophonic set if no proper subset of S is an open monophonic set of G. The upper open monophonic number $om^+(G)$ is the maximum cardinality of a minimal open monophonic set of G. The upper open monophonic numbers of certain standard graphs are determined. It is proved that for a graph G of order n, $om(G) = n$ if and only if $om^+(G) = n$. Graphs G with $om(G) = 2$ are characterized. If a graph G has a minimal open monophonic set S of cardinality 3, then S is also a minimum open monophonic set of G and $om(G) = 3$. For any two positive integers a and b with $4 \leq a \leq b$, there exists a connected graph G with $om(G) = a$ and $om^+(G) = b$.

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1. Introduction

By a graph $G = (V, E)$ we mean a finite, undirected connected graph without loops or multiple edges. The order and size of G are denoted by n and m, respectively. For basic graph theoretic terminology we refer to Harary [4]. The distance $d(u, v)$ between two vertices u and v in a connected graph G is the length of a shortest u-v path in G. An u-v path of length $d(u, v)$ is called an u-v geodesic. It is known that this distance is a metric on the vertex set $V(G)$. For any vertex v of G, the eccentricity $e(v)$ of v is the distance between v and a vertex farthest from v. The minimum eccentricity among the vertices of G is the *radius*, *rad* G and the maximum eccentricity is its *diameter*, *diam* G of G. The *neighborhood* of a vertex v is the set $N(v)$ consisting of all vertices which are adjacent with v. The vertex v is an *extreme vertex* of G if the subgraph induced by its neighbors is complete. For a cut-vertex v in a connected graph G and a component H of $G - v$, the subgraph H and the vertex v together with all edges joining v and $V(H)$ is called a *branch* of G at v. A *geodetic set* of G is a set $S \subseteq V(G)$ such that every vertex of G is contained in a geodesic joining some pair of vertices in S. The *geodetic number* $g(G)$ of G is the cardinality of a minimum geodetic set. A vertex x is said to *lie* on a u-v *geodesic* P if x is a vertex of P and x is called an *internal vertex* of P if $x \neq u, v$. A set S of vertices of a connected graph G is an *open geodetic set* of G if for each vertex v in G, either v is an extreme vertex of G and $v \in S$, or v is an internal vertex of a x-y geodesic for some $x, y \in S$. An open geodetic set of minimum cardinality is a minimum open geodetic set and this cardinality is the *open geodetic number* $og(G)$. It is clear that every open geodetic set is a geodetic set so that $g(G) \leq og(G)$. The geodetic number of a graph was introduced and studied in [1, 2]. The open geodetic number of a graph was introduced and studied in [3, 5, 6] in the name open geodomination in graphs. A chord of a path $u_1, u_2, \ldots, u_n$ in G is an edge u_iu_j with $j \geq i+2$. For two vertices u and v in a connected graph G, a u-v path is called a *monophonic path* if it contains no chords. A *monophonic set* of G is a set $S \subseteq V(G)$ such that every vertex of G is contained in a monophonic path joining some pair of vertices in S. The *monophonic number* $m(G)$ of G is the cardinality of a minimum monophonic set. A set S of vertices in a connected graph G is an *open monophonic set* if for each vertex v in G, either v is an extreme vertex of G and $v \in S$, v is an internal vertex of a x-y monophonic path for some $x, y \in S$. An open monophonic set of minimum cardinality is a *minimum open monophonic* *set* and this cardinality is the *open monophonic number om(G)* of G. An open monophonic set of cardinality $om(G)$ is called $om - set$ of G. The open monophonic number of a graph was introduced and studied in [8]. The *connected open monophonic number* of a graph was introduced and studied in [7].

The following theorems are used in the sequal.

Theorem 1.1. [8] Every extreme vertex of a connected graph G belongs to each open monophonic set of G. In particular, if the set S of all extreme vertices of G is an open monophonic set of G, then S is the unique minimum open monophonic set of G.

Theorem 1.2. [8] If G is a non-trivial connected graph with no extreme vertices, then $om(G) \geq 3$.

Theorem 1.3. [8] If G is a connected graph with a cut-vertex v, then every open monophonic set of G contains at least one vertex from each component of $G - v$.

Theorem 1.4. [8] For any cycle $G = C_n (n \geq 4)$, $om(G) = \{$ 3 if $n \geq 6$
4if $n = 4, 5$.

2. The upper open monophonic number of a graph

Definition 2.1. *An open monophonic set S of vertices in a connected graph G is a minimal open monophonic set if no proper subset of S is an open monophonic set of G. The upper open monophonic number $om^+(G)$ is the maximum cardinality of a minimal open monophonic set of G.*

We illustrate this definition by an example.

Example 2.2. *For the graph G given in Figure 2.1, it is easy to see that no 2-element subset of vertices is an open monophonic set of G. It is easily verified that $S_1 = \{v_1, v_5, v_8\}$, $S_2 = \{v_1, v_2, v_5\}$, $S_3 = \{v_2, v_5, v_7\}$ and $S_3 = \{v_2, v_5, v_8\}$ are the only four minimum open monophonic sets of G so that $om(G) = 3$. It is also easily verified that $T_1 = \{v_1, v_3, v_5, v_6\}$, $T_2 = \{v_2, v_3, v_5, v_6\}$, $T_3 = \{v_3, v_5, v_6, v_7\}$and $T_4 = \{v_3, v_5, v_6, v_8\}$ are the only four minimal open monophonic sets of order 4. It is also verified that there is no minimal open monophonic sets of order greater than 4. Hence $om^+(G) = 4$.*

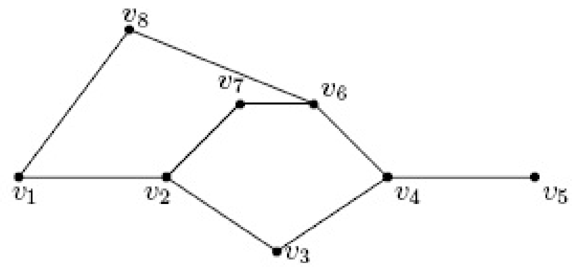

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Figure 2.1

It is clear that every minimum open monophonic set is a minimal open monophonic set of G, and the converse need not be true. For the graph G given in Figure 2.1, $T_1 = \{v_1, v_3, v_5, v_6\}$ is a minimal open monophonic set of G, and not a minimum open monophonic set of G.

Any open monophonic set contains at least two vertices and so $om(G) \geq 2$. The inequality $om(G) \leq om^+(G)$ follows from the fact that every minimum open monophonic set is a minimal open monophonic set. Also, the set of all vertices of G is an open monophonic set of G so $om^+(G) \leq n$. Thus we have the following theorem.

Theorem 2.3. Let G be a connected graph of order n. Then $2 \leq om(G) \leq om^+(G) \leq n$.

We observe that the bounds in Theorem 2.3 are sharp. For any path $P_n (n \geq 2)$, $om(P_n) = 2$. For any tree T of order at least 2, it is clear that the set of all endvertices is the unique minimum open monophonic set so that $om(T) = om^+(T)$. Also, it is easily seen that for the complete graph $K_n (n \geq 2)$, $om^+(K_n) = n$. Now, all the inequalities in Theorem 2.3 can be strict. For the graph G given in Figure 2.1, $om(G) = 3$, $om^+(G) = 4$ and $n = 8$.

Theorem 2.4. For a connected graph G of order at least 2, $om(G) = 2$ if only if there exist two extreme vertices u and v such that every vertex lies on a monophonic path joining u and v.

Proof. If $om(G) = 2$, then let $S = \{u, v\}$ be an open monophonic set of G. Suppose that u or v is not extreme vertex. Assume that u is not extreme. Then u cannot lie as an internal vertex of a monophonic path joining two vertices of S. Hence both u and v are extreme. The converse is clear. □

The following are some examples of graphs illustrating the above theorem.

1. Any Path $P_n (n \geq 2)$

2. Let $P_k : u_1, u_2, \ldots, u_k (k \geq 1)$ be a path of order k. Let $C_r : v_1, v_2, \ldots, v_r, v_1 (r \geq 4)$ be a cycle of order r. Let H be the graph obtained by identifying the vertex u_k of P_k and the vertex v_1 of the C_r. Let $v_i, v_{i+1}, v_{i+2} (i \geq 2)$ be three vertices on C_r and let G be the graph obtained from H by joining the vertices v_i and v_{i+2}. The graph is shown in Figure 2.2. Then u_1 and v_{i+1} are the only extreme vertices of G and $S = \{u_1, v_{i+1}\}$ is an open monophonic set of G so that $om(G) = 2$.

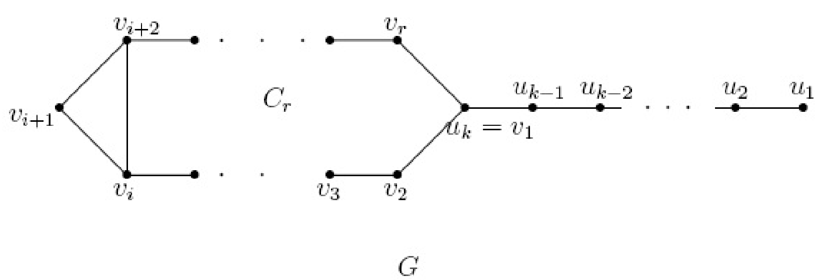

Figure 2.2

3. Let $P_k : u_1, u_2, \ldots, u_k (k \geq 1)$ and $P_l : w_1, w_2, \ldots, w_l (l \geq 1)$ be two paths of order k and l, respectively. Let $C_r : v_1, v_2, \ldots, v_r, v_1 (r \geq 4)$ be a cycle of order r. Let H be the graph obtained by identifying the vertex u_k of P_k and the vertex v_1 of the C_r. Now, let G be the graph obtained from H by identifying the vertex w_1 of P_l and any vertex $v_i (i \neq 2, r)$ of C_r. The graph G is shown in Figure 2.3. Then u_1 and w_l are the only extreme vertices of G and $S = \{u_1, w_l\}$ is an open monophonic set of G so that $om(G) = 2$.

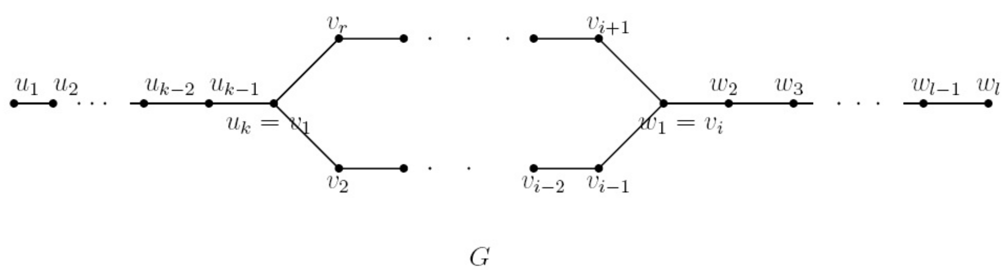

Figure 2.3

4. One may attach a path of any length at the vertex v_{i+1} of the graph G given in Figure 2.2 in Example 2 and get $om(G) = 2$. Many more classes of graphs can be obtained in this manner.

Theorem 2.5. If G is a connected graph with extreme vertices, and if the set S of all extreme vertices is an open monophonic set of G, then $om(G) = om^+(G)$.

Proof. Suppose that G is a graph with extreme vertices and the set all extreme vertices forms an open monophonic set. Since any minimal open monophonic set contains all the extreme vertices, it follows that the minimal open monophonic sets are nothing but the minimum open monophonic sets. Hence $om(G) = om^+(G)$. □

Corollary 2.6. For any Tree T with k endvertices, $om(T) = om^+(T) = k$.

Proof. The set of all endvertices is the unique minimum open monophonic set of G. Hence the result follows. □

Remark 2.7. *It follows from Theorem 2.5 that $om(G) = om^+(G)$ for all the graphs in Figure 2.2 and 2.3 of the above examples.*

Theorem 2.8. Let G be a connected graph of order n. Then $om(G) = n$ if and only if $om^+(G) = n$.

Proof. If $om(G) = n$, then $om^+(G) = n$ follows from Theorem 2.3. Let $om^+(G) = n$. Then the set of all vertices of G is the unique minimal open monophonic set of G. Hence it follows that G contains no proper open monophonic sets so that the set of all vertices is also the minimum open monophonic set so that $om(G) = n$. □

Theorem 2.9. For the complete graph $G = K_n (n \geq 2)$, $om(G) = om^+(G) = n$.

Proof. This follows from Theore 1.1 and 2.5. □

The converse of Theorem 2.9 need not be true. For the graph $G = C_4$, $om(G) = 4$ (Theorem 1.4). Hence by Theorem 2.3, $om^+(G) = 4$. We give yet another example. For the graph G given in Figure 2.4, the set $S = \{v_1, v_2, v_3, v_4, v_5, v_6\}$ is the unique minimum open monophonic set of G and so $om(G) = 6$. Hence by Theorem 2.3, $om^+(G) = 6$.

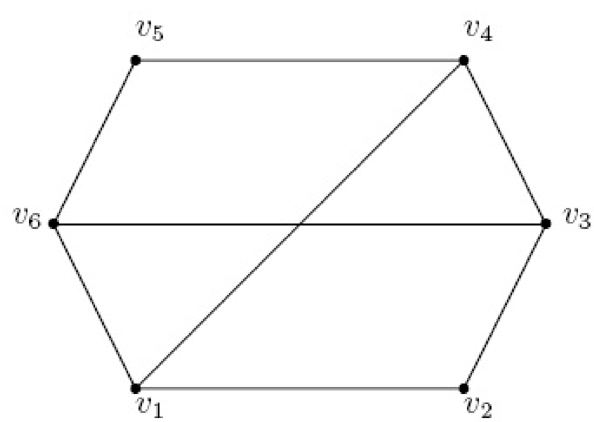

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Figure 2.4

These examples show that there are non-complete graphs G of order n with $om(G) = om^+(G) = n$.

Theorem 2.10. If G is a connected graph of order n with $om(G) = n-1$, then $om^+(G) = n-1$.

Proof. Let $om(G) = n-1$. Then it follows from Theorem 2.3 that $om^+(G) = n$ or $n-1$. If $om^+(G) = n$, then by Theorem 2.8, $om(G) = n$, which contradicts the data. Hence $om^+(G) = n-1$. □

Remark 2.11. *The converse of Theorem 2.10 need not be true. For the graph G given in Figure 2.5, the set $S = \{v_1, v_2, v_3, v_5\}$ is a minimum open monophonic set so that $om(G) = 4$. Also it is easily seen that the set $S' = \{v_1, v_2, v_4, v_5, v_6, v_7, v_8\}$ is a minimal open monophonic set so that $om^+(G) = 7$.*

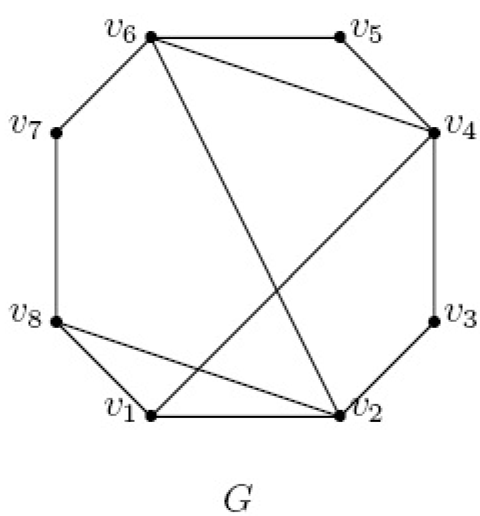

Figure 2.5

Example 2.12. *For the graph $G = C_5 : v_1, v_2, v_3, v_4, v_5, v_1$, it is easily verified that $S = \{v_1, v_2, v_3, v_4\}$ is a minimum open monophonic set and so $om(G) = 4$. Hence by Theorem 2.10, $om^+(G) = 4$.*

3. More results on minimal open monophonic sets

Theorem 3.1. No cut-vertex of a connected graph G belongs to any minimal open monophonic set of G.

Proof. Let S be any minimal open monophonic set of G. Let $v \in S$. We prove that v is not a cut-vertex of G. Suppose that v is a cut-vertex of G. Let $G_1, G_2, \ldots, G_k (k \geq 2)$ be the components of $G - v$. Then v is adjacent to at least one vertex of each G_i for $1 \leq i \leq k$. Let $S' = S - \{v\}$. We show that S' is an open monophonic set of G. Let x be a vertex of G. If x is an extreme vertex of G, then $x \neq v$ and so $x \in S'$. Suppose that x is not an extreme vertex of G. Since S is an open monophonic set of G, x lies as an internal vertex of a u - w monophonic path for some $u, w \in S$. If $v \neq u, w$, then obviously $u, w \in S'$ and S' is an open monophonic set of G. If $v = u$, then $v \neq w$. Assume without loss of generality that $w \in G_1$. By Theorem 1.3, S' contains a vertex w' from $G_i (2 \leq i \leq k)$. Consider w - v monophonic path P (such a path exists since there is at least one w - u geodesic). Let P' be a w - w' monophonic path. Then, since v is a cut-vertex of G, it follows that $P \cup P'$ is a w - w' monophonic path of G. Hence x is an internal vertex of the w - w' monophonic path $P \cup P'$ with $w, w' \in S'$. Thus S' is an open monophonic set of G with $|S'| < |S|$. This is a contradiction to S a minimal open monophonic set. Thus no cut-vertex of G belongs to S. $\square$

Remark 3.2. *Theorem 3.1 can be used to prove that for any tree with k endvertices $om(G) = om^+(G) = k$.*

The next theorem gives an interesting result regarding minimal open geodetic set of cardinality 3.

Theorem 3.3. Let G be a connected graph. If G has a minimal open geodetic set S of cardinality 3, then all the vertices in S are extreme.

We first prove the following lemma and proceed.

Lemma A. If a non-trivial connected graph G contains no extreme vertices, then $og(G) \geq 4$.

Proof. First, we observe that if G is a non-trivial connected graph having no extreme vertices, then the order of G is at least 4. Let S be an open geodetic set of G. If $u \in S$, then there exists vertices v and w such that u lies on a $v - w$ geodesic. Without loss of generality, assume that $d(v, u) \leq d(u, w)$. Then w does not lie on any $u - v$ geodesic. Since, for some $x, y \in S$, w lies in an $x - y$ geodesic, it follows that at least one of x and y is distinct from all of u, v and w. Hence $|S| \geq 4$. Therefore, $og(G) \geq 4$ and the lemma is proved. Now, we prove the theorem.

Let $S = \{u, v, w\}$ be a minimal open geodetic set of G. Then by Lemma A, $og(G) \leq 3$. Suppose that the vertex w is not extreme. We consider three cases.

Case 1. u and v are non-extreme. Then u, v, w are all non-extreme and G has no extreme vertices. Hence by above Lemma A, we see that $og(G) \geq 4$, which is a contradiction.

Case 2. u is extreme and v is not extreme. Since S is an open geodetic set of G, we have v lies as an internal vertex of $u - w$ geodesic and w lies as an internal vertex of $u - v$ geodesic. These in turn, give $d(u, w) > d(u, v)$ and $d(u, v) > d(u, w)$. Hence $d(u, w) > d(u, w)$, which is a contradiction.

Case 3. u and v are extreme. Since S is an open goedetic set of G, we have w lies as an internal vertex of $u - v$ geodesic. Let $d(u, v) = k$ and let P be a $u - v$ geodesic of length k, $d(u, w) = l_1$ and $d(w, v) = l_2$. Then $l_1 + l_2 = k$. Let P' be the $u - w$ subpath P and P'' the $w - v$ subpath of P. We prove that $S' = \{u, v\}$ is an open geodetic set of G. Let x be any vertex of G such that $x \notin S'$. Since $S = \{u, v, w\}$ is a minimal open geodetic set of G with w non-extreme, u and v extreme, it follows that u and v are the only two extreme vertices of G. Hence x is not extreme. Since S is an open geodetic set of G, we have x lies as an internal vertex of $u - v$ geodesic or x lies as an internal vertex of $u - w$ geodesic or x lies as an internal vertex of $v - w$ geodesic. If x lies as an internal vertex of $u - v$ geodesic, there is nothing to prove. If x lies as an internal vertex of $u - w$ geodesic, let Q be a $u - w$ geodesic in which x lies internally. Let R be the $u - v$ walk obtained from Q followed by P''. Then the length of R is k and so R is a $u - v$ geodesic containing x. Thus x lies as an internal vertex of $u - v$ geodesic. Similarly, if x lies as an internal vertex of $v - w$ geodesic, we can prove that x lies as an internal vertex of $u - v$ geodesic. Hence S' is an open geodetic set of G, which contradicts that S is a minimal open geodetic set of G. This completes the proof. □

Remark 3.4. *Theorem 3.3 need not be true in the case of minimal open monophonic sets with cardinality 3. For the cycle* $C_n (n \geq 6)$, *by Theorem*

1.4, $om(G) = 3$*. Note that no vertex of the cycle* $C_n (n \geq 6)$ *is extreme. Also for the graph* G *given in Figure 3.1, the vertices* v_2, v_4 *and* v_5 *are extreme and the set* $S = \{v_2, v_4, v_5\}$ *is the unique minimum open monophonic set of* G *so that* $om(G) = 3$ *and by Theorem 2.5,* $om^+(G) = 3$*.*

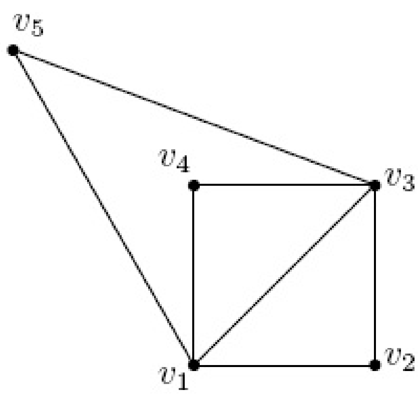

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Figure 3.1

Theorem 3.5. Let G be a connected graph. If G has a minimal open monophonic set S of cardinality 3, then S is also a minimum open monophonic set and so $om(G) = 3$.

Proof. Let $S = \{u, v, w\}$ be a minimal open monophonic set of G. Then $om(G) \leq 3$. First, suppose that G is a graph with no extreme vertices. Hence by Theorem 1.2, $om(G) \geq 3$. Thus $om(G) = 3$. Next, let G be a graph with extreme vertices.

Case 1. u, v, w are extreme vertices. Then S is clearly a minimum open monophonic set so that $om(G) = 3$.

Case 2. u and v are extreme vertices and w is not an extreme vertex. Then it is clear that no 2-element subset of vertices of G is an open monophonic set. Hence S is a minimum open monophonic set of G and so $om(G) = 3$.

Case 3. u is an extreme vertex and v, w are non extreme vertices. We show that no 2-element subset of vertices of G is an open monophonic set of G. Suppose that there exists an open monophonic set T of cardinality 2. Then by Theorem 1.1, $u \in T$. If $T = \{u, v\}$ or $T = \{u, w\}$, then it contradicts that S is a minimal open monophonic set of G. Hence $T = \{u, x\}$, where $x \neq v, w$. Then it is clear that x cannot lie as an internal vertex of a monophonic path joining a pair of vertices of T so that T is not an open monophonic set of G. Hence, it follows that S is a minimum open monophonic set of G and so $om(G) = 3$. □

Corollary 3.6. Let S be an open monophonic set with $|S| = 3$. Then S is minimum if and only if S is minimal.

Proof. If S is a minimum open monophonic set with $|S| = 3$, then it is clear S is a minimal open monophonic set. The converse follows from Theorem 3.5. □

Theorem 3.7. For any two positive integers a and b with $4 \leq a \leq b$, there exists a connected graph G with $om(G) = a$ and $om^+(G) = b$.

Proof. **Case 1.** If $a = b$, let $G = K_{1,a}$. Then for any tree T with a end vertices, $om(G) = om^+(G) = a$.
Case 2. Let $4 \leq a < b$. Let $H = \overline{K_2} + C_{b-a+3}$ with $V(K_2) = \{x, y\}$ in Figure 3.2 obtained from H by adding $a-3$ new vertices $u_1, u_2, u_3, \ldots, u_{a-3}$ and joining each $u_i (1 \leq i \leq a-3)$ with y. It is clear that $S = \{u_1, u_2, u_3, \ldots, u_{a-3}\}$ is not an open monophonic set of G. Also $S \cup \{w, z\}$, where $w, z \notin S$ is not an open monophonic set of G. Let $S' = S \cup \{x, v_i, v_j\}$, where v_i and v_j are non adjacent. Then it is clear that S' is an open monophonic set of G and so $om(G) = a$.

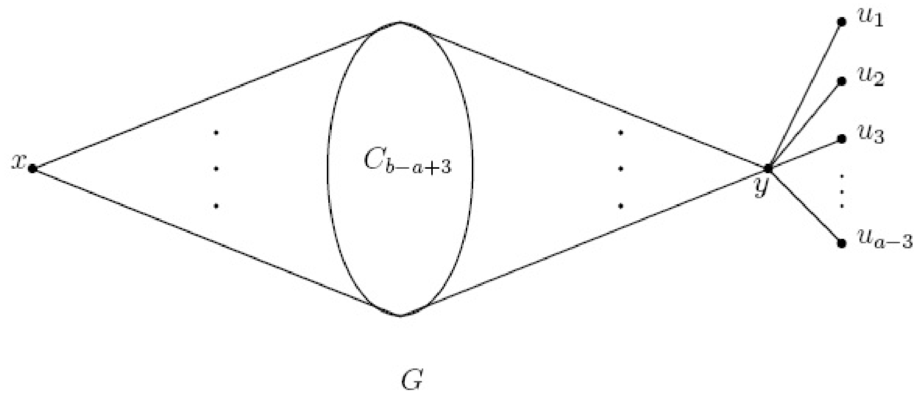

Figure 3.2

We now prove that $om^+(G) = b$. It is clear that $T = S \cup \{v_1, v_2, \ldots, v_{b-a+3}\}$ is an open monophonic set of G. We show that T is a minimal open monophonic set of G. On the contrary, assume that W is a proper subset of T such that W is an open monophonic set of G. Then there exists a vertex $v \in T$ such that $v \notin W$. Since W is an open monophonic set, it contains all its extreme vertices. It is clear that $v = v_j$ for some $j (1 \le j \le b - a + 3)$. Then v_{j+1} does not lie on a monophonic path joining any pair of vertices of W and so W is not an open monophonic set of G, which is a contradiction. Hence T is a minimal open monophonic set of G so that $om^+(G) \ge b$. Now, since y is a cut-vertex of G, y does not belong to any minimal open monophonic set of G. Suppose that $om^+(G) = b + 1$. Let X be a minimal open monophonic set of cardinality $b + 1$. Then $X = V(G) - \{y\}$ and S' is a proper subset of X so that X is not a minimal open monophonic set of G, which is a contradiction. Hence $om^+(G) = b$ $\square$

Conclusion

In view of the results discussed in this paper we leave the following problems as open.

Problem 1 Characterize graphs G of order at least 2 for which $om(G) = om^+(G) = 2$.

Problem 2 Characterize graphs G of order n for which $om(G) = om^+(G) = n$.

Problem 3 Characterize graphs G of order n for which $om(G) = om^+(G) = n - 1$.

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