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The t-pebbling number of Jahangir graph $J_{3,m}$

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Abstract

The t-pebbling number, $f_t(G)$, of a connected graph G, is the smallest positive integer such that from every placement of $f_t(G)$ pebbles, t pebbles can be moved to any specified target vertex by a sequence of pebbling moves, each move removes two pebbles of a vertex and placing one on an adjacent vertex. In this paper, we determine the t-pebbling number for Jahangir graph $J_{3,m}$ and finally we give a conjecture for the t-pebbling number of the graph $J_{n,m}$.

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1. Introduction

We begin by introducing relevant terminology and background on the subject. Here, the term graph refers to a simple graph. A function $\phi : V(G) \rightarrow N \cup \{0\}$ is called a *pebbling*. Let $\phi(v)$ denote the number of pebbles on the vertex v and $\phi(V(A))$ denote the number of pebbles on the vertices of the subgraph A of G. The quantity $\sum_{x \in V(G)} \phi(x)$ is called the *size* of ϕ; the size of ϕ is just the total number of pebbles assigned to vertices. A *pebbling move (step)* consists of removing two pebbles from one vertex and then placing one pebble at an adjacent vertex. For a graph G, if ϕ is a distribution or configuration of pebbles onto the vertices of G and it is possible to move a pebble to the target vertex v, then we say that ϕ is *v-solvable*; otherwise, ϕ is *v-unsolvable*. Then ϕ is *solvable* if it is v-solvable for all $v \in V(G)$, and *unsolvable* otherwise. If $\phi(v) = 1$ or $\phi(u) \geq 2$ where $uv \in E(G)$, then we can easily move one pebble to v. So, we always assume that $\phi(v) = 0$ and $\phi(u) \leq 1$ for all $uv \in E(G)$ when v is the target vertex.

The *t-pebbling number of a vertex v* in a graph G [2], $f_t(v, G)$, is the smallest positive integer m such that however m pebbles are placed on the vertices of the graph, t pebbles can be moved to v in finite number of pebbling moves, each move removes two pebbles of one vertex and placing one on an adjacent vertex. The *t-pebbling number* of G, $f_t(G)$, is defined to be the maximum of the pebbling numbers of its vertices. Thus the *t-pebbling number of a graph G*, $f_t(G)$, is the least m such that, for any configuration of m pebbles to the vertices of G, we can move t pebbles to any vertex by a sequence of moves, each move removes two pebbles of one vertex and placing one on an adjacent vertex. Clearly, $f_1(G) = f(G)$, the *pebbling number* of G.

Fact 1. *([12]) For any vertex v of a graph G, $f(v, G) \geq n$ where $n = |V(G)|$.*

Fact 2. *([12]) The pebbling number of a graph G satisfies*

$$f(G) \geq max\{2^{diam(G)}, |V(G)|\}.$$

Jahangir graph $J_{n,m}$ [11], for $m \geq 3$, is a graph on $nm + 1$ vertices, that is, a graph consisting of a cycle C_{nm} with one additional vertex which is adjacent to m vertices of C_{nm} at distance n to each other on C_{nm}.

For labeling $J_{3,m}$ $(m \geq 3)$, let v_{3m+1} be the label of the center vertex and $v_1, v_2, \cdots, v_{3m}$ be the label of the vertices that are incident clockwise on cycle C_{3m} so that $deg(v_1) = 3$.

With regard to t-pebbling number of graphs, we have found the following theorems:

Theorem 1. *([9]). Let K_n be the complete graph on n vertices where $n \geq 2$. Then $f_t(K_n) = 2t + n - 2$.*

Theorem 2. *([2]). Let $K_1 = \{v\}$. Let $C_{n-1} = (u_1, u_2, \cdots, u_{n-1})$ be a cycle of length $n-1$. Then the t-pebbling number of the wheel graph W_n is $f_t(W_n) = 4t + n - 4$ for $n \geq 5$.*

Theorem 3.

$$f_t(G) = \begin{cases} 2t + n - 2, \text{ if } 2t \leq n - s_1 \\ 4t + s_1 - 2, \text{ if } 2t \geq n - s_1 \end{cases}$$

Theorem 4. *([9]). Let $K_{1,n}$ be an n-star where $n > 1$. Then $f_t(K_{1,n}) = 4t + n - 2$.*

Theorem 5. *([9]). Let C_n denote a simple cycle with n vertices, where $n \geq 3$. Then $f_t(C_{2k}) = t2^k$ and $f_t(C_{2k+1}) = \frac{2^{k+2} - (-1)^{k+2}}{3} + (t-1)2^k$.*

Theorem 6. *([9]). Let P_n be a path on n vertices. Then $f_t(P_n) = t(2^{n-1})$.*

Theorem 7. *([9]). Let Q_n be the n-cube. Then $f_t(Q_n) = t(2^n)$.*

Lourdusamy et. al proved the t-pebbling number of Jahngir graph $J_{2,m}$ for $m \geq 3$ and $t \geq 1$ in [3, 6, 7, 8]. In the next section, we are going to present the pebbling number of Jahangir graph $J_{3,m}$ for $m \geq 3$. In section three, we find the t-pebbling number for $J_{3,m}$. In the last section, we give a conjecture for the t-pebbling number of Jahangir graph $J_{n,m}$, where $2 < n < m$.

2. The pebbling number for Jahangir graph $J_{3,m}$

Consider Jahangir graph $J_{3,m}$ ($m \geq 3$), and we can see that $J_{3,m}$ has m cycles of length five, 5-cycles, that is,

$$S_1 : v_1 v_2 v_3 v_4 v_{3m+1} v_1,$$

$$S_2 : v_4 v_5 v_6 v_7 v_{3m+1} v_4,$$

$$\vdots$$

$$S_m : v_{3m-2}v_{3m-1}v_{3m}v_1v_{3m+1}v_{3m-2}.$$

For $i, j = 1, 2, , m$, $i \neq j$, two cycles S_i and S_j are adjacent if there exists a *common edge* $(v_k v_{3m+1})$ between them. The vertex v_k is called a *common vertex* for S_i and S_j.

Theorem 1. *For Jahangir graph $J_{3,3}$, $f(J_{3,3}) = 17$.*

Proof. Clearly, by Fact 2, $f(J_{3,3}) \geq max\{16, 10\} = 16$. Let $\phi(v_6) = 15$, $\phi(v_9) = 1$ and for all $i \neq 6, 9$, $\phi(v_i) = 0$. If v_2 is a target vertex, then pebbling ϕ is v_2-unsolvable. Thus $f(J_{3,3}) \geq 17$.

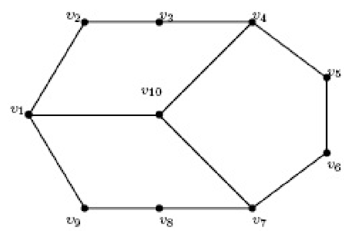

Figure 1. Jahangir graph $J_{3,3}$

Consider the distribution of 17 pebbles on the vertices of $J_{3,3}$.

Case 1: Let v_{10} be the target vertex.
Without loss of generality, let $\phi(V(S_1)) \geq 5$, since $J_{3,3}$ has 17 pebbles and three 5-cycles. Hence, we can move one pebble to v_{10} by Theorem 5.

Case 2: Let v_1 be the target vertex.
If $\phi(V(S_1)) \geq 5$ or $\phi(V(S_3)) \geq 5$, then we can easily move one pebble to v_1. Assume $\phi(v_2) + \phi(v_3) \leq 3$ and $\phi(v_8) + \phi(v_9) \leq 3$ (otherwise one pebble can be moved to v_1). Obviously, $\phi(V(S_2)) \geq 11$, and by Theorem 5, $f_2(C_5) = 9$. Therefore by moving two pebbles to v_{10}, we can move one pebble to v_1.

Case 3: Let v_2 be the target vertex.
If $\phi(V(S_1)) \geq 5$, then we can easily move one pebble to v_2. Since, $\phi(V(J_{3,3})-$

$\{v_2, v_3\}) \geq 16$, either $\phi(V(S_2)) \geq 8$ or $\phi(V(S_3)) \geq 8$.

Case 3.1: Let $\phi(V(S_3)) \geq 8$.
If $\phi(V(S_3)) \geq 9$, then by Theorem 5 we can move one pebble to v_2. Let $\phi(V(S_3)) = 8$. Clearly, $\phi(V(S_2)) \geq 5$. Thus we can move one pebble to v_7 or v_{10} from the pebbles on $V(S_2)$. Now, we have at least 9 pebbles on $V(S_3)$ and hence we can move one pebble to v_2 through v_1 by Theorem 5.

Case 3.2: Let $\phi(V(S_2)) \geq 8$.
Let $\phi(V(S_2)) \geq 9$. Clearly we can move one pebble to v_2 if $\phi(v_1) = 1$ or $\phi(v_3) = 1$ or $\phi(v_8) + \phi(v_9) \geq 4$. So, we assume $\phi(v_1) = 0$, $\phi(v_3) = 0$ and $\phi(v_8) + \phi(v_9) \leq 3$ such that we cannot move one pebble to v_1 from the pebbles on the vertices v_8 and v_9. Now, we have $\phi(V(S_2)) \geq 14$. We assume $\phi(v_4) = 0$ and $\phi(v_{10}) = 0$ (otherwise one pebble could be moved to v_2 by Theorem 5).

Case 3.2.1: Let $\phi(v_8) = 2$ or 3 and $\phi(v_9) = 0$.
If $\phi(v_7) \geq 1$, then we move one pebble to v_7 from v_8 and then we move one pebble to v_{10} from v_7. Now $V(S_2) - \{v_{10}\}$ contains at least 13 pebbles and hence we can easily move one pebble to v_2 through v_{10} by Theorem 5. Assume $\phi(v_7) = 0$. Let $\phi(v_8) = 2$. If $\phi(v_6) \geq 2$, then we move one pebble v_{10} through v_7 from v_6 and v_8 and hence we can easily move one pebble to v_2 through v_{10} by Theorem 5. If $\phi(v_6) \leq 1$, then we can easily move one pebble to v_2 since $\phi(v_5) \geq 8$ and $d(v_2, v_5) = 3$. Let $\phi(v_8) = 3$. If $\phi(v_6) \geq 4$, then we move one pebble v_8 from v_6 and then we move one pebble to v_1. Now $V(S_2)$ contains at least 10 pebbles and hence we can move one more pebble to v_1 through v_{10} (by Theorem 5) and then one pebble can be easily moved to v_2. If $\phi(v_6) \leq 3$, then we can easily move one pebble to v_2 since $\phi(v_5) \geq 8$ and $d(v_2, v_5) = 3$.

Case 3.2.2: Let $\phi(v_8) = \phi(v_9) = 1$.
If $\phi(v_6) + \phi(v_7) \geq 4$, then we can move one pebble to v_1 using the pebbles on v_8 and v_9. Then $\phi(V(S_2)) - 4 \geq 10$ and hence we can move another one pebble to v_1 through v_{10} and then one pebble can be easily moved to v_2. Suppose, $\phi(v_6) + \phi(v_7) \leq 3$, then $\phi(v_5) \geq 8$. Thus we can easily move one pebble to v_2 since $d(v_2, v_5) = 3$.

Case 3.2.3: Let $\phi(v_8) + \phi(v_9) \leq 1$.
Since $\phi(v_8) + \phi(v_9) \leq 1$, we have $\phi(V(S_2)) \geq 16$. If $\phi(v_5) \geq 2$, then we move

one pebble to v_4. Then the number of remaining pebbles on the vertices of S_2 is at least 14 and hence we can move three more pebbles to v_4 by Theorem 5. Thus we can easily move one pebble to v_2 from v_4. Assume $\phi(v_5) \leq 1$. In a similar way, we may assume $\phi(v_7) \leq 1$. This implies that $\phi(v_6) \geq 14$ and clearly we can move one pebble to v_2 by moving seven pebbles to v_5 if $\phi(v_5) = 1$ (or v_7 if $\phi(v_7) = 1$), since, $d(v_2, v_5) = 3$ and $d(v_2, v_7) = 3$. Assume $\phi(v_5) = 0$ and $\phi(v_7) = 0$. Then $\phi(v_6) = 16$ and hence one pebble can be moved to v_2 since $d(v_2, v_6) = 4$.
Therefore 17 pebbles are sufficient to pebble the vertices of $J_{3,3}$ and hence $f(J_{3,3}) = 17$. □

Theorem 2. *For Jahangir graph $J_{3,4}$, $f(J_{3,4}) = 21$.*

Proof. Let $\phi(v_8) = 15$, $\phi(v_5) = 3$, $\phi(v_9) = \phi(v_{12}) = 1$ and $\phi(v_i) = 0$ for all $i \neq 5, 8, 9, 12$. Then we cannot move one pebble to v_2. Thus $f(J_{3,4}) \geq 21$.

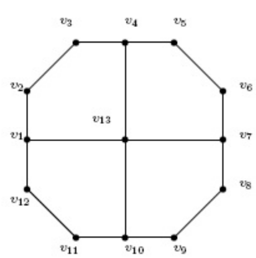

Figure 2. Jahangir graph $J_{3,4}$

Consider the distribution of 21 pebbles on the vertices of $J_{3,4}$.

Case 1: Let v_{13} be the target vertex.
Without loss of generality, let $\phi(V(S_1)) \geq 5$, since $J_{3,4}$ has 21 pebbles and four 5-cycles. Hence, we can move one pebble to v_{13} by Theorem 5.

Case 2: Let v_1 be the target vertex.
If $\phi(V(S_1)) \geq 5$ or $\phi(V(S_4)) \geq 5$, then we can easily move one pebble to v_1. Assume $\phi(v_2) + \phi(v_3) \leq 3$ and $\phi(v_{11}) + \phi(v_{12}) \leq 3$ (otherwise one pebble can be moved to v_1). Thus, $\phi(V(S_2)) + \phi(V(S_3)) \geq 15$. If both

$\phi(V(S_2)) \geq 5$ and $\phi(V(S_3)) \geq 5$, then we can easily move one pebble to v_1. Without loss of generality, let $\phi(V(S_3)) \leq 4$. So, $\phi(V(S_2)) \geq 11$ and hence we can move one pebble to v_1 by moving two pebbles to v_{13}, since $f_2(C_5) = 9$ by Theorem 5.

Case 3: Let v_2 be the target vertex.
If $\phi(V(S_1)) \geq 5$, then clearly we can move one pebble to v_2. So, we have $\phi(V(S_2)) + \phi(V(S_3)) + \phi(V(S_4)) \geq 20$ and hence $\phi(V(S_i)) \geq 7$ for some $i = 2, 3, 4$.

Case 3.1: Let $\phi(V(S_2)) \geq 7$.
Let $\phi(V(S_2)) \geq 9$. If $\phi(v_1) = 1$ or $\phi(v_3) = 1$ or $\phi(v_{11}) + \phi(v_{12}) \geq 4$, then clearly we can move one pebble to v_2. Assume $\phi(v_1) = 0$, $\phi(v_3) = 0$ and $\phi(v_{11}) + \phi(v_{12}) \leq 3$ such that we cannot move one pebble to v_1. We have $\phi(V(S_2)) + \phi(V(S_3)) \geq 18$.

Case 3.1.1: Let $\phi(v_{11}) = 0$ and $\phi(v_{12}) = 2$ or $\phi(v_{11}) = 0$ and $\phi(v_{12}) = 3$.
If $\phi(v_{10}) = 1$ or $\phi(v_9) \geq 2$, then we move one pebble to v_{13}. If $\phi(V(S_3)) - 2 \geq 5$, then we can move one more pebble to v_{13} and hence we are done since $\phi(V(S_2)) \geq 9$. Assume that $\phi(V(S_3)) - 2 \leq 4$. Thus $\phi(V(S_2)) \geq 12$. If $\phi(v_8) \geq 2$, then we move one pebble to v_7. Now, $V(S_2)$ contains at least 13 pebbles and hence we can move three pebbles to v_{13} and so we can move one pebble to v_2. Let $\phi(v_8) \leq 1$. Clearly, $\phi(V(S_2)) \geq 13$ and hence we can move one pebble to v_2 through v_{13}. Let $\phi(v_{10}) = 0$, $\phi(v_9) \leq 1$ and $\phi(v_8) \leq 3$. Thus, $\phi(V(S_2)) \geq 15$. If $\phi(v_{13}) = 1$ or $\phi(v_4) = 1$ or $\phi(v_5) \geq 2$ or $\phi(v_7) \geq 2$, then we can move three additonal pebbles to either v_4 or v_{13}, since $\phi(V(S_2)) - 2 \geq 13$ and hence we can easily move one pebble to v_2. So, we assume that $\phi(v_{13}) = 0$, $\phi(v_4) = 0$, $\phi(v_5) \leq 1$ and $\phi(v_7) \leq 1$ and thus $\phi(v_6) \geq 13$. Let $\phi(v_7) = 1$. If $\phi(V(S_3)) = 4$, then we can move one pebble to v_7 and then we move six pebbles to v_7 from v_6 and hence we can move one pebble to v_2, since $d(v_2, v_7) = 3$. If not, then we can move seven pebbles to v_7 from v_6 and hence we are done. Thus we assume $\phi(v_7) = 0$. For the same reason we assume $\phi(v_5) = 0$ and thus $\phi(v_6) \geq 15$. If $\phi(V(S_3)) \geq 3$, then we can move one pebble to v_7 and then we move seven pebbles to v_7 from v_6 and hence we can move one pebble to v_2, since $d(v_2, v_7) = 3$. If not, then we can move eight pebbles to v_7 from v_6 and hence we are done.

Case 3.1.2: Let $\phi(v_{11}) = 1$ and $\phi(v_{12}) = 1$.
Clearly, we can move one pebble to v_1 using the pebbles on v_{11} and v_{12}

if $\phi(V(S_3)) \geq 7$ and then we move one more pebble to v_1 from $\phi(V(S_2))$ and hence we are done. Assume $\phi(V(S_3)) \leq 6$ and so $\phi(V(S_2)) \geq 13$. If $\phi(V(S_3)) \geq 5$, then also we can easily move one pebble to v_2. Assume $\phi(V(S_3)) \leq 4$ and thus $\phi(V(S_2)) \geq 15$. We do the same thing as we did above when $\phi(V(S_2)) \geq 15$ to put one pebble at v_2. Next, we assume $\phi(V(S_2)) = 7$ or 8. If $\phi(V(S_3)) \geq 5$ and $\phi(V(S_4)) \geq 5$, then we can easily move one pebble to v_2. Assume that $\phi(V(S_3)) \leq 4$ and so $\phi(V(S_4)) \geq 8$. Either we can move one more pebble to v_{10} from $\phi(V(S_3))$ or we should have $\phi(V(S_4)) \geq 9$ and hence we can easily move one pebble to v_2. Assume that $\phi(V(S_4)) \leq 4$ and so $\phi(V(S_3)) \geq 8$. If $\phi(v_{11}) + \phi(v_{12}) \geq 4$ or $\phi(v_3) = 1$, then we can move one pebble to v_2 either through v_1 or v_3. So we assume that $\phi(v_{11}) + \phi(v_{12}) \leq 3$ and $\phi(v_3) = 0$ and hence we can move 'four pebbles to v_{13}' or 'one pebble to v_1 and two pebbles to v_{13}', like we did above. Thus we can easily move one pebble to v_2.

Case 3.2: Let $\phi(V(S_4)) \geq 7$.
Either $\phi(V(S_4)) \geq 9$ or we can add one or two pebbles (if $\phi(V(S_4)) = 8$ or 7) to $V(S_4)$ from $V(S_2)$ and $V(S_3)$. Thus we can easily move one pebble to v_2.

Case 3.3: Let $\phi(V(S_3)) \geq 7$.
Let $\phi(V(S_3)) \geq 9$. Clearly, we can move one pebble to v_2 if $\phi(V(S_4)) \geq 5$ or $\phi(v_1) = 1$ or $\phi(v_{11}) + \phi(v_{12}) \geq 4$. So we assume that $\phi(v_1) = 0$ and $\phi(v_{11}) + \phi(v_{12}) \leq 3$.

Case 3.3.1: Let $\phi(V(S_2)) \geq 7$.
Clearly, we can move one pebble to v_2 by Case 3.1.

Case 3.3.2: Let $\phi(V(S_2)) = 5$ or 6.
Clearly we can move one pebble to v_2 if $\phi(v_3) = 1$. So, we assume $\phi(v_3) = 0$. If $\phi(v_{11}) \geq 2$, then we move one pebble to v_{10} and hence $V(S_3)$ contains 13 pebbles. Thus we can move four pebbles to v_{13} from $V(S_2)$ and $V(S_3)$ and hence we can move one pebble to v_2. Let $\phi(v_{11}) \leq 1$. Thus, $\phi(S_3) \geq 13$ and hence we can move one pebble to v_2 through v_{13}.

Case 3.3.3: Let $\phi(V(S_2)) \leq 4$.
We have $\phi(V(S_3)) \geq 13$. Hence, we can move 'four pebbles to v_{13}' or 'one pebble to v_1 and two pebbles to v_{13}', by considering the cases $\phi(v_{11}) = 2$ or 3 or $\phi(v_{11}) = 1$ and $\phi(v_{12}) = 1$ or $\phi(v_{12}) + \phi(v_{12}) \leq 1$. Otherwise, $\phi(V(S_3)) \geq 17$ and hence we can easily move one pebble to v_2 by Theorem

5.

For the case $\phi(V(S_3)) = 7$ or 8, one could see that we can easily always move a pebble to v_2.

Thus we can always move one pebble to v_2 using 21 pebbles on the vertices of $J_{3,4}$. So, $f(J_{3,4}) = 21$. □

Theorem 3. *For Jahangir graph $J_{3,m}$ ($m \geq 5$), $f(J_{3,m}) = 3m + 10$.*

Proof. Consider the following distribution: $\phi(v_8) = 15$, $\phi(v_9) = \phi(v_{3m}) = \phi(v_{3m-1}) = 1$, $\phi(v_4) = 3$ and $\phi(v_i) = 3$ for all $i \in \{12+3k\}$ $(0 \leq k \leq m-5)$. Then we cannot move one pebble to v_2. Since this distribution contains $15 + 1 + 1 + 1 + 3 + (m - 4)3 = 3m + 9$ pebbles, $f(J_{3,m}) \geq 3m + 10$ for $m \geq 5$.

Now, we consider the distribution of $3m + 10$ pebbles on the vertices of $J_{3,m}$ where $m \geq 5$.

Case 1: Let v_{3m+1} be the target vertex.
If any one of the 5-cycle contains five or more pebbles, then we can easily move one pebble to v_{3m+1}. Consider every 5-cycle contains at most four pebbles only. Since we have placed $3m + 10$ pebbles on the vertices of $J_{3,m}$, ten 5-cycles must contain exactly four pebbles. Without loss of generality, let $\phi(V(S_1)) = 4$. If one of adjacect cycle also has four pebbles or the adjacent vertex of a common vertex from the adjacent cycle contains more than two pebbles, then we can move one pebble to v_{3m+1} through the common vertex v_1 or v_4. If both the adjacent vertices of a common vertex have more than one pebble each, then also we can move one pebble to v_{3m+1}. Otherwise, the graph $J_{3,m}$ must contain at most $3m + 1$ pebbles - which is a contradiction to the total number of pebbles placed on the vertices of $J_{3,m}$.

Case 2: Let v_1 be the target vertex.
If $\phi(V(S_1)) \geq 5$ or $\phi(V(S_m)) \geq 5$, then we can easily move one pebble to v_1. Also if $\phi(v_2 + v_3) \geq 4$ or $\phi(v_{3m} + v_{3m-1}) \geq 4$, then we can move one pebble to v_1. So, we assume $\phi(v_2 + v_3) \leq 3$ and $\phi(v_{3m-1} + v_{3m}) \leq 3$. If $\phi(V(S_i)) \geq 5$ and $\phi(V(S_j))) \geq 5$ for some $i \neq 1$, m, $j \neq 1$, m, then we can move two pebbles to v_{3m+1} and hence one pebble is moved to v_1. Assume $\phi(V(S_i)) \geq 5$ and all other cycles contain at most four pebbles each except S_1 and S_m. Suppose we cannot move one more pebble to v_{3m+1} or we cannot move one pebble to v_1, then the graph $J_{3,m}$ contains at most $3m + 2$ pebbles - a contradiction to the total number of pebbles placed on the vertices of $J_{3,m}$. Next, we assume that every cycle contains at most

four pebbles only. If we have two adjacent cycles with four pebbles each on them, then we can move two pebbles to v_{3m+1} and hence we move one pebble to v_1. Thus we assume only one adjacent copy has four pebbles each on them. We move one pebble to v_{3m+1} from that adjacent cycle. Suppose if we cannot move one more pebble to v_{3m+1} or if we cannot move one pebble to v_1, then the grpah has at most $3m + 2$ pebbles - a contradiction to the total number of pebbles placed on the vertices of $J_{3,m}$. Suppose if there is no such adjacent cycles, then we can move two pebbles to v_{3m+1}, since we have $3m + 10$ pebbles and ten cycles have exactly four pebbles. If we cannot move one pebble to v_1, then the graph $J_{3,m}$ has at most $3m$ pebbles - a contradiction to the total number of pebbles placed on the vertices of $J_{3,m}$.

Case 3: Let v_2 be the target vertex.
If $\phi(V(S_1)) \geq 5$, then clearly we can move one pebble to v_2. Suppose four cycles have five pebbles each on them. Then we can move four pebbles to v_{3m+1} pebbles and hence one pebbles is moved to v_2. Let three cycles only have more than four pebbles. So we can move three pebbles to v_{3m+1}. If we cannot move one more pebble to v_{3m+1} or if we cannot move one pebble to v_2, then the graph $J_{3,m}$ has at most $3m + 8$ pebbles - a contradiction. Let two cycles have more than four pebbles. Suppose if we cannot move one pebble to v_2, then the graph has at most $3m + 6$ pebbles - a contradiction. Let only one cycle has more than four pebbles. Suppose if we cannot move one pebble to v_2, then the graph has at most $3m + 9$ pebbles - a contradiction. Assume every cycle has at most four pebbles only. Suppose if we cannot move one pebble to v_2, then the graph has at most $3m + 5$ pebbles - a contradiction.

Thus we can always move one pebble to v_2 using $3m + 10$ pebbles on the vertices of $J_{3,m}$. So, $f(J_{3,m}) = 3m + 10$. □

3. The t-pebbling number of Jahangir graph $J_{3,m}$

Theorem 1. *For Jahangir graph $J_{3,3}$, $f_t(J_{3,3}) = 16t + 1$.*

Proof. Let $\phi(v_6) = 16(t - 1) + 15$, $\phi(v_9) = 1$ and $\phi(v_i) = 0$ for all $i \neq 6, 9$. Then we cannot move t pebbles to v_2. Thus $f_t(J_{3,3}) > 16t$.

Now, consider the distribution of the $16t + 1$ pebbles on the vertices of $J_{3,3}$. Clearly the result is true for $t = 1$. Assume the result is true for $2 \leq t' < t$. Clearly, the graph $J_{3,3}$ has at least 33 pebbles and hence we can move one pebble to any target vertex at a cost of at most sixteen pebbles.

Then the remaining number of pebbles on the vertices of $J_{3,3}$ is at least $16(t-1)+1$ and hence we can move $t-1$ additional pebbles to that target vertex by induction. Thus $f_t(J_{3,3}) \le 16t+1$. □

Theorem 2. *For Jahangir graph $J_{3,4}$, $f_t(J_{3,4}) = 16t+5$.*

Proof. Let $\phi(v_8) = 16(t-1)+15$, $\phi(v_5) = 3$, $\phi(v_9) = \phi(v_{12}) = 1$ and $\phi(v_i) = 0$ for all $i \neq 5, 8, 9, 12$. Then we cannot move one pebble to v_2. Thus $f(J_{3,4}) > 16t+4$.

Now, consider the distribution of the $16t+5$ pebbles on the vertices of $J_{3,4}$. Clearly the result is true for $t = 1$. Assume the result is true for $2 \le t' < t$. Clearly, the graph $J_{3,4}$ has at least 37 pebbles and hence we can move one pebble to any target vertex at a cost of at most sixteen pebbles. Then the remaining number of pebbles on the vertices of $J_{3,4}$ is at least $16(t-1)+5$ and hence we can move $t-1$ additional pebbles to that target vertex by induction. Thus $f_t(J_{3,4}) \le 16t+5$. □

Theorem 3. *For Jahangir graph $J_{3,m}$ $(m \ge 5)$, $f_t(J_{3,m}) = 16t+3m-6$.*

Proof. Consider the following distribution: $\phi(v_8) = 16(t-1)+15$, $\phi(v_9) = \phi(v_{3m}) = \phi(v_{3m-1}) = 1$, $\phi(v_4) = 3$ and $\phi(v_i) = 3$ for all $i \in \{12+3k\}$ $(0 \le k \le m-5)$. Then we cannot move one pebble to v_2. Since this distribution contains $16(t-1)+3m+9$ pebbles, $f(J_{3,m}) \ge 16t+3m-6$ for $m \ge 5$.

Now, consider the distribution of the $16t+3m-6$ pebbles on the vertices of $J_{3,m}$. Clearly the result is true for $t = 1$. Assume the result is true for $2 \le t' < t$. Clearly, the graph $J_{3,m}$ has at least $3m+24$ pebbles and hence we can move one pebble to any target vertex at a cost of at most sixteen pebbles. Then the remaining number of pebbles on the vertices of $J_{3,m}$ is at least $16(t-1)+3m-6$ and hence we can move $t-1$ additional pebbles to that target vertex by induction. Thus $f_t(J_{3,m}) \le 16t+3m-6$. □

4. An upper bound for the t-pebbling number of Jahangir graph $J_{n,m}$

Here, we present the known results about the pebbling number of Jahangir graphs from [6, 7, 8]. The pebbling number of Jahangir graph $J_{2,m}$ $(m \ge 3)$ is as follows:

Theorem 1. *[6] For Jahangir graph $J_{2,3}$, $f(J_{2,3}) = 8$.*

Theorem 2. *[6] For Jahangir graph $J_{2,4}$, $f(J_{2,4}) = 16$.*

Theorem 3. *[6] For Jahangir graph $J_{2,5}$, $f(J_{2,5}) = 18$.*

Theorem 4. *[6] For Jahangir graph $J_{2,6}$, $f(J_{2,6}) = 21$.*

Theorem 5. *[6] For Jahangir graph $J_{2,7}$, $f(J_{2,7}) = 23$.*

Theorem 6. *[7] For Jahangir graph $J_{2,m}$ where $m \geq 8$, $f(J_{2,m}) = 2m+10$.*

The t-pebbling number of Jahangir graph $J_{2,m}$ $(m \geq 3)$ is as follows:

Theorem 7. *[8] For Jahangir graph $J_{2,3}$, $f_t(J_{2,3}) = 8t$.*

Theorem 8. *[8] For Jahangir graph $J_{2,4}$, $f_t(J_{2,4}) = 16t$.*

Theorem 9. *[8] For Jahangir graph $J_{2,5}$, $f_t(J_{2,5}) = 16t + 2$.*

Theorem 10. *[8] For Jahangir graph $J_{2,m}$, $f_t(J_{2,m}) = 16(t-1) + f(J_{2,m})$ where $m \geq 6$.*

From the above results and the results from this paper, we can conclude that $f_t(J_{n,m}) \geq t(2^k)$, where $k = 2^{2\lfloor \frac{n}{2} \rfloor + 2}$ is the diameter of $J_{n,m}$ for $3 \leq n < m$ (for $n = 2$, we take $m \geq 4$).

After seeing the behaviour of Jahangir graph $J_{n,m}$, we give the following conjecture for the t-pebbling number of $J_{n,m}$.

Conjecture 1. *For Jahangir graph $J_{n,m}$ $(3 \leq n < m)$,*

$$f_t(J_{n,m}) \leq \begin{cases} t(2^k) + (m-2)(2^{\lfloor \frac{n}{2} \rfloor} - 1) & \textit{if } n \textit{ is even} \\ t(2^k) + (m-3)(2^{\lfloor \frac{n}{2} \rfloor + 1} - 1) + n & \textit{if } n \textit{ is odd,} \end{cases}$$

where $k = 2^{2\lfloor \frac{n}{2} \rfloor + 2}$ is the diameter of $J_{n,m}$.

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