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# Spectrum and fine spectrum of the upper triangular matrix $U(r, s)$ over the sequence space 

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#### Abstract

Fine spectra of various matrix operators on different sequence spaces have been investigated by several authors. Recently, some authors have determined the approximate point spectrum, the defect spectrum and the compression spectrum of various matrix operators on different sequence spaces. Here in this article we have determined the spectrum and fine spectrum of the upper triangular matrix $U(r, s)$ on the sequence space cs. In a further development, we have also determined the approximate point spectrum, the defect spectrum and the compression spectrum of the operator $U(r, s)$ on the sequence space cs.


Keywords and Phrases : Spectrum of an operator ; matrix mapping; sequence space.

AMS Classification No.: 47 A10, $47 B 37$.

## 1. Introduction

By $w$, we denote the space of all real or complex valued sequences. Throughout the paper $c, c_{0}, b v, c s, b s, \ell_{1}, \ell_{\infty}$ represent the spaces of all convergent, null, bounded variation, convergent series, bounded series, absolutely summable and bounded sequences respectively. Also $b v_{0}$ denotes the sequence space $b v \cap c_{0}$.

Okutoyi [23] determined the spectrum of the Cesàro operator $C_{1}$ on the sequence space $b v_{0}$. The fine spectra of the Cesàro operator $C_{1}$ over the sequence space $b v_{p},(1 \leq p<\infty)$ was determined by Akhmedov and Başar [2]. Altay and Başar [3, 4] determined the fine spectrum of the difference operator $\Delta$ and the generalized difference operator $B(r, s)$ on the sequence spaces $c_{0}$ and $c$. Furkan, Bilgiç and Kayaduman [14] have determined the fine spectrum of the generalized difference operator $B(r, s)$ over the sequence spaces $\ell_{1}$ and $b v$. The spectrum and fine spectrum of the Zweier Matrix on the sequence spaces $\ell_{1}$ and $b v$ were studied by Altay and Karakus [5]. Altun [6, 7] determined the fine spectra of triangular Toeplitz operators and tridiagonal symmetric matrices over some sequence spaces. Fine spectra of operator $B(r, s, t)$ over the sequence spaces $\ell_{1}$ and $b v$ and generalized difference operator $B(r, s)$ over the sequence spaces $\ell_{p}$ and $b v_{p}$, $(1 \leq p<\infty)$ were studied by Bilgiç and Furkan [11, 12]. Akhmedov and El-Shabrawy [1] determined the fine spectrum of the operator $\Delta_{a, b}$ on the sequence space $c$. Panigrahi and Srivastava $[24,25]$ studied the spectrum and fine spectrum of the second order difference operator $\Delta_{u v}^{2}$ on the sequence space $c_{0}$ and generalized second order forward difference operator $\Delta_{u v w}^{2}$ on the sequence space $\ell_{1}$. Fine spectrum of the generalized difference operator $\Delta_{v}$ on the sequence space $\ell_{1}$ was investigated by Srivastava and Kumar [28]. Fine spectra of upper triangular double-band matrix $U(r, s)$ over the sequence spaces $c_{0}$ and $c$ were studied by Karakaya and Altun [20]. Recently, Karaisa and Başar [19] have determined the spectrum and fine spectrum of the upper traiangular matrix $A(r, s, t)$ over the sequence space $\ell_{p},(0<p<\infty)$. In a further development, they have also determined the approximate point spectrum, defect spectrum and compression spectrum of the operator $A(r, s, t)$ on the sequence space $\ell_{p},(0<p<\infty)$. The approximate point spectrum, defect spectrum and compression spectrum of the operator $B(r, s)$ on the sequence spaces $c_{0}, c, \ell_{p}$ and $b v_{p},(1<p<\infty)$ were studied by Başar, Durna and Yildirim [9].

The notion of matrix transformations over sequence space has been studied from various aspects. Banach algebra of matrix maps have been investigated by Rath and Tripathy [26]. Besides the above listed workers, the spectrum and fine spectrum for various matrix operators has been investigated by Tripathy and Pal [29, 30], Tripathy and Saikia [31] and many others in the recent years.

In this paper, we shall determine the spectrum and fine spectrum of the upper triangular matrix $U(r, s)$ on the sequence space $c s$. Also,we will determine the approximate point spectrum, the defect spectrum and the compression spectrum of the operator $U(r, s)$ on the sequence space $c s$. Clearly, cs $=\left\{x=\left(x_{n}\right) \in w: \lim _{n \rightarrow \infty} \sum_{i=0}^{n} x_{i} \quad\right.$ exists $\}$ is a Banach space with respect to the norm $\|x\|_{c s}=\sup _{n}\left|\sum_{i=0}^{n} x_{i}\right|$.

## 2. Preliminaries and Background

Let $X$ and $Y$ be Banach spaces and $T: X \rightarrow Y$ be a bounded linear operator. By $R(T)$, we denote the range of $T$, i. e.

$$
R(T)=\{y \in Y: y=T x, x \in X\} .
$$

By $B(X)$, we denote the set of all bounded linear operators on $X$ into itself. If $T \in B(X)$, then the adjoint $T^{*}$ of $T$ is a bounded linear operator on the dual $X^{*}$ of $X$ defined by $\left(T^{*} f\right)(x)=f(T x)$, for all $f \in X^{*}$ and $x \in X$. Let $X \neq\{\theta\}$ be a complex normed linear space, where $\theta$ is the zero element and $T: D(T) \rightarrow X$ be a linear operator with domain $D(T) \subseteq X$. With $T$, we associate the operator

$$
T_{\lambda}=T-\lambda I,
$$

where $\lambda$ is a complex number and $I$ is the identity operator on $D(T)$. If $T_{\lambda}$ has an inverse which is linear, we denote it by $T_{\lambda}^{-1}$, that is

$$
T_{\lambda}^{-1}=(T-\lambda I)^{-1},
$$

and call it the resolvent operator of $T$. A regular value $\lambda$ of $T$ is a complex number such that
(R1) $T_{\lambda}^{-1}$ exists,
(R2) $T_{\lambda}^{-1}$ is bounded,
(R3) $T_{\lambda}^{-1}$ is defined on a set which is dense in $X$ i. e. $\overline{R\left(T_{\lambda}\right)}=X$.
The resolvent set of $T$, denoted by $\rho(T, X)$, is the set of all regular values $\lambda$ of $T$. Its complement $\sigma(T, X)=\mathbf{C}-\rho(T, X)$ in the complex plane $\mathbf{C}$ is called the spectrum of $T$. Furthermore, the spectrum $\sigma(T, X)$ is partitioned into three disjoint sets as follows:

The point(discrete) spectrum $\sigma_{p}(T, X)$ is the set such that $T_{\lambda}^{-1}$ does not exist. Any such $\lambda \in \sigma_{p}(T, X)$ is called an eigenvalue of $T$.

The continuous spectrum $\sigma_{c}(T, X)$ is the set such that $T_{\lambda}^{-1}$ exists and satisfies ( $R 3$ ), but not ( $R 2$ ), that is, $T_{\lambda}^{-1}$ is unbounded.

The residual spectrum $\sigma_{r}(T, X)$ is the set such that $T_{\lambda}^{-1}$ exists (and may be bounded or not), but does not satisfy ( $R 3$ ), that is, the domain of $T_{\lambda}^{-1}$ is not dense in $X$.

From Goldberg [17], if $X$ is a Banach space and $T \in B(X)$, then there are three possibilities for $R(T)$ and $T^{-1}$ :
(I) $R(T)=X$,
(II) $R(T) \neq \overline{R(T)}=X$
(III) $\overline{R(T)} \neq X$
and
(1) $T^{-1}$ exists and is continuous,
(2) $T^{-1}$ exists but is discontinuous,
(3) $T^{-1}$ does not exist.

Applying Goldberg [17] classification to $T_{\lambda}$, we have three possibilities for $T_{\lambda}$ and $T_{\lambda}^{-1}$;
(I) $T_{\lambda}$ is surjective,
(II) $R\left(T_{\lambda}\right) \neq \overline{R\left(T_{\lambda}\right)}=X$,
(III) $\overline{R\left(T_{\lambda}\right)} \neq X$,
and
(I) $T_{\lambda}$ is injective and $T_{\lambda}^{-1}$ is continuous,
(II) $T_{\lambda}$ is injective but $T_{\lambda}^{-1}$ is discontinuous,
(III) $T_{\lambda}$ is not injective.

If these possibilities are combined in all possible ways, nine different states are created which may be shown as in the Table 2.1.

These are labelled by: $I_{1}, I_{2}, I_{3}, I I_{1}, I I_{2}, I I_{3}, I I I_{1}, I I I_{2}$ and $I I I_{3}$. If $\lambda$ is a complex number such that $T_{\lambda} \in I_{1}$ or $T_{\lambda} \in I_{2}$, then $\lambda$ is in the resolvent set $\rho(T, X)$ of $T$. The further classification gives rise to the fine spectrum of $T$. If an operator is in state $I I_{2}$ for example, then $R(T) \neq \overline{R(T)}=X$ and $T^{-1}$ exists but is discontinuous and we write $\lambda \in I I_{2} \sigma(T, X)$. The state $I I_{1}$ is impossible as if $T_{\lambda}$ is injective, then from Kreyszig [[22], Problem 6, p. 290] $T_{\lambda}^{-1}$ is bounded and hence continuous if and only if $R\left(T_{\lambda}\right)$ is closed.

Again, following Appell et. al. [8], we define the three more subdivisions of the spectrum called as the approximate point spectrum, defect spectrum and compression spectrum.

Given a bounded linear operator $T$ in a Banach space $X$, we call a sequence $\left(x_{k}\right)$ in $X$ as a Weyl sequence for $T$ if $\left\|x_{k}\right\|=1$ and $\left\|T x_{k}\right\| \rightarrow 0$ as $k \rightarrow \infty$.

The approximate point spectrum of $T$, denoted by $\sigma_{a p}(T, X)$, is defined as the set
(2.1) $\sigma_{a p}(T, X)=\{\lambda \in \mathbf{C}$ : there exists a Weyl sequence for $T-\lambda I\}$

The defect spectrum of $T$, denoted by $\sigma_{\delta}(T, X)$,is defined as the set

$$
\begin{equation*}
\sigma_{\delta}(T, X)=\{\lambda \in \mathbf{C}: T-\lambda I \quad \text { is not surjective }\} \tag{2.2}
\end{equation*}
$$

The two subspectra given by equations (2.1) and (2.2) form a (not necessarily disjoint) subdivisions

$$
\begin{equation*}
\sigma(T, X)=\sigma_{a p}(T, X) \cup \sigma_{\delta}(T, X) \tag{2.3}
\end{equation*}
$$

of the spectrum. There is another subspectrum

$$
\sigma_{c o}(T, X)=\{\lambda \in \mathbf{C}: \overline{R(T-\lambda I)} \neq X\}
$$

which is often called the compression spectrum of $T$. The compression spectrum gives rise to another (not necessarily disjoint) decomposition

$$
\begin{equation*}
\sigma(T, X)=\sigma_{a p}(T, X) \cup \sigma_{c o}(T, X) \tag{2.4}
\end{equation*}
$$

Clearly, $\sigma_{p}(T, X) \subseteq \sigma_{a p}(T, X)$ and $\sigma_{c o}(T, X) \subseteq \sigma_{\delta}(T, X)$. Moreover, it is easy to verify that $\sigma_{r}(T, X)=\sigma_{c o}(T, X) \backslash \sigma_{p}(T, X)$ and $\sigma_{c}(T, X)=$ $\sigma(T, X) \backslash\left[\sigma_{p}(T, X) \cup \sigma_{c o}(T, X)\right]$.

By the definitions given above, we can illustrate the subdivisions spectrum in the Table 2.2.

Proposition 2.1. [Appell et al. [8], Proposition 1.3, p. 28] Spectra and subspectra of an operator $T \in B(X)$ and its adjoint $T^{*} \in B\left(X^{*}\right)$ are related by the following relations:
a) $\sigma\left(T^{*}, X^{*}\right)=\sigma(T, X)$.
b) $\sigma_{c}\left(T^{*}, X^{*}\right) \subseteq \sigma_{a p}(T, X)$.
c) $\sigma_{a p}\left(T^{*}, X^{*}\right)=\sigma_{\delta}(T, X)$.
d) $\sigma_{\delta}\left(T^{*}, X^{*}\right)=\sigma_{a p}(T, X)$.
e) $\sigma_{p}\left(T^{*}, X^{*}\right)=\sigma_{c o}(T, X)$.
f) $\sigma_{c o}\left(T^{*}, X^{*}\right) \supseteq \sigma_{p}(T, X)$.
g) $\sigma(T, X)=\sigma_{a p}(T, X) \cup \sigma_{p}\left(T^{*}, X^{*}\right)=\sigma_{p}(T, X) \cup \sigma_{a p}\left(T^{*}, X^{*}\right)$.

The relations (c)-(f) show that the approximate point spectrum is in a certain sense dual to defect spectrum, and the point spectrum dual to the compression spectrum. The equality (g) implies, in particular, that $\sigma(T, X)=\sigma_{a p}(T, X)$ if X is a Hilbert space and T is normal. Roughly speaking, this shows that normal (in particular, self-adjoint) operators on Hilbert spaces are most similar to matrices in finite dimensional spaces (Appell et al. [8]).

Let $\lambda$ and $\mu$ be two sequence spaces and $A=\left(a_{n k}\right)$ be an infinite matrix of real or complex numbers $a_{n k}$, where $n, k \in \mathbf{N}_{0}=\{0,1,2, \ldots\}$. Then, we
say that $A$ defines a matrix mapping from $\lambda$ into $\mu$, and we denote it by $A: \lambda \rightarrow \mu$, if for every sequence $x=\left(x_{k}\right) \in \lambda$, the sequence $A x=\left\{(A x)_{n}\right\}$, the $A$-transform of $x$, is in $\mu$, where

$$
\begin{equation*}
(A x)_{n}=\sum_{k=0}^{\infty} a_{n k} x_{k}, n \in \mathbf{N}_{0} \tag{2.5}
\end{equation*}
$$

By $(\lambda: \mu)$, we denote the class of all matrices such that $A: \lambda \rightarrow \mu$. Thus, $A \in(\lambda: \mu)$ if and only if the series on the right hand side of equation (2.5) converges for each $n \in \mathbf{N}_{0}$ and every $x \in \lambda$ and we have $A x=\left\{(A x)_{n}\right\}_{n \in \mathbf{N}_{0}} \in \mu$ for all $x \in \lambda$.

The upper triangular matrix $U(r, s)$ is an infinite matrix of the form

$$
U(r, s)=\left[\begin{array}{ccccc}
r & s & 0 & 0 & \cdots \\
0 & r & s & 0 & \cdots \\
0 & 0 & r & s & \cdots \\
0 & 0 & 0 & r & \cdots \\
\vdots & \vdots & \vdots & \vdots & \ddots
\end{array}\right]
$$

where $s \neq 0$.
The following results will be used in order to establish the results of this article.

Lemma 2.2. [Wilansky [32] Example 6B, Page 130] The matrix $A=\left(a_{n k}\right)$ gives rise to a bounded linear operator $T \in B(c s)$ from $c s$ to itself if and only if:
(i) $\stackrel{\sup }{m} \sum_{k}\left|\sum_{n=1}^{m}\left(a_{n k}-a_{n, k-1}\right)\right|<\infty$.
(ii) $\sum_{n} a_{n k}$ is convergent for each k .

Lemma 2.3. [Goldberg [17], Page 59] $T$ has a dense range if and only if $T^{*}$ is one to one.

Lemma 2.4. [Goldberg [17], Page 60] $T$ has a bounded inverse if and only if $T^{*}$ is onto.

## 3. Spectrum and fine spectrum of the operator $U(r, s)$ over the sequence space $c s$

The following result will be used for establishing some results of this section.
Lemma 3.1. [Akhmedov and El-Shabrawy [1], Lemma 2.1] Let ( $c_{n}$ ) and $\left(d_{n}\right)$ be two sequences of complex numbers such that $\lim _{n \rightarrow \infty} c_{n}=c$ and $|c|<1$. Define the sequence $\left(z_{n}\right)$ of complex numbers such that $z_{n+1}=$ $c_{n+1} z_{n}+d_{n+1}$ for all $n \in \mathbf{N}_{0}$. Then
(i) if $\left(d_{n}\right)$ is bounded, then $\left(z_{n}\right)$ is bounded.
(ii) if $\left(d_{n}\right)$ is convergent then $\left(z_{n}\right)$ is convergent.
(iii) if $\left(d_{n}\right)$ is a a null sequence, then $\left(z_{n}\right)$ is a null sequence.

Theorem 3.2. $U(r, s): c s \rightarrow c s$ is a bounded linear operator and

$$
\|U(r, s)\|_{(c s: c s)} \leq|r|+|s|
$$

Proof. From Lemma 2.2, it is easy to show that $U(r, s): c s \rightarrow c s$ is a bounded linear operator.

Now,

$$
\begin{aligned}
|U x|=|U(r, s) x| & =\left|\sum_{i=1}^{n}\left(r x_{i-1}+s x_{i}\right)\right| \\
& \leq|r|\left|\sum_{i=1}^{n} x_{i-1}\right|+|s|\left|\sum_{i=1}^{n} x_{i}\right| \\
& \leq(|r|+|s|)\|x\|_{c s}
\end{aligned}
$$

And hence, $\|U(r, s)\|_{(c s: c s)} \leq|r|+|s|$
Theorem 3.3. The point spectrum of the operator $U(r, s)$ over $c s$ is given by

$$
\sigma_{p}(U(r, s), c s)=\{\alpha \in \mathbf{C}:|\alpha-r|<|s|\} .
$$

Proof. Let $\alpha$ be an eigenvalue of the operator $U(r, s)$. Then there exists $x \neq \theta=(0,0,0,0, \ldots)$ in $c s$ such that $U(r, s) x=\alpha x$. Then, we have

$$
\begin{aligned}
r x_{0}+s x_{1} & =\alpha x_{0} \\
r x_{1}+s x_{2} & =\alpha x_{1} \\
r x_{2}+s x_{3} & =\alpha x_{2} \\
& \cdots \\
r x_{n}+s x_{n+1} & =\alpha x_{n}, \quad n \geq 0
\end{aligned}
$$

Then, we have

$$
\begin{aligned}
x_{1} & =\frac{\alpha-r}{s} x_{0} \\
x_{2} & =\frac{\alpha-r}{s} x_{1}=\left(\frac{\alpha-r}{s}\right)^{2} x_{0} \\
x_{3} & =\frac{\alpha-r}{s} x_{2}=\left(\frac{\alpha-r}{s}\right)^{3} x_{0} \\
& \cdots \\
x_{n} & =\left(\frac{\alpha-r}{s}\right)^{n} x_{0}, \quad n \geq 1
\end{aligned}
$$

Since, $x=\left(x_{n}\right) \in c s$, so $\lim _{n \rightarrow \infty} \sum_{i=0}^{n} x_{i}=\lim _{n \rightarrow \infty} \sum_{i=0}^{n}\left(\frac{\alpha-r}{s}\right)^{n} x_{0}$ exists if and only if
$|\alpha-r|<|s|$. Hence, $\sigma_{p}(U(r, s), c s)=\{\alpha \in \mathbf{C}:|\alpha-r|<|s|\}$.
If $T: c s \rightarrow c s$ is a bounded linear operator represented by a matrix $A$, then it is known that the adjoint operator $T^{*}: c s^{*} \rightarrow c s^{*}$ is defined by the transpose $A^{t}$ of the matrix $A$. It should be noted that the dual space $c s^{*}$ of $c s$ is isometrically isomorphic to the Banach space $b v$ of all bounded variation sequences normed by $\|x\|_{b v}=\sum_{n=0}^{\infty}\left|x_{n+1}-x_{n}\right|+\lim _{n \rightarrow \infty}\left|x_{n}\right|$.

Theorem 3.4. The point spectrum of the operator $U(r, s)^{*}$ over $c s^{*}$ is given by

$$
\sigma_{p}\left(U(r, s)^{*}, c s^{*} \cong b v\right)=\phi .
$$

Proof. Let $\alpha$ be an eigenvalue of the operator $U(r, s)^{*}$. Then there exists $x \neq \theta=(0,0,0, \ldots)$ in $b v$ such that $U(r, s) x=\alpha x$. Then, we have

$$
r x_{0}=\alpha x_{0}
$$

$$
\begin{array}{r}
s x_{0}+r x_{1}=\alpha x_{1} \\
s x_{1}+r x_{1}=\alpha x_{2} \\
\cdots \\
s x_{n}+r x_{n+1}=\alpha x_{n}, \quad n \geq 1
\end{array}
$$

If $x_{n_{0}}$ is the first non-zero entry of the sequence $\left(x_{n}\right)$, then $\alpha=r$. Then from the relation $s x_{n_{0}}+r x_{n_{0}+1}=\alpha x_{n_{0}+1}$, we have $s x_{n_{0}}=0$. But $s \neq 0$ and hence, $x_{n_{0}}=0$, a contradiction. Hence, $\sigma_{p}\left(U(r, s)^{*}, c s^{*} \cong b v\right)=\phi$.

Theorem 3.5. For any $\alpha \in \mathbf{C}, U(r, s)-\alpha I$ has a dense range.

Proof. By Theorem 3.4, $\sigma_{p}\left(U(r, s)^{*}, c s^{*} \cong b v\right)=\phi$.
Hence, $U(r, s)^{*}-\alpha I$ i.e. $(U(r, s)-\alpha I)^{*}$ is one to one for all $\alpha \in \mathbf{C}$. So, by applying Lemma 2.3, we get the required result.

Corollary 3.6. The residual spectrum of the operator $U(r, s)$ over $c s$ is given by

$$
\sigma_{r}(U(r, s), c s)=\phi
$$

Proof. $\quad$ Since, $U(r, s)-\alpha I$ has a dense range for all $\alpha \in \mathbf{C}$, so $\sigma_{r}(U(r, s), c s)=$ $\phi$.

Theorem 3.7. The continuous spectrum and the spectrum of the operator $U(r, s)$ over $c s$ are respectively given by

$$
\sigma_{c}(U(r, s), c s)=\{\alpha \in \mathbf{C}:|\alpha-r|=|s|\}
$$

and

$$
\sigma(U(r, s), c s)=\{\alpha \in \mathbf{C}:|\alpha-r| \leq|s|\} .
$$

Proof. Let $y=\left(y_{n}\right) \in b v$ be such that $(U(r, s)-\alpha I)^{*} x=y$ for some $x=\left(x_{n}\right)$. Then we have following system of linear equations:

$$
\begin{aligned}
(r-\alpha) x_{0} & =y_{0} \\
s x_{0}+(r-\alpha) x_{1} & =y_{1} \\
s x_{1}+(r-\alpha) x_{2} & =y_{2}
\end{aligned}
$$

$$
s x_{n-1}+(r-\alpha) x_{n}=y_{n}+1, \quad n \geq 1
$$

Solving these equations we get,

$$
\begin{aligned}
x_{0} & =\frac{1}{r-\alpha} y_{0} \\
x_{1} & =\frac{1}{r-\alpha}\left(y_{1}-s x_{0}\right)=\frac{1}{r-\alpha} y_{1}-\frac{s}{(r-\alpha)^{2}} y_{0} \\
x_{2} & =\frac{1}{r-\alpha}\left(y_{2}-s x_{1}\right)=\frac{1}{r-\alpha} y_{2}-\frac{s}{(r-\alpha)^{2}} y_{1}+\frac{s^{2}}{(r-\alpha)^{3}} y_{0} \\
x_{3} & =\frac{1}{r-\alpha}\left(y_{3}-s x_{2}\right)=\frac{1}{r-\alpha} y_{3}-\frac{s}{(r-\alpha)^{2}} y_{2}+\frac{s^{2}}{(r-\alpha)^{3}} y_{1}-\frac{s^{3}}{(r-\alpha)^{4}} y_{0} \\
\ldots & =\frac{1}{r-\alpha} \sum_{i=0}^{k}\left(-\frac{s}{r-\alpha}\right)^{k-i} y_{i}
\end{aligned}
$$

Also $x_{k+1}=\frac{1}{r-\alpha} \sum_{i=0}^{k+1}\left(-\frac{s}{r-\alpha}\right)^{k+1-i} y_{i}$.
Now,

$$
\begin{aligned}
\left|x_{k+1}-x_{k}\right|= & \frac{1}{|r-\alpha|}\left|\sum_{i=0}^{k+1}\left(-\frac{s}{r-\alpha}\right)^{k+1-i} y_{i}-\sum_{i=0}^{k}\left(-\frac{s}{r-\alpha}\right)^{k-i} y_{i}\right| \\
= & \frac{1}{|r-\alpha|} \left\lvert\,\left\{\left(-\frac{s}{r-\alpha}\right)^{k+1} y_{0}+\left(-\frac{s}{r-\alpha}\right)^{k} y_{1}+\cdots+y_{k+1}\right\}\right. \\
& \left.-\left\{\left(-\frac{s}{r-\alpha}\right)^{k} y_{0}+\left(-\frac{s}{r-\alpha}\right)^{k-1} y_{1}+\cdots+y_{k}\right\} \right\rvert\, \\
= & \frac{1}{|r-\alpha|} \left\lvert\,\left(-\frac{s}{r-\alpha}\right)^{k+1} y_{0}+\left(-\frac{s}{r-\alpha}\right)^{k}\left(y_{1}-y_{0}\right)+\right. \\
& \left.\left(-\frac{s}{r-\alpha}\right)^{k-1}\left(y_{2}-y_{1}\right)+\cdots+\left(y_{k+1}-y_{k}\right) \right\rvert\, \\
\leq & \frac{1}{|r-\alpha|}\left[\left|\frac{s}{r-\alpha}\right|^{k+1}\left|y_{0}\right|+\left|\frac{s}{r-\alpha}\right|^{k}\left|y_{1}-y_{0}\right|+\right.
\end{aligned}
$$

$$
\left.\left|\frac{s}{r-\alpha}\right|^{k-1}\left|y_{2}-y_{1}\right|+\cdots+\left|y_{k+1}-y_{k}\right|\right]
$$

Since $y=\left(y_{n}\right) \in b v$, therefore $\lim _{n \rightarrow \infty} \sum_{i=0}^{n}\left|y_{i+1}-y_{i}\right|$ exists and

$$
\|y\|_{b v}=\sum_{n=0}^{\infty}\left|y_{n+1}-y_{n}\right|+\lim _{n \rightarrow \infty}\left|y_{n}\right| .
$$

Therefore $\left|y_{n+1}-y_{n}\right| \leq\|y\|_{b v}$ and $\left|y_{n}\right| \leq\|y\|_{b v}$ for all $n \in \mathbf{N}_{0}$. Hence,

$$
\begin{aligned}
& \lim _{n \rightarrow \infty} \sum_{k=0}^{n}\left|x_{k+1}-x_{k}\right| \\
& \leq \frac{1}{|r-\alpha|}\left[\begin{array}{l}
\lim _{n \rightarrow \infty} \sum_{k=0}^{n}\left|\frac{s}{r-\alpha}\right|^{k+1}\left|y_{0}\right|+\lim _{n \rightarrow \infty} \sum_{k=0}^{n}\left|\frac{s}{r-\alpha}\right|^{k}\left|y_{1}-y_{0}\right| \\
+\lim _{n \rightarrow \infty} \sum_{k=0}^{n}\left|\frac{s}{r-\alpha}\right|^{k-1}\left|y_{2}-y_{1}\right|+\cdots+\lim _{n \rightarrow \infty} \sum_{k=0}^{n}\left|y_{k+1}-y_{k}\right|
\end{array}\right] \\
& \leq \frac{1}{r-\alpha \mid}\left[\begin{array}{l}
\lim _{n \rightarrow \infty} \sum_{k=0}^{n}\left|\frac{s}{r-\alpha}\right|^{k+1}+\lim _{n \rightarrow \infty} \sum_{k=0}^{n}\left|\frac{s}{r-\alpha}\right|^{k}+\lim _{n \rightarrow \infty} \sum_{k=0}^{n}\left|\frac{s}{r-\alpha}\right|^{k-1} \\
+\cdots+\left|\frac{s}{r-\alpha}\right|+1
\end{array}\right]\|y\|_{b v}
\end{aligned}
$$

Let $\alpha \in \mathbf{C}$ such that $|r-\alpha|>|s|$. Then $\lim _{n \rightarrow \infty} \sum_{k=0}^{n}\left|x_{k+1}-x_{k}\right|<\infty$ and hence $x=\left(x_{n}\right)$ is in $b v$. Therefore the operator $(U(r, s)-\alpha I)^{*}$ is onto if $|r-\alpha|>|s|$. Then by Lemma 2.4, $U(r, s)-\alpha I$ has a bounded inverse if $|r-\alpha|>|s|$.

Hence,

$$
\sigma_{c}(U(r, s), c s) \subseteq\{\alpha \in \mathbf{C}:|r-\alpha| \leq|s|\}
$$

Now

$$
\sigma(U(r, s), c s)=
$$

$$
\sigma_{p}(U(r, s), c s) \cup \sigma_{r}(U(r, s), c s) \cup \sigma_{c}(U(r, s), c s) \subseteq\{\alpha \in \mathbf{C}:|r-\alpha| \leq|s|\} .
$$

By Theorem 3.3, we have

$$
\{\alpha \in \mathbf{C}:|r-\alpha|<|s|\}=\sigma_{p}(U(r, s), c s) \subset \sigma(U(r, s), c s)
$$

Since $\sigma(U(r, s), c s)$ is a compact set, so it is closed and hence,

$$
\overline{\{\alpha \in \mathbf{C}:|r-\alpha|<|s|\}} \subset \overline{\sigma(U(r, s), c s)}=\sigma(U(r, s), c s)
$$

and therefore, $\{\alpha \in \mathbf{C}:|r-\alpha| \leq|s|\} \subset \sigma(U(r, s), c s)$.
Hence, $\sigma(U(r, s), c s)=\{\alpha \in \mathbf{C}:|\alpha-r| \leq|s|\}$.
Since $\sigma(U(r, s), c s)$ is disjoint union of $\sigma_{p}(U(r, s), c s), \sigma_{r}(U(r, s), c s)$ and $\sigma_{c}(U(r, s), c s)$,
therefore $\sigma_{c}(U(r, s), c s)=\{\alpha \in \mathbf{C}:|\alpha-r|=|s|\}$.

Theorem 3.8. If $|\alpha-r|<|s|$, then $\alpha \in I_{3} \sigma(U(r, s), c s)$ (See Table 2.2).
Proof. Let $|\alpha-r|<|s|$. Then by Theorem 3.3, $\alpha \in \sigma_{p}(U(r, s), c s)$. So $\alpha$ satisfies Goldberg's condition 3.

To get the result we need to show that $U(r, s)-\alpha I$ is onto when $|\alpha-r|<$ $|s|$.

Let $y=\left(y_{n}\right) \in c s$ be such that $(U(r, s)-\alpha I) x=y$ for some $x=\left(x_{n}\right)$.
Then $(r-\alpha) x_{n}+s x_{n+1}=y_{n} ; \quad n \geq 0$.
Now,

$$
\begin{aligned}
s x_{n+1} & =(\alpha-r) x_{n}+y_{n} \\
& \Rightarrow x_{n+1}=\frac{(\alpha-r)}{s} x_{n}+\frac{1}{s} y_{n} \\
& \Rightarrow \sum_{n=0}^{k} x_{n+1}=\frac{(\alpha-r)}{s} \sum_{n=0}^{k} x_{n}+\frac{1}{s} \sum_{n=0}^{k} y_{n} \\
& \Rightarrow \sum_{n=0}^{k+1} x_{n}=\frac{(\alpha-r)}{s} \sum_{n=0}^{k} x_{n}+\left(\frac{1}{s} \sum_{n=0}^{k} y_{n}+x_{0}\right)
\end{aligned}
$$

Let $c_{k}=\frac{\alpha-r}{s}, z_{k}=\sum_{n=0}^{k} x_{n}$ and $d_{k+1}=\frac{1}{s} \sum_{n=0}^{k} y_{n}+x_{0}$.
Then $z_{k+1}=c_{k+1} z_{k}+d_{k+1}$. Now, $\lim _{k \rightarrow \infty} c_{k}=\frac{\alpha-r}{s}$ and $\left|\frac{\alpha-r}{s}\right|<1$.
Also, as $y=\left(y_{n}\right) \in c s$, so $\lim _{n \rightarrow \infty} \sum_{n=0}^{k} y_{n}$ exists and hence, $\left(d_{k}\right) \in c$.
Hence, by Lemma 3.1 (ii), the sequence $\left(z_{k}\right)=\left(\sum_{n=0}^{k} x_{n}\right)$ is also convergent and so,
$x=\left(x_{n}\right) \in c s$. Therefore, $U(r, s)-\alpha I$ is onto. So, $\alpha \in I$ (See Table 2.2).

Hence the result.

Theorem 3.9. The approximate point spectrum of the operator $U(r, s)$ over $c s$ is given by

$$
\sigma_{a p}(U(r, s), c s)=\{\alpha \in \mathbf{C}:|\alpha-r| \leq|s|\} .
$$

Proof. From Table 2,

$$
\sigma_{a p}(U(r, s), c s)=\sigma(U(r, s), c s) \backslash I I I_{1} \sigma(U(r, s), c s) .
$$

Also $\sigma_{r}(U(r, s), c s)=I I I_{1} \sigma(U(r, s), c s) \cup I I I_{2} \sigma(U(r, s), c s)$ (See Table 2)

By Corollary 3.6, $\sigma_{r}(U(r, s), c s)=\phi$ and so, $I I I_{1} \sigma(U(r, s), c s)=\phi$.
Hence, from Theorem 3.6, $\sigma_{a p}(U(r, s), c s)=\{\alpha \in \mathbf{C}:|\alpha-r| \leq|s|\}$.
Theorem 3.10. The compression spectrum of the operator $U(r, s)$ over $c s$ is given by

$$
\sigma_{c o}(U(r, s), c s)=\phi .
$$

Proof. By Proposition 3.6(e), we get

$$
\sigma_{p}\left(U(r, s)^{*}, c s^{*}\right)=\sigma_{c o}(U(r, s), c s) .
$$

Using Theorem 3.4, we get the required result.
Theorem 3.11. The defect spectrum of the operator $U(r, s)$ over $c s$ is given by

$$
\sigma_{\delta}(U(r, s), c s)=\{\alpha \in \mathbf{C}:|\alpha-r|=|s|\} .
$$

Proof. From Table 2.2,we have

$$
\sigma_{\delta}(U(r, s), c s)=\sigma(U(r, s), c s) \backslash I_{3} \sigma(U(r, s), c s) .
$$

Using Theorem 3.7 and Theorem 3.8, we get the required result.
Corollary 3.12. The following statements hold:
(i) $\sigma_{p}\left(U(r, s)^{*}, c s^{*} \cong b v\right)=\{\alpha \in \mathbf{C}:|\alpha-r|=|s|\}$.
(ii) $\sigma_{\delta}\left(U(r, s)^{*}, c s^{*} \cong b v\right)=\{\alpha \in \mathbf{C}:|\alpha-r| \leq|s|\}$.

Proof. Using Proposition ??(c) and (d), we get

$$
\begin{aligned}
\sigma_{p}\left(U(r, s)^{*}, c s^{*} \cong b v\right) & =\sigma_{\delta}(U(r, s), c s) \\
\sigma_{\delta}\left(U(r, s)^{*}, c s^{*} \cong b v\right) & =\sigma_{a p}(U(r, s), c s)
\end{aligned}
$$

Using Theorem 3.9 and Theorem 3.11, we get the required results.

## 4. Conclusion

In this article we have determined the spectrum and fine spectrum of the matrix operator $U(r, s)$ over the sequence space $c s$. In a further development, we have introduced the concept of the approximate point spectrum, defect spectrum and compression spectrum of the operator $U(r, s)$ on the sequence space $U(r, s)$. This is a new development and the work can be applied to investigate the spectra and fine spectra of some other matrix operators too.

## References

[1] A. M. Akhmedov and S. R. El-Shabrawy, On the fine spectrum of the operator $\Delta_{a, b}$ over the sequence space $c$, Comput. Math. Appl., 61 (10), pp. 2994-3002, (2011).
[2] A. M. Akhmedov and F. Başar, The fine spectra of the Cesàro operator $C_{1}$ over the sequence space $b v_{p}(1 \leq p<\infty)$, Math. J. Okayama Univ. 50, pp. 135-147, (2008).
[3] B. Altay and F. Başar, On the fine spectrum of the difference operator $\Delta$ on $c_{0}$ and $c$, Inf. Sci., 168, pp. 217-224, (2004).
[4] B. Altay and F. Başar, On the fine spectrum of the generalized difference operator $B(r, s)$ over the sequence spaces $c_{0}$ and $c$, Int. J. Math. Math. Sci., 2005 : 18, pp. 3005-3013, (2005).
[5] B. Altay and M. Karakuş, On the spectrum and the fine spectrum of the Zweier matrix as an operator on some sequence spaces, Thai J. Math., 3 (2), pp. 153-162, (2005).
[6] M. Altun, On the fine spectra of triangular Toeplitz operators, Appl. Math. Comput., 217, pp. 8044-8051, (2011).
[7] M. Altun, Fine spectra of tridiagonal symmetric matrices, Int. J. Math. Math. Sci., Article ID 161209, (2011).
[8] J. Appell, E. Pascale, A. Vignoli, Nonlinear Spectral Theory, Walter de Gruyter, Berlin, New York, (2004).
[9] F. Başar, N. Durna, M. Yildirim, Subdivisions of the spectra for generalized difference operator over certain sequence spaces, Thai J. Math., 9 (2), pp. 279-289, (2011).
[10] F. Başar, Summability Theory and Its Applications, Bentham Science Publishers,e-books, Monographs, Istanbul, (2012).
[11] H. Bilgiç and H.Furkan, On the fine spectrum of operator $B(r, s, t)$ over the sequence spaces $\ell_{1}$ and $b v$, Math Comput. Model., 45, pp. 883-891, (2007).
[12] H. Bilgiç and H.Furkan, On the fine spectrum of the generalized difference operator $B(r, s)$ over the sequence spaces $\ell_{1}$ and $b v_{p}(1<p<\infty)$, Nonlinear Anal., 68, pp. 499-506, (2008).
[13] J. P. Cartlitdge, Weighted mean matrices as operators on $\ell^{p}$, Ph. D. dissertation, Indiana University, Indiana, (1978).
[14] H. Furkan, H. Bilgiç and K. Kayaduman, On the fine spectrum of the generalized difference operator $B(r, s)$ over the sequence spaces $\ell_{1}$ and $b v$, Hokkaido Math. J. 35, pp. 893-904, (2006).
[15] H. Furkan, H. Bilgiç and B. Altay, On the fine spectrum of operator $B(r, s, t)$ over $c_{0}$ and $c$, Comput. Math. Appl., 53, pp. 989-998, (2007).
[16] H. Furkan, H. Bilgiç and F. Başar, On the fine spectrum of operator $B(r, s, t)$ over the sequence spaces $\ell_{p}$ and $b v_{p},(1<p<\infty)$, Comput. Math. Appl., 60, pp. 2141-2152, (2010).
[17] S. Goldberg, Unbounded Linear Operators-Theory and Applications, Dover Publications, Inc, New York.
[18] M. Gonzalez, The fine spectrum of the Cesàro operator in $\ell^{p}$, Arch. Math., 44, pp. 355-358, (1985).
[19] A. Karaisa and F. Başar, Fine spectrum of upper triangular tripleband matrices over the sequence space $\ell_{p}(0<p<\infty$, Abst. Appl. Anal., Article ID 342682, (2013).
[20] V. Karakaya and M. Altun, Fine spectra of upper triangular doubleband matrices, J. Comp. Appl. Math., 234, pp. 1387-1394, (2010).
[21] K. Kayaduman and H. Furkan, The fine spectra of the difference operator $\Delta$ over the sequence spaces $\ell_{1}$ and $b v$,Int. Math. Forum, 1 (24), pp. 1153-1160, (2006).
[22] E. Kreysziag, Introductory Functional Analysis with Application, John Wiley and Sons, (1989).
[23] J. I. Okutoyi, On the spectrum of $C_{1}$ as an operator on $b v_{0}$, J. Austral. Math. Soc. (Series A) 48, pp. 79-86, (1990).
[24] B. L. Panigrahi and P. D. Srivastava, Spectrum and the fine spectrum of the generalised second order difference operator $\Delta_{u v}^{2}$ on sequence space $c_{0}$, Thai J. Math., Vol. 9, No. 1, pp. 57-74, (2011).
[25] B. L. Panigrahi and P. D. Srivastava, Spectrum and fine spectrum of the generalized second order forward difference operator $\Delta_{u v w}^{2}$ on the sequence space $\ell_{1}$, Demonstratio Mathematica, Vol. XLV, No. 3, (2012).
[26] D. Rath and B. C. Tripathy, On the Banach algebra of triangular conservative matrices of operators, J. Math. Anal. Appl.; 197(3), pp. 743-751, (1994).
[27] B. E. Rhoades, The fine spectra for weighted mean operators, Pac. J. Math., 104 (1), pp. 219-230, (1983).
[28] P. D. Srivastava and S. Kumar, Fine spectrum of the generalized difference operator $\Delta_{v}$ on the sequence space $\ell_{1}$, Thai J. Math., 8 (2), pp. 221-233, (2010).
[29] B. C. Tripathy and A. Paul, Spectra of the operator $B(f, g)$ on the vector valued sequence space $c_{0}(X)$, Bol. Soc. Parana. Mat., 31 (1), pp. 105-111, (2013).
[30] B. C. Tripathy and A. Paul, The Spectrum of the operator $D(r, 0,0, s)$ over the sequence space $c_{0}$ and $c$; Kyungpook Math. J., 53 (2), pp. 247256, (2013).
[31] B. C. Tripathy and P. Saikia, On the spectrum of the Cesàro operator $C_{1}$ on $\overline{b v} \cap \ell_{\infty}$, Math. Slovaca, 63 (3), pp. 563-572, (2013).
[32] A. Wilansky, Summability Through Functional Analysis, North Holland, (1984).

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|  | I | II | III |
| :---: | :---: | :---: | :---: |
| 1 | $\rho(T, X)$ |  | $\sigma_{r}(T, X)$ |
| 2 | $\sigma_{c}(T, X)$ | $\sigma_{c}(T, X)$ | $\sigma_{r}(T, X)$ |
| 3 | $\sigma_{p}(T, X)$ | $\sigma_{p}(T, X)$ | $\sigma_{p}(T, X)$ |

Table 2.1: Subdivisions of spectrum of a linear operator

|  |  | 1 | 2 | 3 |
| :---: | :---: | :---: | :---: | :---: |
|  |  | $T_{\lambda}^{-1}$ exists and is bounded | $T_{\lambda}^{-1}$ exists and is not bounded | $T_{\lambda}^{-1}$ does not exist |
| I | $R(T-\lambda I)=X$ | $\lambda \in \rho(T, X)$ |  | $\begin{gathered} \lambda \in \sigma_{p}(T, X) \\ \lambda \in \sigma_{a p}(T, X) \\ \hline \end{gathered}$ |
| II | $\overline{R(T-\lambda I)}=X$ | $\lambda \in \rho(T, X)$ | $\begin{gathered} \lambda \in \sigma_{c}(T, X) \\ \lambda \in \sigma_{a p}(T, X) \\ \lambda \in \sigma_{\delta}(T, X) \\ \hline \end{gathered}$ | $\begin{gathered} \lambda \in \sigma_{p}(T, X) \\ \lambda \in \sigma_{a p}(T, X) \\ \lambda \in \sigma_{\delta}(T, X) \\ \hline \end{gathered}$ |
| III | $\overline{R(T-\lambda I)} \neq X$ | $\begin{aligned} & \hline \lambda \in \sigma_{r}(T, X) \\ & \lambda \in \sigma_{\delta}(T, X) \\ & \lambda \in \sigma_{c o}(T, X) \end{aligned}$ | $\begin{gathered} \lambda \in \sigma_{r}(T, X) \\ \lambda \in \sigma_{a p}(T, X) \\ \lambda \in \sigma_{\delta}(T, X) \\ \lambda \in \sigma_{c o}(T, X) \end{gathered}$ | $\begin{gathered} \lambda \in \sigma_{p}(T, X) \\ \lambda \in \sigma_{a p}(T, X) \\ \lambda \in \sigma_{\delta}(T, X) \\ \lambda \in \sigma_{c o}(T, X) \\ \hline \end{gathered}$ |

Table 2.2: Subdivisions of spectrum of a linear operator

