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Functional equations of Cauchy's and d'Alembert's Type on Compact Groups

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Abstract

Using the non-abelian Fourier transform, we find the central continuous solutions of the functional equation

$$\sum_{k=0}^{n-1} f(x\sigma^k(y)) = nf(x)f(y) \qquad x, y \in G,$$

where G is an arbitrary compact group, $n \in \mathbf{N}\setminus\{0\}$ and σ is a continuous automorphism of G, such that $\sigma^n = I$. We express the solutions in terms of the unitary (group) characters of G.

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1. Introduction

Let G be a group, $n \in \mathbf{N} \setminus \{0\}$ and σ an automorphism of G, such that $\sigma^n = I$, where I denotes the identity map. We consider the functional equation

$$\sum_{k=0}^{n-1} f(x\sigma^k(y)) = nf(x)f(y), \qquad x, y \in G, \tag{1.1}$$

where $f : G \to \mathbf{C}$ is the function to determine. This equation has been solved on abelian groups. See Shin'ya [[13], Corollary 3.12] and Stetkær [[17], Theorem 14.9]. The functional equation (1.1) is a generalization of Cauchy's and d'Alembert's functional equations. In fact, Cauchy's functional equation

$$f(xy) = f(x)f(y), \qquad x, y \in G, \tag{1.2}$$

results from (1.1) by taking $\sigma = I$. In the particular case where $n = 2$, (1.1) reduces to functional equation

$$f(xy) + f(x\sigma(y)) = 2f(x)f(y), \qquad x, y \in G, \tag{1.3}$$

When G is abelian and $\sigma = -I$, the functional equation (1.3) becomes d'Alembert's functional equation

$$f(x + y) + f(x - y) = 2f(x)f(y), \quad x, y \in G. \tag{1.4}$$

This explains the choice of the title of this paper. At present the theory of d'Alembert's equation is extensively developed (cf. [2] -[11]; [14] -[20]).
Because of the importance of this types of equations, it is worthwhile to provide a solution for functional equation (1.1) in the case of f being a central function and the group G being compact and possibly non-Abelian. Our approach uses harmonic analysis and representation theory on compact groups. The idea of using Fourier analysis for solving it goes back to [5].

Throughout the rest of this paper, $\mathcal{B}(\mathcal{H})$ is the algebra of all bounded linear operators on Hilbert space $\mathcal{H}$, the group G is always assumed to be compact and σ is a continuous automorphism. By solutions (resp. representations), we always mean continuous solutions (resp. continuous representations).

2. Preliminaries

In this section, we set up some notation and conventions and briefly review some fundamental facts in Fourier analysis which will be used later.

Let G be a compact group with the normalized Haar measure dx. Let $\hat{G}$ stand for the set of equivalence classes of continuous irreducible unitary representations of G. It is known that for $[\pi] \in \hat{G}$, π is finite dimensional. We denote its dimension by d_π. Consider $\varepsilon_\pi = span\{\pi_{ij} : i, j = 1, ..., d_\pi\}$ the linear span of matrix-valued representative of $[\pi]$. For $f \in L^2(G)$, the Fourier transform of f is defined by

$$\hat{f}(\pi) = d_\pi \int_G f(x)\pi(x)^{-1}dx \in M_{d_\pi}(\mathbf{C}),$$

for all $[\pi] \in \hat{G}$, where $M_{d_\pi}(\mathbf{C})$ is the space of all $d_\pi \times d_\pi$ complex matrix.

As usual, left and right regular representations of G in $L^2(G)$ are defined by

$(\mathrm{L}_y f)(x) = f(y^{-1}x), \quad (R_y f)(x) = f(xy),$

respectively, for all $f \in L^2(G)$ and $x, y \in G$. A crucial property of the Fourier transform is that it converts the regular representations of G into matrix multiplications. The following properties are useful.

i The Fourier inversion formula is given by

$$f(x) = \sum_{[\pi]\in\hat{G}} tr(\hat{f}(\pi)\pi(x)).$$

ii The following identities hold:

$$\widehat{L_y f}(\pi) = \hat{f}(\pi)\pi(y)^{-1}, \quad \widehat{R_y f}(\pi) = \pi(y)\hat{f}(\pi),$$

for all $x, y \in G$.

A function $f : G \to \mathbf{C}$ is central if

$$f(xy) = f(yx),$$

for all $x, y \in G$.

For more information about harmonic analysis we refer to [12].

3. Main result

In this section, we shall solve (1.1) on compact groups by using harmonic analysis of such groups. That two representations π and ρ of G are equivalent is denoted $\pi \simeq \rho$.

Lemma 3.1. *Let G be a compact group, and π a unitary irreducible representation of G. If $f : G \to \mathbf{C}$ is a central function, then $\hat{f}(\pi)$ is a multiple of the identity operator.*

Proof.

$$\pi(y)\hat{f}(\pi) = d_\pi \int_G f(x)\pi(y)\pi(x^{-1})dx$$

$$= d_\pi \int_G f(x)\pi((xy^{-1})^{-1})dx$$

$$= d_\pi \int_G f(xy)\pi(x^{-1})dx$$

$$= d_\pi \int_G f(yx)\pi(x^{-1})dk$$

$$= d_\pi \hat{f}(\pi)\pi(y),$$

for every $y \in G$. By Schur's lemma, $\hat{f}(\pi)$ is a scalar multiple of the identity operator. $\square$

This following lemma is inspired by the Small Dimension Lemma from [20].

Lemma 3.2. *Let G be a compact group and π a unitary irreducible representation of G. Suppose for every $x \in G$ there is $c_x \in \mathbf{C}$ such that*

$$\sum_{k=0}^{n-1} \pi(\sigma^k(x)) = c_x I_{d_\pi}, x \in G. \tag{3.1}$$

Then $d_\pi = 1$.

Proof. Consider the set $S = \{k \in \{1, ..., n-1\} \mid \pi \simeq \pi \circ \sigma^k\}$. We will consider two cases, S is empty or not empty. In the first case, from (3.1) we get

$$\pi_{ij}(x) + \sum_{k=1}^{n-1} \pi_{ij}(\sigma^k(x)) = 0 \quad for\ i \neq j, 1 \leq i, j \leq d_\pi,\ x \in G. \tag{3.2}$$

Since $S = \emptyset$ we have $\varepsilon_\pi \perp \varepsilon_{\pi \circ \sigma^k}$ for all $k = 1, ..., n-1$. Hence $\pi_{ij} = 0$ for $i \neq j$, so π is a diagonal matrix. Since π is irreducible this implies that $d_\pi = 1$.

In the second case, if $S \neq \emptyset$, then

$$S = \{s_0, 2s_0, ..., Ns_0\} \quad and \quad n = (N+1)s_0, \tag{3.3}$$

where $s_0 = \min S$ and $N = cardS$. Indeed, let $k \in S$, there exists $(q, r) \in \mathbf{N} \times \mathbf{N}$ such that $k = qs_0 + r$ and $0 \leq r < s_0$. From $\pi \simeq \pi \circ \sigma^{s_0}$ we arrive at $\pi \circ \sigma^r \simeq \pi \circ \sigma^{qs_0 + r}$, so $\pi \circ \sigma^r \simeq \pi \circ \sigma^k$. This implies that $\pi \simeq \pi \circ \sigma^r$. Since $0 \leq r < s_0$ and $s_0 = \min S$, we have $r = 0$. Then S is contained in the set of integer multiples of s_0. An additional simple inductive argument is needed to show that S has the form $S = \{s_0, 2s_0, ..., Ns_0\}$. Furthermore, $\pi \simeq \pi \circ \sigma^{s_0}$ is equivalent to $\pi \simeq \pi \circ \sigma^{n-s_0}$. From $\pi \simeq \pi \circ \sigma^{n-s_0}$ we infer that $n - s_0 \in S$. Since $n - s_0 + s_0 = n \notin S$ we see that $n - s_0$ is the biggest element in $S = \{s_0, 2s_0, ..., Ns_0\}$, i.e., $n - s_0 = Ns_0$ and $n = (N+1)s_0$.

Let $(\mathcal{H}, \langle , \rangle)$ denote the complex Hilbert space on which the representation π acts. Since $\pi \simeq \pi \circ \sigma^{s_0}$, there exists a unitary operator $T \in \mathcal{B}(\mathcal{H})$ such that

$$\pi \circ \sigma^{s_0}(x) = T^* \pi(x) T,$$

for all $x \in G$, from which induction gives the more general formula so

$$\pi \circ \sigma^{ks_0}(x) = (T^k)^* \pi(x)) T^k,$$

for all $x \in G$ and any k = 0, 1, ... $\mathcal{H}$ has an orthonormal basis $(e_1, e_2, \ldots, e_{d_\pi})$ consisting of eigenvectors of T (by the spectral theorem for normal operators applied to T). We write $Te_i = \lambda_i e_i$, where $\lambda_i \in \mathbf{C}$ for $i = 1; 2, \ldots, d_\pi :$ Actually $|\lambda_i| = 1$, T being unitary.

For any $i = 1; 2, \ldots, d_\pi$, $k \in S$ and $x \in G$ we compute that

$$\begin{aligned} (\pi \circ \sigma^{ks_0})_{ii}(x) &= \left\langle (\pi \circ \sigma^{ks_0})(x) e_i, e_i \right\rangle = \left\langle (T^k)^* \pi(x)) T^k e_i, e_i \right\rangle \\ &= \left\langle T^k \pi(x) T^k e_i, T^k e_i \right\rangle = \left\langle \lambda_i^k \pi(x) e_i, \lambda_i^k e_i \right\rangle \\ &= \lambda_i^k \overline{\lambda_i^k} \left\langle \pi(x) e_i, e_i \right\rangle = |\lambda_i|^{2k} \pi_{ii}(x) = \pi_{ii}(x). \end{aligned}$$

It is easy to see from (3.1) that

$$\pi_{ii}(x)+\sum_{k=1}^{N}\pi_{ii}(\sigma^{ks_0}(x))+\sum_{k\in\overline{S}}\pi_{ii}(\sigma^k(x))=c_x, \tag{3.4}$$

for all $i=1,...,d_\pi$ and $x\in G$, where $\overline{S}$ is the complement of S in $\{1,...,n-1\}$. Since $\pi_{ii}\circ\sigma^{ks_0}=\pi_{ii}$ for all $k=1,...,N$. From (3.4) we obtain

$$(N+1)\pi_{ii}(x)+\sum_{k\in\overline{S}}\pi_{ii}(\sigma^k(x))=c_x,$$

for all $i=1,...,d_\pi,\ x\in G$. Then $d_\pi=1$. Indeed, if $d_\pi>1$, then for all $i=2,...,d_\pi$ we have

$$(N+1)\pi_{ii}+\sum_{k\in\overline{S}}\pi_{ii}\circ\sigma^k=(N+1)\pi_{11}+\sum_{k\in\overline{S}}\pi_{11}\circ\sigma^k,$$

so

$$(N+1)(\pi_{ii}-\pi_{11})=\sum_{k\in\overline{S}}(\pi_{11}-\pi_{ii})\circ\sigma^k. \tag{3.5}$$

Since π is not equivalent to $\pi\circ\sigma^k$ for any $k\in\overline{S}$, we have $\varepsilon_\pi\perp\varepsilon_{\pi\circ\sigma^k}$ for all $k\in\overline{S}$. (3.5) implies that $\pi_{ii}=\pi_{11}$ for all $i=2,...,d_\pi$. But if we use Schur's orthogonality relations which say $\frac{1}{d_\pi}\pi_{ij}$ is an orthonormal basis, we get a contradiction. Then $d_\pi=1$. $\square$

Theorem 3.3. *Let G be a compact group and σ is a continuous automorphism of G such that $\sigma^n=I$. If f is a central continuous non-zero solution of the functional equation (1.1), then there is a unitary character χ of G such that*

$$f=\frac{1}{n}\sum_{k=0}^{n-1}\chi\circ\sigma^k.$$

Proof. Suppose that f satisfies (1.1). Rewrite (1.1) as $\sum_{k=0}^{n-1}R_{\sigma^k(y)}f=nf(y)f$,
for all $y\in G$. Taking the Fourier transform to the above equation and using the identities given in section 2, we have

$$\sum_{k=0}^{n-1}\pi(\sigma^k(y))\hat{f}(\pi)=nf(y)\hat{f}(\pi) \tag{3.6}$$

Since $f \neq 0$, there exists $[\pi] \in \hat{G}$ with $\hat{f}(\pi) \neq 0$. But f is central so $\hat{f}(\pi) = \lambda_\pi I$, and so

$$\sum_{k=0}^{n-1} \pi(\sigma^k(y)) = nf(y)I, \tag{3.7}$$

for all $y \in G$. By applying Lemma 3.2, we conclude from (3.7) that $d_\pi = 1$. From $d_\pi = 1$ we see that π is a character, say $\pi = \chi$ where χ is a unitary character of G. Then (3.7) implies

$$f = \frac{1}{n} \sum_{k=0}^{n-1} \chi \circ \sigma^k.$$

□

Corollary 3.4. *Let G be a compact group and σ is a continuous automorphism of G such that $\sigma^n = I$. If f is a central continuous non-zero solution of the functional equation (1.3), then there is a unitary character χ of G such that*

$$f = \frac{\chi + \chi \circ \sigma}{2}.$$

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