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Ramanujan's fifth order and tenth order mock theta functions - a generalization

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Abstract

A generalization of Ramanujan's fifth order and tenth order mock theta functions is given. It is shown that these belong to the family of F_q-functions. Using the properties of F_q-functions, relationship is given between these generalized fifth order mock theta functions of the first group with the generalized functions of the second group. The same is done for the generalized functions of the tenth order. q-Integral representation and multibasic expansions are also given.

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1. Introduction

Early in 1920, January 12, 1920 to be exact, three months before his death, S. Ramanujan wrote his last letter to G.H. Hardy [7, pp.354-355]. He wrote "I discovered very interesting functions recently which I call 'Mock' θ-functions. Unlike the 'False' θ-functions (studied partially by Prof. Rogers in his paper [8]) they enter into mathematics as beautifully as the ordinary θ-functions. I am sending you with this letter some examples". He then gave a list of seventeen functions, four of order three, ten of order five and three of order seven together with the identities that they satisfy.

G.N. Watson was the first to write papers on the mock theta functions [9,10]. The first of these is Watson's Presidential Address to the London Mathematical Society in 1935. The title was "The Final Problem: An Account of the Mock Theta Functions". In these two papers Watson proved most of the identities found in the letter of Ramanujan. The first paper considers only the third order mock theta functions. It also gives three new third order mock theta functions.

Mock theta functions of order five and seven were difficult. The fifth order mock theta functions are in two groups and Ramanujan claimed that no relation exists between mock theta functions of the first group with mock theta functions of the second group. We have given a generalization of the fifth order mock theta functions. Our aim is to give relations between the generalized functions of the first group with the generalized functions of the second group. A generalization of the Ramanujan's tenth order mock theta functions and relations between these generalized functions is also given. In section 3, we show that these generalized functions belong to the family of F_q-functions.

By using the difference operator we develop relations between the generalized functions of the two groups, this is done in section 4.

In section 5 we represent these functions as q-Integrals and in section 6 multibasic expansion is given for these generalized functions. In section 7 we show that the generalized tenth order mock theta functions are F_q-functions and in section 8 relations among these generalized functions are given. In section 9 we represent these generalized tenth order mock theta function as q-Integrals and in section 10 we give multibasic expansion for these generalized functions.

Definition of F_q-functions

The functions which satisfy the functional equation

$$D_{q,z}F(z,\alpha) = F(z,\alpha+1), \tag{1.1}$$

where

$$zD_{q,z}F(z,\alpha) = F(z,\alpha) - F(zq,\alpha) \tag{1.2}$$

are called F_q-functions.

Throughout the paper we use the following usual basic hypergeometric notations :

For $\left|q^k\right| < 1$,
$(a;q^k)_n = (1-a)(1-aq^k).......(1-aq^{k(n-1)}), n \geq 1$
$(a;q^k)_0 = 1,$
$(a;q^k)_\infty = \prod_{j=0}^{\infty}(1-aq^{kj}).$

For convenience we shall write

$$(a_1, a_2, ..., a_m; q^k)_n = (a_1;q^k)_n(a_2;q^k)_n............(a_m;q^k)_n.$$

When k=1, we usually write $(a)_n$ and $(a)_\infty$ insteat of $(a;q)_n$ and $(a;q)_\infty$, respectively.

2. Definition of fifth order and tenth order mock theta functions and their generalization

Ramanujan's ten fifth order mock theta functions are as follows :

$f_0(q) := \sum_{n=0}^{\infty} \frac{q^{n^2}}{(-q)_n}$, $\qquad \varphi_0(q) := \sum_{n=0}^{\infty} q^{n^2}(-q;q^2)_n,$

$\psi_0(q) := \sum_{n=0}^{\infty} q^{(n+1)(n+2)/2}(-q)_n$, $\qquad F_0(q) := \sum_{n=0}^{\infty} \frac{q^{2n^2}}{(q;q^2)_n},$

$\chi_0(q) := \sum_{n=0}^{\infty} \frac{q^n}{(q^{n+1})_n},$

and

$f_1(q) := \sum_{n=0}^{\infty} \frac{q^{n^2+n}}{(-q)_n}$, $\qquad \varphi_1(q) : = \sum_{n=0}^{\infty} q^{(n+1)^2}(-q;q^2)_n,$

$\psi_1(q) := \sum_{n=0}^{\infty} q^{\frac{n^2+n}{2}}(-q)_n,$ $\qquad F_1(q) := \sum_{n=0}^{\infty} \frac{q^{2n^2+2n}}{(q;q^2)_{n+1}},$

$\chi_1(q) := \sum_{n=0}^{\infty} \frac{q^n}{(q^{n+1})_{n+1}}.$

Ramanujan's four tenth order mock theta functions are as follows :

$\Phi(q) := \sum_{n=0}^{\infty} \dfrac{q^{n(n+1)/2}}{(q;q^2)_{n+1}}$, $\Psi(q) := \sum_{n=0}^{\infty} \dfrac{q^{(n+1)(n+2)/2}}{(q;q^2)_{n+1}}$,

$X(q) := \sum_{n=0}^{\infty} \dfrac{(-1)^n q^{n^2}}{(-q;q)_{2n}}$, $\chi(q) := \sum_{n=0}^{\infty} \dfrac{(-1)^n q^{(n+1)^2}}{(-q;q)_{2n+1}}$.

We define generalization of Ramanujan's ten fifth order mock theta functions as follows :

$$f_0(t,z,\alpha;q) := \frac{1}{(t)_\infty}\sum_{n=0}^{\infty}\frac{(t)_n q^{n^2-3n+n\alpha} z^{2n}}{(-z;q)_n}, \tag{2.1}$$

$$\varphi_0(t,z,\alpha;q) := \frac{1}{(t)_\infty}\sum_{n=0}^{\infty}\frac{(t)_n q^{n^2+n+n\alpha}}{z^{2n}}\left(-q^3/z^2;q^2\right)_n, \tag{2.2}$$

$$\psi_0(t,z,\alpha;q) := \frac{1}{(t)_\infty}\sum_{n=0}^{\infty}\frac{(t)_n q^{\frac{n(n+1)}{2}+n\alpha}}{z^n}\left(-q^2/z;q\right)_{n-1}, \tag{2.3}$$

$$F_0(t,z,\alpha;q) := \frac{1}{(t)_\infty}\sum_{n=0}^{\infty}\frac{(t)_n q^{2n^2-5n+n\alpha} z^{4n}}{(z^2/q;q^2)_n}, \tag{2.4}$$

$$\chi_0(t,z,\alpha;q) := \frac{1}{(t)_\infty}\sum_{n=0}^{\infty}\frac{(t)_n q^{n\alpha}(z;q)_n}{(z^2q^{-1};q)_{2n}}, \tag{2.5}$$

$$f_1(t,z,\alpha;q) := \frac{1}{(t)_\infty}\sum_{n=0}^{\infty}\frac{(t)_n q^{n^2-2n+n\alpha} z^{2n}}{(-z;q)_n}, \tag{2.6}$$

$$\varphi_1(t,z,\alpha;q) := \frac{q^5}{z^4}\frac{1}{(t)_\infty}\sum_{n=0}^{\infty}\frac{(t)_n q^{n^2+3n+n\alpha}}{z^{2n}}(-q^3/z^2;q^2)_n, \tag{2.7}$$

$$\psi_1(t,z,\alpha;q) := \frac{q}{z}\frac{1}{(t)_\infty}\sum_{n=0}^{\infty}\frac{(t)_n q^{n(n+1)/2+n\alpha}}{z^n}(-q^2/z;q)_n, \tag{2.8}$$

$$F_1(t,z,\alpha;q) := \frac{1}{(t)_\infty}\sum_{n=0}^{\infty}\frac{(t)_n q^{2n^2-3n+n\alpha} z^{4n}}{(z^2/q;q^2)_{n+1}}, \tag{2.9}$$

$$\chi_1(t,z,\alpha;q) := \frac{1}{(t)_\infty}\sum_{n=0}^{\infty}\frac{(t)_n q^{n\alpha}(z;q)_n}{(z^2q^{-1};q)_{2n+1}}. \tag{2.10}$$

Setting $t = 0$, $\alpha = 1$ and $z = q$ these functions reduce to fifth order mock theta functions of Ramanujan.

We define a generalization of four tenth order mock theta functions of Ramanujan :

$$\Phi(t,z,\alpha;q) := \frac{1}{(t)_\infty}\sum_{n=0}^{\infty}\frac{(t)_n q^{n(n-3)/2+n\alpha}z^n}{(z^2/q;q^2)_{n+1}}, \tag{2.11}$$

$$\Psi(t,z,\alpha;q) := \frac{z}{(t)_\infty}\sum_{n=0}^{\infty}\frac{(t)_n q^{n(n-1)/2+n\alpha}z^n}{(z^2/q;q^2)_{n+1}}, \tag{2.12}$$

$$X(t,z,\alpha;q) := \frac{1}{(t)_\infty}\sum_{n=0}^{\infty}\frac{(-1)^n(t)_n q^{n^2-3n+n\alpha}z^{2n}}{(-z^2/q;q)_{2n}}, \tag{2.13}$$

$$\chi(t,z,\alpha;q) := \frac{1}{(t)_\infty}\sum_{n=0}^{\infty}\frac{(-1)^n(t)_n q^{n^2-n+1+n\alpha}z^{2n}}{(-z^2/q;q)_{2n+1}}. \tag{2.14}$$

Setting $t=0, \alpha=1$ and $z=q$, these functions redeuce to Ramanujan's tenth order mock theta functions.

3. Generalized fifth order mock theta functions are F_q-functions

We show that our generalized functions satisfy the F_q-equation (1.1) and so are F_q-functions.

Theorem 1

The generalized ten fifth order mock theta functions $f_0(t,z,\alpha;q)$, $\varphi_0(t,z,\alpha;q)$, $\psi_0(t,z,\alpha;q)$, $F_0(t,z,\alpha;q)$, $\chi_0(t,z,\alpha;q)$, $f_1(t,z,\alpha;q)$, $\varphi_1(t,z,\alpha;q)$, $\psi_1(t,z,\alpha;q)$, $F_1(t,z,\alpha;q)$, $\chi_1(t,z,\alpha;q)$ are F_q -functions.

We give a detailed proof for the function $f_0(t,z,\alpha;q)$ only, the proof for the other functions are similar, so omitted.

Proof

Applying the difference oprator $D_{q,t}$ to $f_0(t,z,\alpha;q)$, we have

$$tD_{q,t}f_0(t,z,\alpha;q) = f_0(t,z,\alpha;q) - f_0(tq,z,\alpha;q)$$

$$= \frac{1}{(t)_\infty}\sum_{n=0}^{\infty}\frac{(t)_n q^{n^2-3n+n\alpha}z^{2n}}{(-z;q)_n} - \frac{1}{(tq)_\infty}\sum_{n=0}^{\infty}\frac{(tq)_n q^{n^2-3n+n\alpha}z^{2n}}{(-z;q)_n}$$

$$= \frac{1}{(t)_\infty}\sum_{n=0}^{\infty}\frac{(t)_n q^{n^2-3n+n\alpha} z^{2n}}{(-z;q)_n} - \frac{1}{(t)_\infty}\sum_{n=0}^{\infty}\frac{(t)_n q^{n^2-3n+n\alpha} z^{2n}(1-tq^n)}{(-z;q)_n}$$

$$= \frac{t}{(t)_\infty}\sum_{n=0}^{\infty}\frac{(t)_n q^{n^2-3n+n(\alpha+1)} z^{2n}}{(-z;q)_n}.$$

So

$$D_{q,t} f_0(t,z,\alpha;q) = f_0(t,z,\alpha+1;q).$$

Hence $f_0(t,z,\alpha;q)$ is a F_q-function.
Similarly all other functions in Theorem 1 are F_q-functions.

4. Relations between the generalized fifth order mock theta functions of the first group with the generalized fifth order mock theta function of the second group

In proving the relationship we shall use the property that they are F_q-functions, as proved in section 3.

Theorem 2
(*i*) $D_{q,t} f_0(t,z,\alpha;q) = f_1(t,z,\alpha;q)$,
(*ii*) $D^2_{q,t}\varphi_0(t,z,\alpha;q) = \frac{z^4}{q^5}\varphi_1(t,z,\alpha;q)$,
(*iii*) $\chi_0(t,z,\alpha;q) = \chi_1(t,z,\alpha;q) - \frac{z^2}{q} D^2_{q,t}\chi_1(t,z,\alpha;q)$,
(*iv*) $\psi_1(t,z,\alpha;q) = \frac{q}{z}\psi_0(t,z,\alpha;q) + \frac{q^2}{z^2} D_{q,t}\psi_0(t,z,\alpha;q)$,
(*v*) $F_0(t,z,\alpha;q) = \frac{1}{(t)_\infty}\sum_{n=0}^{\infty}\dfrac{(t)_n q^{2n^2-5n+n\alpha} z^{4n}}{(z^2/q;q^2)_{n+1}} - \frac{z^2}{q}F_1(t,z,\alpha;q)$.

Proof of (i)

$$D_{q,t} f_0(t,z,\alpha;q) = f_0(t,z,\alpha+1;q)$$

$$= \frac{1}{(t)_\infty}\sum_{n=0}^{\infty}\frac{(t)_n q^{n^2-2n+n\alpha} z^{2n}}{(-z;q)_n}$$

$$= f_1(t,z,\alpha;q),$$

which proves Theorem 2 (i)

Proof of (ii)

$$D^2_{q,t}\varphi_0(t,z,\alpha;q) = \varphi_0(t,z,\alpha+2;q)$$

$$= \frac{1}{(t)_\infty}\sum_{n=0}^{\infty}\frac{(t)_n q^{n^2+3n+n\alpha}}{z^{2n}}(-q^3/z^2;q^2)_n$$

$$= \frac{z^4}{q^5}\varphi_1(t,z,\alpha;q),$$

which proves Theorem 2 (ii).

Proof of (iii)

$$\chi_0(t,z,\alpha;q) = \frac{1}{(t)_\infty}\sum_{n=0}^{\infty}\frac{(t)_n q^{n\alpha}(z;q)_n(1-z^2q^{2n-1})}{(z^2q^{-1};q)_{2n+1}}$$

$$= \frac{1}{(t)_\infty}\sum_{n=0}^{\infty}\frac{(t)_n q^{n\alpha}(z;q)_n}{(z^2q^{-1};q)_{2n+1}} - \frac{z^2}{q}\frac{1}{(t)_\infty}\sum_{n=0}^{\infty}\frac{(t)_n q^{n\alpha+2n}(z;q)_n}{(z^2q^{-1};q)_{2n+1}}$$
$$= \chi_1(t,z,\alpha;q) - \tfrac{z^2}{q}D^2_{q,t}\chi_1(t,z,\alpha;q),$$

which proves Theorem 2 (iii).

Proof of (iv)

$$\psi_1(t,z,\alpha;q) = \frac{q}{z}\frac{1}{(t)_\infty}\sum_{n=0}^{\infty}\frac{(t)_n q^{n(n+1)/2+n\alpha}(-q^2/z;q)_n}{z^n}$$

$$= \frac{q}{z}\frac{1}{(t)_\infty}\sum_{n=0}^{\infty}\frac{(t)_n q^{n(n+1)/2+n\alpha}(-q^2/z;q)_{n-1}(1+q^{n+1}/z)}{z^n}$$

$$= \frac{q}{z}\,\frac{1}{(t)_\infty}\sum_{n=0}^{\infty}\frac{(t)_n q^{n(n+1)/2+n\alpha}(-q^2/z;q)_{n-1}}{z^n} + \frac{q^2}{z^2}\frac{1}{(t)_\infty}$$

$$\times\sum_{n=0}^{\infty}\frac{(t)_n q^{n(n+1)/2+n\alpha+n}(-q^2/z;q)_{n-1}}{z^n}$$

$$= \frac{q}{z}\psi_0(t,z,\alpha;q) + \frac{q^2}{z^2}D_{q,t}\psi_0(t,z,\alpha;q),$$

which proves Theorem 2 (iv).

Proof of (v)

$$F_0(t,z,\alpha;q)=\frac{1}{(t)_\infty}\sum_{n=0}^{\infty}\frac{(t)_n q^{2n^2-5n+n\alpha}z^{4n}}{(z^2/q;q^2)_n}$$

$$=\frac{1}{(t)_\infty}\sum_{n=0}^{\infty}\frac{(t)_n q^{2n^2-5n+n\alpha}z^{4n}(1-z^2q^{2n-1})}{(z^2/q;q^2)_{n+1}}$$

$$=\frac{1}{(t)_\infty}\sum_{n=0}^{\infty}\frac{(t)_n q^{2n^2-5n+n\alpha}z^{4n}}{(z^2/q;q^2)_{n+1}}-\frac{z^2}{q}\frac{1}{(t)_\infty}\sum_{n=0}^{\infty}\frac{(t)_n q^{2n^2-3n+n\alpha}z^{4n}}{(z^2/q;q^2)_{n+1}}$$

$$=\frac{1}{(t)_\infty}\sum_{n=0}^{\infty}\frac{(t)_n q^{2n^2-5n+n\alpha}z^{4n}}{(z^2/q;q^2)_{n+1}}-\frac{z^2}{q}F_1(t,z,\alpha;q),$$

which proves Theorem 2 (v).

5. q-Integral representation for the generalized fifth order mock theta functions

The q-integral was defined by Thomae and Jackon [5, p.19] as $\int_0^1 f(t)d_qt=(1-q)\sum_{n=0}^{\infty}f(q^n)q^n$.

Limiting case of q-beta integral [5, p.19 (1.11.7)] is

$$\frac{1}{(q^x;q)_\infty}=\frac{(1-q)^{-1}}{(q;q)_\infty}\int_0^1 t^{x-1}(tq;q)d_qt. \tag{5.1}$$

We use extensively (5.1) to give q-integral representation for the generalized functions defined in section 2.

Theorem 3

the following equalities hold :

(i) $f_0(q^t,z,\alpha;q)=\dfrac{(1-q)^{-1}}{(q;q)_\infty}\int_0^1 w^{t-1}(wq;q)_\infty f_0(0,z,aw;q)d_qw$

(ii) $\varphi_0(q^t,z,\alpha;q)=\dfrac{(1-q)^{-1}}{(q;q)_\infty}\int_0^1 w^{t-1}(wq;q)_\infty \varphi_0(0,z,aw;q)d_qw$

(iii) $\psi_0(q^t,z,\alpha;q)=\dfrac{(1-q)^{-1}}{(q;q)_\infty}\int_0^1 w^{t-1}(wq;q)_\infty \psi_0(0,z,aw;q)d_qw$

(iv) $F_0(q^t,z,\alpha;q) = \dfrac{(1-q)^{-1}}{(q;q)_\infty} \int_0^1 w^{t-1}(wq;q)_\infty F_0(0,z,aw;q)d_qw$

(v) $\chi_0(q^t,z,\alpha;q) = \dfrac{(1-q)^{-1}}{(q;q)_\infty} \int_0^1 w^{t-1}(wq;q)_\infty \chi_0(0,z,aw;q)d_qw$

(vi) $f_1(q^t,z,\alpha;q) = \dfrac{(1-q)^{-1}}{(q;q)_\infty} \int_0^1 w^{t-1}(wq;q)_\infty f_1(0,z,aw;q)d_qw$

(vii) $\varphi_1(q^t,z,\alpha;q) = \dfrac{(1-q)^{-1}}{(q;q)_\infty} \int_0^1 w^{t-1}(wq;q)_\infty \varphi_1(0,z,aw;q)d_qw$

(viii) $\psi_1(q^t,z,\alpha;q) = \dfrac{(1-q)^{-1}}{(q;q)_\infty} \int_0^1 w^{t-1}(wq;q)_\infty \psi_1(0,z,aw;q)d_qw$

(ix) $F_1(q^t,z,\alpha;q) = \dfrac{(1-q)^{-1}}{(q;q)_\infty} \int_0^1 w^{t-1}(wq;q)_\infty F_1(0,z,aw;q)d_qw$

(x) $\chi_1(q^t,z,\alpha;q) = \dfrac{(1-q)^{-1}}{(q;q)_\infty} \int_0^1 w^{t-1}(wq;q)_\infty \chi_1(0,z,aw;q)d_qw.$

A detailed proof for $f_0(q^t,z,\alpha;q)$ is given. The proofs of the other functions are similar, so omitted.

Proof

By definition

$$f_0(t,z,\alpha;q) = \frac{1}{(t)_\infty}\sum_{n=0}^{\infty}\frac{(t)_n q^{n^2-3n+n\alpha}z^{2n}}{(-z;q)_n}$$

Replacing t by q^t and q^α by a, we have

$$\begin{aligned} f_0(q^t,z,\alpha;q) &= \frac{1}{(q^t)_\infty}\sum_{n=0}^{\infty}\frac{(q^t)_n q^{n^2-3n+n\alpha}z^{2n}}{(-z;q)_n} \\ &= \sum_{n=0}^{\infty}\frac{q^{n^2-3n+n\alpha}z^{2n}}{(-z;q)_n(q^{n+t})_\infty} \\ &= \sum_{n=0}^{\infty}\frac{q^{n^2-3n+n\alpha}z^{2n}}{(-z;q)_n}\frac{(1-q)^{-1}}{(q;q)_\infty}\quad \int_0^1 w^{n+t-1}(wq;q)_\infty d_qw, \end{aligned}$$

by (5.1)

$$= \frac{(1-q)^{-1}}{(q;q)_\infty}\int_0^1 w^{t-1}(wq;q)_\infty d_qw \sum_{n=0}^{\infty}\frac{q^{n^2-3n}z^{2n}}{(-z;q)_n}(aw)^n d_qw. \tag{5.2}$$

But

$$f_0(0,z,\alpha;q)=\sum_{n=0}^{\infty}\frac{q^{n^2-3n+n\alpha}z^{2n}}{(-z;q)_n},$$

and since $q^\alpha = a$,

$f_0(0,z,a;q)=\sum_{n=0}^{\infty}\dfrac{(a)^n q^{n^2-3n}z^{2n}}{(-z;q)_n}.$

Hence

$$f_0(0,z,aw;q)=\sum_{n=0}^{\infty}\frac{q^{n^2-3n}z^{2n}(aw)^n}{(-z;q)_n}. \tag{5.3}$$

By (5.3), (5.2) can be written as

$$f_0(q^t,z,\alpha;q)=\frac{(1-q)^{-1}}{(q;q)_\infty}\int_0^1 w^{t-1}(wq;q)_\infty f_0(0,z,aw;q)d_qw,$$

which proves Theorem 3(i). The proof of all the other fuctions is similar, so omitted.

6. Multibasic expansion of generalized fifth order mock theta functions

Using the summation formula [5, (3.6.7), p.71] and [6, Lemma 10, p.57], we have the multibasic expansion

$$\sum_{k=0}^{\infty}\frac{(1-ap^kq^k)(1-bp^kq^{-k})(a,b;p)_k(c,a/bc;q)_kq^k}{(1-a)(1-b)(q,aq/b;q)_k(ap/c,bcp;p)_k}\sum_{m=0}^{\infty}\alpha_{m+k}$$

$$=\sum_{m=0}^{\infty}\frac{(ap,bp;p)_m(cq,aq/bc;q)_m}{(ap/c,bcp;p)_m(q,aq/b;q)_m}\alpha_m. \tag{6.1}$$

Corollary 1

Letting $q\to q^2$ and $c\to\infty$ in (6.1), we have

$$\sum_{k=0}^{\infty}\frac{(1-ap^kq^{2k})(1-bp^kq^{-2k})(a,b;p)_kq^{k^2+k}}{(1-a)(1-b)(q^2,aq^2/b;q^2)_kb^kp^{\frac{k^2+k}{2}}}\sum_{m=0}^{\infty}\alpha_{m+k}$$

$$(6.2)\qquad = \sum_{m=0}^{\infty} \frac{(ap, bp; p)_m q^{m^2+m}}{(q^2, aq^2/b; q^2)_m b^m p^{\frac{m^2+m}{2}}} \alpha_m.$$

Corollary 2
Letting $q \to q^3$and $c \to \infty$ in (6.1), we have

$$\sum_{k=0}^{\infty} \frac{(1-ap^k q^{3k})(1-bp^k q^{-3k})(a,b;p)_k q^{\frac{3k^2+3k}{2}}}{(1-a)(1-b)(q^3, aq^3/b; q^3)_k b^k p^{\frac{k^2+k}{2}}} \sum_{m=0}^{\infty} \alpha_{m+k}$$

$$(6.3)\qquad = \sum_{m=0}^{\infty} \frac{(ap, bp; p)_m q^{\frac{3m^2+3m}{2}}}{(q^3, aq^3/b; q^3)_m b^m p^{\frac{m^2+m}{2}}} \alpha_m.$$

Corollary 3
Letting $q \to q^5$and $c \to \infty$ in (6.1), we have

$$\sum_{k=0}^{\infty} \frac{(1-ap^k q^{5k})(1-bp^k q^{-5k})(a,b;p)_k q^{\frac{5k^2+5k}{2}}}{(1-a)(1-b)(q^5, aq^5/b; q^5)_k b^k p^{\frac{k^2+k}{2}}} \sum_{m=0}^{\infty} \alpha_{m+k}$$

$$(6.4)\qquad = \sum_{m=0}^{\infty} \frac{(ap, bp; p)_m q^{\frac{5m^2+5m}{2}}}{(q^5, aq^5/b; q^5)_m b^m p^{\frac{m^2+m}{2}}} \alpha_m.$$

We use following standard notation for multibasic hypergeometric series

$$\phi[^{a_1,\ldots\ldots,a_r:c_{1,1},\ldots\ldots\ldots,c_{1,r_1}:\ldots\ldots:c_{m,1},\ldots\ldots\ldots,c_{m,r_m}}_{b_1,\ldots\ldots,b_s:e_{1,1},\ldots\ldots\ldots,e_{1,s_1}:\ldots\ldots:e_{m,1},\ldots\ldots\ldots,e_{m,s_m}}; q, q_1,\ldots\ldots,q_m; z]$$

$$=\sum_{n=0}^{\infty} \frac{(a_1,\ldots\ldots,a_r;q)_n}{(q,b_1,\ldots\ldots,b_s;q)_n} z^n [(-1)^n q^{\frac{n^2-n}{2}}]^{1+s-r} \prod_{j=1}^{m} \frac{(c_{j,1},\ldots\ldots,c_{j,r_j};q_j)_n}{(e_{j,1},\ldots\ldots,e_{j,s_j};q_j)_n} [(-1)^n q_j^{\frac{n^2-n}{2}}]^{s_j-r_j}.$$

Theorem 4
The generalized functions have the following expressions as the multi-basic hypergeometric series :

(i) $\mathrm{f}_0(t,z,\alpha;q) = \dfrac{1}{(t)_\infty} \sum_{k=0}^{\infty} \dfrac{(1-tq^{4k-1})(1-q^{-2k+2})(t;q)_{k-1} q^{k^2-3k+k\alpha} z^{2k}}{(1-q^{k+2})(z;q)_k}$

$$\times\phi\left[\begin{smallmatrix} q,0,0:tq^{3k},q^{3k+3} \\ q^{k+3},-zq^{k}:0,0 \end{smallmatrix};q,q^3;z^2q^{\alpha-2}\right]$$

(ii)

$$\varphi_0(t,z,\alpha;q)=\frac{1}{(t)_\infty}\sum_{k=0}^{\infty}\frac{(1-tq^{4k-1})(1-q^{-2k+1})(t;q)_{k-1}(-q^3/z^2;q^2)_k q^{k^2+k+k\alpha}}{(1-q^{k+1})z^{2k}}$$

$$\times\phi\left[\begin{smallmatrix} q,0:-q^{2k+3}/z^2:tq^{3k+1},q^{3k+3} \\ q^{k+2}:0:0,0 \end{smallmatrix};q,q^2,q^3;z^{-2}q^{\alpha+1}\right]$$

(iii)

$$\psi_0(t,z,\alpha;q)=\frac{1}{(t)_\infty}\sum_{k=0}^{\infty}\frac{(1-tq^{3k-1})(1-q^{-k+1})(t;q)_{k-1}(-q^2/z;q)_{k-1}q^{\frac{k(k+1)}{2}+k\alpha}}{(1-q^{k+1})z^{k}}$$

$$\times\phi\left[\begin{smallmatrix} q,-q^{k+1}/z:tq^{2k},q^{2k+2} \\ q^{k+2}:0,0 \end{smallmatrix};q,q^2;z^{-1}q^{\alpha+1}\right]$$

(iv)

$$F_0(t,z,\alpha;q)=\frac{1}{(t)_\infty}\sum_{k=0}^{\infty}\frac{(1-tq^{6k-1})(1-q^{-4k+4})(t;q)_{k-1}q^{2k^2-5k+k\alpha}z^{4k}}{(1-q^{k+4})(z^2/q;q^2)_k}$$

$$\times\phi\left[\begin{smallmatrix} q,0:0:tq^{5k},q^{5k+5} \\ q^{k+5}:z^2q^{2k-1}:0,0 \end{smallmatrix};q,q^2,q^5;z^4q^{\alpha-3}\right]$$

(v)

$$f_1(t,z,\alpha;q)=\frac{1}{(t)_\infty}\sum_{k=0}^{\infty}\frac{(1-tq^{4k-1})(1-q^{-2k+1})(t;q)_{k-1}q^{k^2-2k+k\alpha}z^{2k}}{(1-q^{k+1})(-z;q)_k}$$

$$\times\phi\left[\begin{smallmatrix} q,0,0:tq^{3k+1},q^{3k+3} \\ q^{k+2},-zq^{k}:0,0 \end{smallmatrix};q,q^3;z^2q^{\alpha-2}\right]$$

(vi) $\varphi_1(t,z,\alpha;q)$

$$=\frac{z^2}{q^2}\frac{1}{(t)_\infty}\sum_{k=0}^{\infty}\frac{(1-tq^{4k-1})(1-q^{-2k+1})(t;q)_{k-1}(-q^3/z^2;q^2)_{k-1}q^{k^2+k+k\alpha}}{(1-q^{k+1})z^{2k}}$$

$$\times\phi\left[\begin{smallmatrix} q,0:-q^{2k+1}/z^2:tq^{3k+1},q^{3k+3} \\ q^{k+2}:0:0,0 \end{smallmatrix};q,q^2,q^3;z^{-2}q^{\alpha+1}\right]$$

(vii)

$$\psi_1(t,z,\alpha;q) = \frac{q}{z}\frac{1}{(t)_\infty}\sum_{k=0}^{\infty}\frac{(1-tq^{3k-1})(1-q^{-k+1})(t;q)_{k-1}(-q^2/z;q)_k q^{\frac{k(k+1)}{2}+k\alpha}}{(1-q^{k+1})(z;q)_k}$$

$$\times\phi\left[{}^{q,-q^{k+2}/z:tq^{2k},q^{2k+2}}_{q^{k+2}:0,0};q,q^2;z^{-1}q^{\alpha+1}\right]$$

(viii)

$$F_1(t,z,\alpha;q) = \frac{1}{(t)_\infty}\sum_{k=0}^{\infty}\frac{(1-tq^{6k-1})(1-q^{-4k+4})(t;q)_{k-1}q^{2k^2-3k+k\alpha}z^{4k}}{(1-q^{k+4})(z^2/q;q^2)_{k+1}}$$

$$\times\phi\left[{}^{q,0:0:tq^{5k},q^{k+5}}_{q^{k+5}:z^2q^{2k+1}:0,0};q,q^2,q^5;z^4q^{\alpha-1}\right].$$

We shall give the proof of (i) only, for others we will state the value of the parameters.

Proof of (i)

Taking $a=t/q, b=q^2, p=q$ and $\alpha_m = \dfrac{q^{m\alpha-2m}(q^3;q^3)_m(t;q^3)_m z^{2m}}{(q^3;q)_m(-z;q)_m}$ in (6.3),
we have

$$\sum_{k=0}^{\infty}\frac{(1-tq^{4k-1})(1-q^{-2k+2})(t/q,q^2;q)_k q^{\frac{3k^2+3k}{2}}}{(1-t/q)(1-q^2)(t,q^3;q^3)_k q^{\frac{k^2+5k}{2}}}$$
$$\times\sum_{m=0}^{\infty}\frac{(q^3;q^3)_{m+k}(t;q^3)_{m+k}z^{2(m+k)}q^{(m+k)\alpha-2(m+k)}}{(q^3;q)_{m+k}(-z;q)_{m+k}}$$
$$= \sum_{m=0}^{\infty}\frac{q^{m^2-3m+m\alpha}(t;q)_m z^{2m}}{(-z;q)_m} \tag{6.5}$$

The right hand side of (6.5) is equal to

$$(t,q)_\infty f_0(t,z,\alpha;q).$$

The left hand side of (6.5) is equal to

$$\sum_{k=0}^{\infty}\frac{(1-tq^{4k-1})(1-q^{-2k+2})(t;q)_{k-1}(q^2;q)_k(q^3;q^3)_k(t;q^3)_k q^{k^2-3k+k\alpha}z^{2k}}{(1-q^2)(q^3,t;q^3)_k(q^3;q)_k(-z;q)_k}$$

$$\times \sum_{m=0}^{\infty} \frac{q^{m\alpha-2m}(q^{3k+3};q^3)_m(tq^{3k};q^3)_m z^{2m}}{(q^{k+3};q)_m(-zq^k;q)_m}$$

$$= \sum_{k=0}^{\infty} \frac{(1-tq^{4k-1})(1-q^{-2k+2})(t;q)_{k-1}q^{k^2-3k+k\alpha}z^{2k}}{(1-q^{k+2})(-z;q)_k}$$
$$\times \left[\begin{smallmatrix} q,0,0:tq^{3k},q^{3k+3} \\ q^{k+3},-zq^k:0,0 \end{smallmatrix}; q, q^3; z^2q^{\alpha-2}\right],$$

which proves Theorem 4 (i).

Proof of (ii)

Take $a = t/q, b = q, p = q$ and
$\alpha_m = \dfrac{q^{m\alpha+m}(q^3;q^3)_m(tq;q^3)_m(-q^2/z^2;q^2)_m}{(q^2;q)_m z^{2m}}$ in (6.3).

Proof of (iii)

Take $a = t/q, b = q, p = q$ and $\alpha_m = \dfrac{q^{m\alpha+m}(q^2;q^2)_m(t;q^2)_m(-q^2/z;q)_{m-1}}{(q^2;q)_m z^m}$ in (6.2).

Proof of (iv)

Take $a = t/q, b = q^4, p = q$ and $\alpha_m = \dfrac{q^{m\alpha-3m}(q^5;q^5)_m(t;q^5)_m z^{4m}}{(q^5;q)_m(z^2/q;q^2)_m}$ in (6.4).

Proof of (v)

Take $a = t/q, b = q, p = q$ and $\alpha_m = \dfrac{q^{m\alpha-2m}(q^3;q^3)_m(tq;q^3)_m z^{2m}}{(q^2;q)_m(-z;q)_m}$ in (6.3).

Proof of (vi)

Take $a = t/q, b = q, p = q$ and
$\alpha_m = \dfrac{q^{m\alpha+m}(q^3;q^3)_m(tq;q^3)_m(-q^3/z^2;q^2)_{m-1}}{(q^2;q)_m z^{2m}}$ in (6.3).

Proof of (vii)

Take $a = t/q, b = q, p = q$ and $\alpha_m = \dfrac{q^{m\alpha+m}(q^2;q^2)_m(t;q^2)_m(-q^2/z;q)_m}{(q^2;q)_m z^m}$ in (6.2).

Proof of (viii)

Take $a = t/q, b = q^4, p = q$ and $\alpha_m = \dfrac{q^{m\alpha-m}(q^5;q^5)_m(t;q^5)_m z^{4m}}{(q^5;q)_m(z^2/q;q^2)_{m+1}}$ in (6.4).

7. Generalized tenth order mock theta functions are F_q-functions

We show that the four generalized tenth order mock theta functions are F_q-functions.

Theorem 5
The generalized four tenth order mock theta functions
$\Phi(t,z,\alpha;q), \Psi(t,z,\alpha;q), X(t,z,\alpha;q)$, and $\chi(t,z,\alpha;q)$ are F_q-functions.

We shall give a detailed proof for the function $\Phi(t,z,\alpha;q)$, the proof for the other functions are similar, so omitted.

Proof
Apply the difference oprator $D_{q,t}$ to $\Phi(t,z,\alpha;q)$, we have
$tD_{q,t}\ \Phi(t,z,\alpha;q) = \Phi(t,z,\alpha;q) - \Phi(tq,z,\alpha;q)$
$= \dfrac{1}{(t)_\infty}\sum_{n=0}^{\infty}\dfrac{(t)_n q^{n(n-3)/2+n\alpha}z^n}{(z^2/q;q^2)_{n+1}} - \dfrac{1}{(tq)_\infty}\sum_{n=0}^{\infty}\dfrac{(tq)_n q^{n(n-3)/2+n\alpha}z^n}{(z^2/q;q^2)_{n+1}}$
$= \dfrac{1}{(t)_\infty}\sum_{n=0}^{\infty}\dfrac{(t)_n q^{n(n-3)/2+n\alpha}z^n}{(z^2/q;q^2)_{n+1}} - \dfrac{1}{(t)_\infty}\sum_{n=0}^{\infty}\dfrac{(t)_n q^{n(n-3)/2+n\alpha+n}z^n(1-tq^n)}{(z^2/q;q^2)_{n+1}}$
$= \dfrac{t}{(t)_\infty}\sum_{n=0}^{\infty}\dfrac{(t)_n q^{n(n-3)/2+n(\alpha+1)}z^n}{(z^2/q;q)_{n+1}}.$

So
$$D_{q,t}\Phi(t,z,\alpha;q), = \Phi(t,z,\alpha+1;q).$$

Hence $\Phi(t,z,\alpha;q)$ is a F_q-function. Similarly the other three functions are also F_q-functions.

8. Relations between generalized mock theta function of tenth order

For proving the relationship between these generalized functions we shall use the property that they are F_q-functions.

Theorem 6

(i) $D_{q,t}\ \Phi(t,z,\alpha;q) = \frac{1}{z}\Psi(t,z,\alpha;q)$.

(ii) $X(t,z,\alpha;q) = \dfrac{1}{(t)_\infty}\sum_{n=0}^{\infty}\dfrac{(-1)^n(t)_n q^{n^2-3n+n\alpha}z^{2n}}{(-z^2/q;q)_{2n+1}} + \dfrac{z^2}{q^2}\chi(t,z,\alpha;q)$.

Proof of (i)

$$\begin{aligned}
D_{q,t}\Phi(t,z,\alpha;q) &= \Phi(t,z,\alpha+1;q)\\
&= \frac{1}{(t)_\infty}\sum_{n=0}^{\infty}\frac{(t)_n q^{n(n-3)/2+n\alpha+n}z^n}{(z^2/q;q^2)_{n+1}}\\
&= \frac{1}{(t)_\infty}\sum_{n=0}^{\infty}\frac{(t)_n q^{n(n-1)/2+n\alpha}z^n}{(z^2/q;q^2)_{n+1}}\\
&= \tfrac{1}{z}\Psi(t,z,\alpha;q),
\end{aligned}$$

which proves Theorem 6(i).

Proof of (ii)

$$\begin{aligned}
X(t,z,\alpha;q) &= \frac{1}{(t)_\infty}\sum_{n=0}^{\infty}\frac{(-1)^n(t)_n q^{n^2-3n+n\alpha}z^{2n}(1+z^2q^{2n-1})}{(-z^2/q;q)_{2n+1}}\\
&= \frac{1}{(t)_\infty}\sum_{n=0}^{\infty}\frac{(-1)^n(t)_n q^{n^2-3n+n\alpha}z^{2n}}{(-z^2/q;q)_{2n+1}} + \tfrac{z^2}{q}\frac{1}{(t)_\infty}\\
&\quad\times\sum_{n=0}^{\infty}\frac{(-1)^n(t)_n q^{n^2-n+n\alpha}z^{2n}}{(-z^2/q;q)_{2n+1}}\\
&= \frac{1}{(t)_\infty}\sum_{n=0}^{\infty}\frac{(-1)^n(t)_n q^{n^2-3n+n\alpha}z^{2n}}{(-z^2/q;q)_{2n+1}} + \tfrac{z^2}{q^2}\chi(t,z,\alpha;q),
\end{aligned}$$

which proves Theorem 6(ii).

9. q-Integral representation for the generalized tenth order mock theta functions

We give q-integral representation for the four generalized tenth order mock theta functions.

Theorem 7

The following equalities hold :

(i) $\Phi(q^t,z,\alpha;q) = \frac{(1-q)^{-1}}{(q;q)_\infty}\int_0^1 w^{t-1}(wq;q)_\infty\Phi(0,z,aw;q)d_qw$

(ii) $\Psi(q^t,z,\alpha;q) = \frac{(1-q)^{-1}}{(q;q)_\infty}\int_0^1 w^{t-1}(wq;q)_\infty\Psi(0,z,aw;q)d_qw$

(iii) $X(q^t,z,\alpha;q) = \frac{(1-q)^{-1}}{(q;q)_\infty}\int_0^1 w^{t-1}(wq;q)_\infty X(0,z,aw;q)d_qw$

(iv) $\chi(q^t,z,\alpha;q)=\frac{(1-q)^{-1}}{(q;q)_\infty}\int_0^1 w^{t-1}(wq;q)_\infty\chi(0,z,aw;q)d_qw.$

We give a detailed proof for $\Phi(q^t,z,\alpha;q)$ only and omit the proof for the other functions, as they are similar.

Proof

By definition

$$\Phi(t,z,\alpha;q)=\frac{1}{(t)_\infty}\sum_{n=0}^{\infty}\frac{(t)_n q^{n(n-3)/2+n\alpha}z^n}{(z^2/q;q^2)_{n+1}}$$

Replacing t by q^t and q^α by a, we have

$$\begin{aligned}\Phi(q^t,z,\alpha;q)&=\frac{1}{(q^t)_\infty}\sum_{n=0}^{\infty}\frac{(q^t)_n q^{n(n-3)/2+n\alpha}z^n}{(z^2/q;q^2)_{n+1}}\\&=\sum_{n=0}^{\infty}\frac{q^{n(n-3)/2+n\alpha}z^n}{(z^2/q;q^2)_{n+1}(q^{n+t})_\infty}\\&=\sum_{n=0}^{\infty}\frac{q^{n(n-3)/2+n\alpha}z^n}{(z^2/q;q^2)_{n+1}}\frac{(1-q)^{-1}}{(q;q)_\infty}\int_0^1 w^{n+t-1}(wq;q)_\infty d_qw\end{aligned}$$

by (5.1)

$$=\frac{(1-q)^{-1}}{(q;q)_\infty}\int_0^1 w^{t-1}(wq;q)_\infty\sum_{n=0}^{\infty}\frac{q^{n(n-3)/2+n\alpha}z^n}{(z^2/q;q^2)_{n+1}}(aw)^n d_qw. \tag{9.1}$$

But

$$\Phi(0,z,\alpha;q)=\sum_{n=0}^{\infty}\frac{q^{n(n-3)/2+n\alpha}z^n}{(z^2/q;q^2)_{n+1}},$$

and since $q^\alpha=a$,

$$\Phi(0,z,\alpha;q)=\sum_{n=0}^{\infty}\frac{(a)^n q^{n(n-3)/2}z^n}{(z^2/q;q^2)_{n+1}}$$

Hence

$$\Phi(0,z,aw;q)=\sum_{n=0}^{\infty}\frac{q^{n(n-3)/2}(aw)^n}{(z^2/q;q^2)_{n+1}}. \tag{9.2}$$

By (9.2), (9.1) can be written as

$$\Phi(q^t,z,\alpha;q)=\frac{(1-q)^{-1}}{(q;q)_\infty}\int_0^1 w^{t-1}(wq;q)_\infty\Phi(0,z,aw;q)d_qw,$$

which proves Theorem 7 (i). The proof for the other three functions being similar, so omitted.

10. Multibasic expansions for generalized tenth order mock theta functions

We now give multibasic expansions for our generalized functions.

Theorem 8

The generalized functions have the following expressions as a multibasic hypergeometric series:

$(i)\quad \Phi(t,z,\alpha;q) =$

$$\frac{(1-z^2/q)^{-1}}{(t;q)_\infty}\sum_{k=0}^{\infty}\frac{(1-tq^{3k-1})(1-q^{-k+2})(t;q)_{k-1}q^{\frac{k(k-3)}{2}+k\alpha}z^k}{(1-q^{k+2})(z^2q;q^2)_k} \times\phi\left[{q,0:tq^{2k-1},q^{2k+2} \atop q^{k+3}:z^2q^{2k+1},0};q,q^2;zq^\alpha\right]$$

$(ii)\quad \Psi(t,z,\alpha;q) =$

$$\frac{z(1-z^2/q)^{-1}}{(t;q)_\infty}\sum_{k=0}^{\infty}\frac{(1-tq^{3k-1})(1-q^{-k+1})(t;q)_{k-1}q^{\frac{k(k-1)}{2}+k\alpha}z^k}{(1-q^{k+1})(z^2q;q^2)_k} \times\phi\left[{q,0:tq^{2k},q^{2k+2} \atop q^{k+2}:z^2q^{2k+1},0};q,q^2;zq^\alpha\right]$$

$$(iii)\quad X(t,z,\alpha;q) = \frac{1}{(t;q)_\infty}\sum_{k=0}^{\infty}\frac{(-1)^k(1-q^{-2k+2})(t;q)_kq^{k^2-3k+k\alpha}z^{2k}}{(1-q^{k+2})(-z^2/q;q)_{2k}} \times\phi\left[{q,tq^k:0,0:q^{3k+3} \atop q^{k+3}:-z^2q^{2k-1},-z^2q^{2k}:0};q,q^2,q^3;-z^2q^{\alpha-2}\right]$$

$$(iv)\quad \chi(t,z,\alpha;q) = \frac{q}{(t;q)_\infty}\sum_{k=0}^{\infty}\frac{(-1)^k(1-q^{1-2k})(t;q)_kq^{k^2-k+k\alpha}z^{2k}}{(1-q^{2k+1})(-z^2/q;q)_{2k+1}} \times\phi\left[{q,tq^k:0,0:q^{3k+3} \atop q^{k+2}:-z^2q^{2k},-z^2q^{2k+1}:0};q,q^2,q^3;-z^2q^{\alpha-1}\right].$$

Proof of (i)

Take $a=t/q, b=q^2, p=q$ and $\alpha_m = \dfrac{q^{m\alpha}(q^2;q^2)_m(t/q;q^2)_m z^m}{(q^3;q)_m(z^2q;q^2)_m}$ in (6.2).

Proof of (ii)

Take $a = t/q, b = q, p = q$ and $\alpha_m = \dfrac{q^{m\alpha}(q^2;q^2)_m(t;q^2)_m z^m}{(q^2;q)_m(z^2q;q^2)_m}$ in (6.2).

Proof of (iii)

Let $a \to 0, b = q^2, p = q$ and $\alpha_m = \dfrac{(-q^{\alpha-2})^m(q^3;q^3)_m(t;q)_m z^{2m}}{(q^3;q)_m(-z^2/q;q^2)_{2m}}$ in (6.3).

Proof of (iv)

Let $a \to 0, b = q, p = q$ and $\alpha_m = \dfrac{(-q^{\alpha-1})^m(q^3;q^3)_m(t;q)_m z^{2m}}{(q^2;q)_m(-z^2/q;q)_{2m+1}}$ in (6.3).

Conclusion

Seventh order mock theta functions were more diffcult, however, we generalized them and found interesting properities. The paper has already been communicated for publication.

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