Proyecciones Journal of Mathematics
Vol. 34, N^o 3, pp. 229-241, September 2015.
Universidad Católica del Norte
Antofagasta - Chile
DOI: 10.4067/S0716-09172015000300003

Stability in delay Volterra difference equations of neutral type

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Received : March 2015. Accepted : June 2015

Abstract

Sufficient conditions for the zero solution of a certain class of neutral Volterra difference equations with variable delays to be asymptotically stable are obtained. The Banach's fixed point theorem is employed in proving our results.

2000 Mathematics Subject Classification: *349A30, 39A70.*

Keywords : *Banach's Fixed point theorem, Volterra difference equation, asymptotic stability.*

1. Introduction

The study of the stability of the zero solution of difference equations has gained the attention of many mathematicians lately, see [1], [2], [3], [5], [7], [9], [11] and [12]. In this paper we consider the nonlinear difference equation with variable delays

$$\begin{aligned}\Delta x(n) &= -\sum_{j=1}^{N} a_j(n)x(n-\tau_j(n)) + \sum_{j=1}^{N} \Delta Q_j(n, x(n-\tau_j(n))) \\ &\quad + \sum_{j=1}^{N} \sum_{s=n-\tau_j(n)}^{n-1} k_j(n,s) f_j(s, x(s)), \end{aligned} \tag{1.1}$$

with the initial condition

$$x(n) = \psi(n) \text{ for } n \in [m(n_0), n_0] \cap \mathbf{Z},$$

where $\psi : [m(n_0), n_0] \cap \mathbf{Z} \to \mathbf{R}$ is a bounded sequence and for $n_0 \geq 0$,

$$m_j(n_0) = \inf\{n - \tau_j(n),\ n \geq n_0\},\ m(n_0) = \min\{m_j(n_0),\ 1 \leq j \leq N\}.$$

Here Δ denotes the forward difference operator. That is, $\Delta x(n) = x(n+1) - x(n)$ for any sequence $\{x(n) : n \in \mathbf{Z}^+\}$. We assume throughout this paper that $a_j : \mathbf{Z}^+ \to \mathbf{R}$, $k_j : \mathbf{Z}^+ \times ([m_j(n_0), \infty) \cap \mathbf{Z}) \to \mathbf{R}$, $f_j : \mathbf{Z}^+ \times \mathbf{R} \to \mathbf{R}$, $Q_j : \mathbf{Z}^+ \times \mathbf{R} \to \mathbf{R}$ and $\tau_j : \mathbf{Z}^+ \to \mathbf{Z}^+$, for $j = 1, ..., N$. Special cases of (1.1) have been considered by a number of researchers in recent times.

For instance, Raffoul in [7] considered the equation

$$\Delta x(n) = -a(n)x(n-\tau), \tag{1.2}$$

where τ is a positive constant. The first author in [11], extended the results obtained in [7] to the equation

$$\Delta x(n) = -\sum_{j=1}^{N} a_j(n)x(n-\tau_j(n)). \tag{1.3}$$

The first author also in [12] considered the the following nonlinear Volterra difference equation

$$x(n+1) \quad = \quad a(n)x(n) + c(n)\Delta x(n-\tau(n)) + \sum_{s=n-\tau(n)}^{n-1} k(n,s)q(x(s)). \tag{1.4}$$

Ardjouni and Djoudi in [1] considered the nonlinear Volterra difference equation with variable delays

$$x(n+1) \quad = \quad a(n)x(n-\tau_1(n)) + c(n)\Delta x(n-\tau_2(n)) + \sum_{s=n-\tau_2(n)}^{n-1} k(n,s)q(x(s)). \tag{1.5}$$

Moreover, Ardjouni and Djoudi in [2] considered the difference equations with variable delays

$$\Delta x(n) \quad = \quad -\sum_{j=1}^{N} a_j(n)x(n-\tau_j(n)) + \sum_{j=1}^{N} c_j(n)\Delta x(n-\tau_j(n)). \tag{1.6}$$

Motivated by the above mentioned papers, we obtain in this paper sufficient conditions for the zero solution of (1.1) to be asymptotically stable.

2. Stability

Let $n_0 \in \mathbf{Z} \cap [0, \infty)$, be fixed. We let $D(n_0)$ be the set of bounded sequences $\psi : [m(n_0), n_0] \cap \mathbf{Z} \to \mathbf{R}$, with the norm $|\psi|_0 = \max\{|\psi(n)| : n \in [m(n_0), n_0] \cap \mathbf{Z}\}$. Also, let $(\mathbf{B}, ||.||)$ be the Banach space of bounded sequences $x : [m(n_0), \infty) \cap \mathbf{Z} \to \mathbf{R}$ with the maximum norm $||.||$.
In this paper we assume that for j=1,...,N,

$$|Q_j(n,x) - Q_j(n,y)| \leq L_1||x-y||, \tag{2.1}$$

and

$$|f_j(n,x) - f_j(n,y)| \leq L_2||x-y|| \tag{2.2}$$

for some positive constants L_1 and L_2. Also, for j=1,...,N,

$$f_j(n,0)=0, \quad Q_j(n,0)=0, \tag{2.3}$$

and

$$n-\tau_j(n)\to\infty \text{ as } n\to\infty. \tag{2.4}$$

Lemma 2.1 Let $h_j : [m(n_0),\infty)\cap\mathbf{Z}\to\mathbf{R}$ be an arbitrary sequence, for $j = 1,...,N$. Suppose that $H(n)=1-\sum_{j=1}^{N}h_j(n)\neq 0$, for all $n\in[n_o,\infty)\cap\mathbf{Z}$. Then x is a solution of equation (1.1) if and only if

$$\begin{aligned} x(n) \;=\; & \Big[x(n_0)-\sum_{j=1}^{N}Q_j(n_0,x(n_0-\tau_j(n_0)))-\sum_{j=1}^{N}\sum_{s=n_0-\tau_j(n_0)}^{n_0-1}h_j(s)x(s)\Big] \\ & \prod_{u=n_0}^{n-1}H(u) \\ & +\sum_{j=1}^{N}Q_j(n,x(n-\tau_j(n)))+\sum_{j=1}^{N}\sum_{s=n-\tau_j(n)}^{n-1}h_j(s)x(s) \\ & +\sum_{s=n_0}^{n-1}\Big[\sum_{j=1}^{N}\{h_j(n-\tau_j(n))-a_j(n)\}x(n-\tau_j(n)) \\ & -[1-H(s)]\sum_{j=1}^{N}Q_j(s,x(s-\tau_j(s)))-[1-H(s)] \\ & \sum_{j=1}^{N}\sum_{r=s-\tau_j(s)}^{s-1}h_j(r)x(r) \\ & +\sum_{j=1}^{N}\sum_{u=s-\tau_j(s)}^{s-1}k_j(s,u)f_j(u,x(u))\Big]\prod_{u=s+1}^{n-1}H(u). \end{aligned} \tag{2.5}$$

Proof. Rewrite (1.1) as

$$\Delta x(n) \;=\; -\sum_{j=1}^{N}h_j(n)x(n)+\Delta_n\sum_{j=1}^{N}\sum_{s=n-\tau_j(n)}^{n-1}h_j(s)x(s)$$

$$+\sum_{j=1}^{N}\{h_j(n-\tau_j(n))-a_j(n)\}x(n-\tau_j(n))$$
$$+\sum_{j=1}^{N}\Delta Q_j(n,x(n-\tau_j(n)))$$
$$+\sum_{j=1}^{N}\sum_{s=n-\tau_j(n)}^{n-1}k_j(n,s)f_j(s,x(s)),$$

where Δ_n denotes the difference taken with respect to n.

The above equation is equivalent to

$$\begin{aligned} x(n+1) &= H(n)x(n)+\Delta_n\sum_{j=1}^{N}\sum_{s=n-\tau_j(n)}^{n-1}h_j(s)x(s) \\ &\quad +\sum_{j=1}^{N}\{h_j(n-\tau_j(n))-a_j(n)\}x(n-\tau_j(n)) \\ &\quad +\sum_{j=1}^{N}\Delta Q_j(n,x(n-\tau_j(n))) \\ &\quad +\sum_{j=1}^{N}\sum_{s=n-\tau_j(n)}^{n-1}k_j(n,s)f_j(s,x(s)). \end{aligned} \tag{2.6}$$

Rewrite equation (2.6) as

$$\begin{aligned} \Delta_n\Big[\prod_{u=n_0}^{n-1}H(u)^{-1}x(u)\Big] &= \Big[\Delta_n\sum_{j=1}^{N}\sum_{s=n-\tau_j(n)}^{n-1}h_j(s)x(s) \\ &\quad +\sum_{j=1}^{N}\{h_j(n-\tau_j(n))-a_j(n)\}x(n-\tau_j(n)) \\ &\quad +\sum_{j=1}^{N}\Delta Q_j(n,x(n-\tau_j(n))) \end{aligned} \tag{2.7}$$

$$+\sum_{j=1}^{N}\sum_{s=n-\tau_j(n)}^{n-1}k_j(n,s)f_j(s,x(s))\Big]\prod_{u=n_0}^{n}H(u)^{-1}.$$

Summing (2.7) from n_0 to $n-1$ we obtain

$$\begin{aligned}\sum_{s=n_0}^{n-1}\Delta_s\Big[\prod_{u=n_0}^{s-1}H(u)^{-1}x(s)\Big] \;=\; & \sum_{s=n_0}^{n-1}\Big[\Delta_s\sum_{j=1}^{N}\sum_{r=s-\tau_j(s)}^{s-1}h_j(r)x(r)\\ &+\sum_{j=1}^{N}\{h_j(n-\tau_j(n))-a_j(n)\}x(n-\tau_j(n))\\ &+\sum_{j=1}^{N}\Delta Q_j(n,x(n-\tau_j(n)))\\ &+\sum_{j=1}^{N}\sum_{u=s-\tau_j(s)}^{s-1}k_j(s,u)f_j(u,x(u))\Big]\prod_{u=n_0}^{s}H(u)^{-1}.\end{aligned}$$

Consequently, we have

$$\begin{aligned}&\prod_{u=n_0}^{n-1}H(u)^{-1}x(n)-\prod_{u=n_0}^{n_0-1}H(u)^{-1}x(n_0)\\ &=\sum_{s=n_0}^{n-1}\Big[\Delta_s\sum_{j=1}^{N}\sum_{r=s-\tau_j(s)}^{s-1}h_j(r)x(r)\\ &+\sum_{j=1}^{N}\{h_j(n-\tau_j(n))-a_j(n)\}x(n-\tau_j(n))\\ &+\sum_{j=1}^{N}\Delta Q_j(n,x(n-\tau_j(n)))\\ &+\sum_{j=1}^{N}\sum_{u=s-\tau_j(s)}^{s-1}k_j(s,u)f_j(u,x(u))\Big]\prod_{u=n_0}^{s}H(u)^{-1}.\end{aligned}\tag{2.8}$$

Dividing both sides of (2.8) by $\prod_{u=n_0}^{n-1}H(u)^{-1}$ we obtain

$$\begin{aligned}x(n) \;=\; & x(n_0)\prod_{u=n_0}^{n-1}H(u)+\sum_{s=n_0}^{n-1}\Big[\Delta_s\sum_{j=1}^{N}\sum_{r=s-\tau_j(s)}^{s-1}h_j(r)x(r)\\ &+\sum_{j=1}^{N}\{h_j(n-\tau_j(n))-a_j(n)\}x(n-\tau_j(n))\\ &+\sum_{j=1}^{N}\Delta Q_j(n,x(n-\tau_j(n)))\end{aligned}$$

$$+\sum_{j=1}^{N}\sum_{u=s-\tau_j(s)}^{s-1} k_j(s,u) f_j(u,x(u))\Big]\prod_{u=s+1}^{n-1} H(u). \tag{2.9}$$

Using the summation by parts formula, we obtain

$$\begin{aligned}
&\sum_{s=n_0}^{n-1}\prod_{u=s+1}^{n-1} H(u)\Big[\sum_{j=1}^{N}\Delta Q_j(n, x(n-\tau_j(n)))\Big]\\
&=\sum_{j=1}^{N} Q_j(n, x(n-\tau_j(n))) - \sum_{j=1}^{N} Q_j(n_0, x(n_0-\tau_j(n_0)))\prod_{u=n_0}^{n-1} H(u)\\
&-\sum_{s=n_0}^{n-1}\sum_{j=1}^{N} Q_j(s, x(s-\tau_j(s)))[1-H(s)]\prod_{u=s+1}^{n-1} H(u),
\end{aligned} \tag{2.10}$$

and

$$\begin{aligned}
&\sum_{s=n_0}^{n-1}\prod_{u=s+1}^{n-1} H(u)\Big[\Delta_s \sum_{j=1}^{N}\sum_{r=s-\tau_j(s)}^{s-1} h_j(r)x(r)\Big]\\
&=\sum_{j=1}^{N}\sum_{s=n-\tau_j(n)}^{n-1} h_j(s)x(s) - \prod_{u=n_0}^{n-1} H(u)\sum_{j=1}^{N}\sum_{s=n_0-\tau_j(n_0)}^{n_0-1} h_j(s)x(s)\\
&-\sum_{s=n_0}^{n-1}[1-H(s)]\prod_{u=s+1}^{n-1} H(u)\sum_{j=1}^{N}\sum_{r=s-\tau_j(s)}^{s-1} h_j(r)x(r).
\end{aligned} \tag{2.11}$$

Substituting (2.10) and (2.11) into (2.9) gives the desired results.

Theorem 2.1 Suppose (2.1), (2.2), (2.3) and (2.4) hold and let $h_j : [m(n_0),\infty)\cap \mathbf{Z}\to \mathbf{R}$ be an arbitrary sequence, for $j=1,...,N$, such that $H(n) = 1-\sum_{j=1}^{N} h_j(n)\neq 0$, for all $n\in[n_o,\infty)\cap\mathbf{Z}$. Suppose further that there exist a constant $\alpha\in(0,1)$ such that

$$\begin{aligned}
&NL_1 + \sum_{j=1}^{N}\sum_{s=n-\tau_j(n)}^{n-1} |h_j(s)|\\
&+\sum_{s=n_0}^{n-1}\Big[\sum_{j=1}^{N}|h_j(n-\tau_j(n)) - a_j(n)|
\end{aligned}$$

$$+|1-H(s)|NL_1+|1-H(s)|\sum_{j=1}^{N}\sum_{r=s-\tau_j(s)}^{s-1}|h_j(r)|$$

$$+L_2\sum_{j=1}^{N}\sum_{u=s-\tau_j(s)}^{s-1}|k_j(s,u)|\Big]\Big|\Big|\prod_{u=s+1}^{n-1}H(u)\Big|\leq\alpha. \tag{2.12}$$

Moreover, assume that there exist a positive constant G such that

$$\Big|\prod_{u=n_0}^{n-1}H(u)\Big|\leq G, \tag{2.13}$$

and

$$\prod_{u=n_0}^{n-1}H(u)\to 0 \text{ as } n\to\infty. \tag{2.14}$$

Then the zero solution of (1.1) is asymptotically stable.

Proof. Let $\epsilon>0$ be given. Choose $\delta>0$ such that

$$\delta G[1+\alpha]+\epsilon\alpha\leq\epsilon.$$

Let $\psi\in D(n_0)$ such that $|\psi(n)|\leq\delta$ and define

$$S=\{\varphi\in B:\ \varphi(n)=\psi(n)\ if\ n\in[m(n_0),n_0]\cap\mathbf{Z},$$
$$||\varphi||\leq\epsilon \text{ and } \varphi(n)\to 0 \text{ as } n\to\infty\}.$$

Then $(S,||.||)$ is a complete metric space where, $||.||$ is the maximum norm.

Define the mapping $P:S\to S$ by

$$(P\varphi)(n)=\psi(n) \text{ for } n\in[m(n_0),n_0]\cap\mathbf{Z},$$

and

$$(P\varphi)(n)=\Big[\psi(n_0)-\sum_{j=1}^{N}Q_j(n_0,\psi(n_0-\tau_j(n_0)))-\sum_{j=1}^{N}\sum_{s=n_0-\tau_j(n_0)}^{n_0-1}h_j(s)\psi(s)\Big]$$
$$\prod_{u=n_0}^{n-1}H(u)$$

$$
\begin{aligned}
&+\sum_{j=1}^{N} Q_j(n,\varphi(n-\tau_j(n))) + \sum_{j=1}^{N}\sum_{s=n-\tau_j(n)}^{n-1} h_j(s)\varphi(s) \\
&+\sum_{s=n_0}^{n-1}\Big[\sum_{j=1}^{N}\{h_j(n-\tau_j(n))-a_j(n)\}\varphi(n-\tau_j(n)) \\
&-[1-H(s)]\sum_{j=1}^{N} Q_j(s,\varphi(s-\tau_j(s))) - [1-H(s)]\sum_{j=1}^{N}\sum_{r=s-\tau_j(s)}^{s-1} h_j(r)\varphi(r) \\
&+\sum_{j=1}^{N}\sum_{u=s-\tau_j(s)}^{s-1} k_j(s,u)f_j(u,\varphi(u))\Big]\prod_{u=s+1}^{n-1} H(u),\ n\ge n_0.
\end{aligned}
$$

(2.15)

Clearly, $P\varphi$ is continuous. We first show that $P: S\to S$. Using (2.15) we obtain

$$
\begin{aligned}
|(P\varphi)(n)| &\le \delta G[1+\alpha] + \Big\{NL_1 + \sum_{j=1}^{N}\sum_{s=n-\tau_j(n)}^{n-1}|h_j(s)| \\
&\quad + \sum_{s=n_0}^{n-1}\Big[\sum_{j=1}^{N}|h_j(n-\tau_j(n))-a_j(n)| \\
&\quad + |1-H(s)|NL_1 + |1-H(s)|\sum_{j=1}^{N}\sum_{r=s-\tau_j(s)}^{s-1}|h_j(r)| \\
&\quad + L_2\sum_{j=1}^{N}\sum_{u=s-\tau_j(s)}^{s-1}|k_j(s,u)|\Big]\Big|\prod_{u=s+1}^{n-1}H(u)\Big|\Big\}||\varphi|| \\
&\le \delta G[1+\alpha]+\alpha\epsilon \\
&\le \epsilon.
\end{aligned}
$$

We next show that $(P\varphi)(n)\to 0$ as $n\to\infty$. The first term on the right hand side of (2.15) goes to zero in view of condition (2.14). Since $\varphi(n)\to 0$ and $n-\tau_j(n)\to\infty$ as $n\to\infty$, we have that $Q_j(n,\varphi(n-\tau_j(n)))\to Q_j(n,0)=0$ as $n\to\infty$ for $j=1,...,N$. Thus showing that the second term on the right hand side of (2.15) goes to zero as $n\to\infty$.

Let $\varphi \in S$ be fixed. The fact that $\varphi(n) \to 0$ and $n - \tau_j(n) \to \infty$ as $n \to \infty$, implies that, given $\epsilon_1 > 0$ there exists $N_1 > n - \tau_j(n)$ for $j = 1, ..., N$ such that $|\varphi(s)| \leq \epsilon_1$ for $s \geq N_1$. Thus

$$\begin{aligned}\Big|\sum_{j=1}^{N}\sum_{s=n-\tau_j(n)}^{n-1} h_j(s)\varphi(s)\Big| &\leq \epsilon_1 \sum_{j=1}^{N}\sum_{s=n-\tau_j(n)}^{n-1} |h_j(s)| \\ &\leq \alpha\epsilon_1 < \epsilon_1.\end{aligned}$$

Thus showing that the third term on the right hand side of (2.15) goes to zero as $n \to \infty$. We next show that the last term on the right hand side of (2.15) goes to zero as $n \to \infty$. Since $\varphi(n) \to 0$ and $n - \tau_j(n) \to \infty$ as $n \to \infty$, for each $\epsilon_2 > 0$, there exists $N_2 > n_0$ such that $s \geq N_2$ implies $|\varphi(s - \tau_j(s))| < \epsilon_2$ for $j = 1, ..., N$. Thus for $n \geq N_2$, the last term on the right hand side of (2.15) satisfies

$$\begin{aligned}&\Big|\sum_{s=n_0}^{n-1}\Big[\sum_{j=1}^{N}\{h_j(n-\tau_j(n)) - a_j(n)\}\varphi(n-\tau_j(n)) \\ &-[1-H(s)]\sum_{j=1}^{N} Q_j(s, \varphi(s-\tau_j(s))) - [1-H(s)]\sum_{j=1}^{N}\sum_{r=s-\tau_j(s)}^{s-1} h_j(r)\varphi(r) \\ &+\sum_{j=1}^{N}\sum_{u=s-\tau_j(s)}^{s-1} k_j(s,u) f_j(u, \varphi(u))\Big]\prod_{u=s+1}^{n-1} H(u)\Big| \\ &\leq \sum_{s=n_0}^{N_2-1}\Big[\sum_{j=1}^{N}|h_j(s-\tau_j(s)) - a_j(s)||\varphi(s-\tau_j(s))| \\ &+|1-H(s)|L_1\sum_{j=1}^{N}|\varphi(s-\tau_j(s))| + |1-H(s)|\sum_{j=1}^{N}\sum_{r=s-\tau_j(s)}^{s-1}|h_j(r)||\varphi(r)| \\ &+L_2\sum_{j=1}^{N}\sum_{u=s-\tau_j(s)}^{s-1}|k_j(s,u)||\varphi(u)|\Big]\Big|\prod_{u=s+1}^{n-1} H(u)\Big| \\ &+\sum_{s=N_2}^{n-1}\Big[\sum_{j=1}^{N}|h_j(s-\tau_j(s)) - a_j(s)||\varphi(s-\tau_j(s))| \\ \\ &+|1-H(s)|L_1\sum_{j=1}^{N}|\varphi(s-\tau_j(s))| + |1-H(s)|\sum_{j=1}^{N}\sum_{r=s-\tau_j(s)}^{s-1}|h_j(r)||\varphi(r)|\end{aligned}$$

$$
\begin{aligned}
&+ L_2 \sum_{j=1}^{N} \sum_{u=s-\tau_j(s)}^{s-1} |k_j(s,u)||\varphi(u)|\Big]\Big| \prod_{u=s+1}^{n-1} H(u)\Big| \\
&\leq \max_{\sigma \geq m(n_0)} \varphi(\sigma) \sum_{s=n_0}^{N_2-1} \Big[\sum_{j=1}^{N} |h_j(s-\tau_j(s)) - a_j(s)| \\
&+ |1-H(s)|L_1 N + |1-H(s)| \sum_{j=1}^{N} \sum_{r=s-\tau_j(s)}^{s-1} |h_j(r)| \\
&+ L_2 \sum_{j=1}^{N} \sum_{u=s-\tau_j(s)}^{s-1} |k_j(s,u)|\Big]\Big| \prod_{u=s+1}^{n-1} H(u)\Big| \\
&+ \epsilon_2 \sum_{s=N_2}^{n-1} \Big[\sum_{j=1}^{N} |h_j(s-\tau_j(s)) - a_j(s)| \\
&+ |1-H(s)|L_1 N + |1-H(s)| \sum_{j=1}^{N} \sum_{r=s-\tau_j(s)}^{s-1} |h_j(r)| \\
&+ L_2 \sum_{j=1}^{N} \sum_{u=s-\tau_j(s)}^{s-1} |k_j(s,u)|\Big]\Big| \prod_{u=s+1}^{n-1} H(u)\Big| \\
&\leq \epsilon_2 + \epsilon_2 \alpha < 2\epsilon_2.
\end{aligned}
$$

Thus showing that the last term on the right hand side of (2.15) goes to zero as $n \to \infty$. Therefore, $(P\varphi) \to 0$ as $n \to \infty$. It therefore follows that P maps S into S.

We finally show that P is a contraction. Let $\varphi, \eta \in S$. Then

$$
\begin{aligned}
|(P\varphi)(n) - (P\eta)(n)| \quad \leq \quad & \Big\{ NL_1 + \sum_{j=1}^{N} \sum_{s=n-\tau_j(n)}^{n-1} |h_j(s)| \\
& + \sum_{s=n_0}^{n-1} \Big[\sum_{j=1}^{N} |h_j(n-\tau_j(n)) - a_j(n)| \\
& + |1-H(s)|NL_1 + |1-H(s)| \sum_{j=1}^{N} \sum_{r=s-\tau_j(s)}^{s-1} |h_j(r)| \\
& + L_2 \sum_{j=1}^{N} \sum_{u=s-\tau_j(s)}^{s-1} |k_j(s,u)|\Big]\Big| \prod_{u=s+1}^{n-1} H(u)\Big|\Big\} ||\varphi - \eta|| \\
\leq \quad & \alpha ||\varphi - \eta||.
\end{aligned}
$$

This shows that P is a contraction. Therefore, by the contraction mapping principle, P has a unique fixed point in S which solves (1.1) and for any $\varphi \in S$, $||P\varphi|| \leq \epsilon$. This shows that the zero solution of (1.1) is stable. Moreover, $(P\varphi) \to 0$ as $n \to \infty$, showing that the zero solution of (1.1) is asymptotically stable. This completes the proof.

References

[1] A. Ardjouni, and A. Djoudi; *Stability in nonlinear neutral Volterra difference equations with variable delays*, Journal of Nonlinear Evolution Equations and Applications, No. 7, pp. 89-100, (2013).

[2] A. Ardjouni, and A. Djoudi; *Stability in linear neutral difference equations with variable delays*, Mathematica Bohemica, No. 3, pp. 245-258, (2013).

[3] S. Elaydi, *Periodicity and stability of linear Volterra difference systems*, Journal of Mathematical Analysis and Applications, 181, pp. 483-492, (1994).

[4] S. Elaydi, *An Introduction to Difference Equations*, Springer, New York, (1999).

[5] M. Islam and E. Yankson, *Boundedness and stability in nonlinear delay difference equations employing fixed point theory*, Electronic Journal of Qualitative Theory of Differential Equations, No. 26, pp. 1-18, (2005).

[6] W. G. Kelly and A. C. Peterson, *Difference Equations : An Introduction with Applications,* Academic Press, (2001).

[7] Y. N. Raffoul, *Stability and periodicity in discrete delay equations*, Journal of Mathematical Analysis and Applications, 324, No. 2, pp. 1356-1362, (2006).

[8] Y. N. Raffoul, *Periodicity in general delay nonlinear difference equations using fixed point theory*, Journal of Difference Equations and Applications, 10, No. 1315, pp. 1229-1242, (2004).

[9] Y. N. Raffoul, *General theorems for stability and boundedness for nonlinear functional discrete systems*, Journal of Mathematical Analysis and Applications, 279, pp. 639-650, (2003).

[10] D. R. Smart, *Fixed point theorems* ; Cambridge Tracts in Mathematics, No. 66. Cambridge University Press, London-New York, (1974).

[11] E. Yankson, *Stability in discrete equations with variable delays*, Electronic Journal of Qualitative Theory of Differential Equations, No. 8, pp. 1-7, (2009).

[12] E. Yankson, *Stability of Volterra difference delay equations*, Electronic Journal of Qualitative Theory of Differential Equations, No. 20, pp. 1-14, (2006).

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