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Holomorphically proyective Killing fields with vectorial fields associated in kahlerian manifolds

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Abstract

Taking into account the harmonic and scalar curvatures in the study of Killing transformations between spacial complex (Einstenian, Peterson-Codazzi, Recurrent) and kaehlerian M spaces with almost complex J structure, we prove that there exists an holomorphically proyective transformation between M spaces and complex spaces.

Keywords: *Holomorphically proyective, Killing fields, kaehlerian manifolds.*

MSC 2010: *53A20, 53B35,53C15.*

1. Introduction and preliminaries

By the end of the 20^{th} century researchers started to link the concept of proyectivity with the phenomenon of complex manifolds specially in terms of their hollomorphic properties. Then, kaehlerian and hermithian manifolds as well as complex hyper surfaces and other manifolds were considered embedded into special transformations. At this point a vast number of publications arose in relation with the concepts of compact K manifolds, proyective infinitesimal transformations in Riemmanian manifolds with additive curvature properties and hollomorphic proyective equivalences and others. Based on [1] [2] and [3], this research studies Kaehler holomorphically proyective manifolds with almost complex structures by using the geometric properties of the harmonic and scalar curvatures evaluated over Killing vectorial fields. Two important applications result from this, the Einstenian and the constant curvature spaces.

Considering (M, g, J) as a kaehlerian manifold of $2n \geq 4$ dimension with a $g = (g_{ij})$ Riemannian metric and an almost-complex structure $J = J_{ij}$ where $J_{ij} = -J_{ji}$ and with a Riemannian curvature tensor of $R_{kji}{}^{h} = \partial_k \Gamma^h_{ji} - \partial_j \Gamma^h_{ki} + \Gamma^h_{ka}\Gamma^a_{ji} - \Gamma^h_{ja}\Gamma^a_{ki}$ the $S_{ji} = R_{aji}{}^{a}$ then the Ricci tensor and the $r = g^{ba}S_{ba}$ scalar curvature satisfy the following proprieties:

$$\begin{array}{lll} i) S_{ji} = J_i^b J_i^a S_{ba} & ii) H_{ji} = J_j^a S_{ai} & iii) H_{ji} + H_{ij} = 0 \\ iv) H_{ji} = J_i^b J_i^a H_{ba} & v) J_i^a J_a^h = -\delta_i^h & vi) J_{\underset{\sim}{j}}^a = -i\delta_j^a \\ vii) g_{ij} = J_i^a J_j^b g_{ab} & viii) \nabla_j \widetilde{F}_i = -\nabla_i \widetilde{F}_j & ix) \widetilde{F}_i := J_i{}^{a} F_a, \end{array}$$

$$where$$

$J^{ji} = g^{ja} J_a{}^{i}$, $H_{ji} = J_a{}^{i} S_{ai}$. The Lie operator derivative in the vectorial field direction X for $R_{kji}{}^{h}$ and ${}_{hji}$ is represented respectively by,

$$L_X R^h_{kji} = \nabla_k L_X \Gamma^h_{ji} - \nabla_j L_X \Gamma^h_{ki} \;\; y \;\; L_X \Gamma^h_{ji} = \nabla_j \nabla_i X^h + R^h_{aji} X^a.$$

If X is a vectorial field then
$i)$ X is a Killing field if satisfies

$$L_X g_{ji} = 0, \; \forall \;\; i, j = \overline{i, n}. \tag{1.1}$$

$ii)$ X is an holomorphically projective transformation when

$$L_X \Gamma^h_{ji} = \delta^h_i F_i + \delta^h_j F_i - J_j{}^{h} J^a_i F_a - J_i{}^{h} J^a_j F_a, \tag{1.2}$$

where $F = (F^i)$ is a particular vector associated to X.

Two metric $g = (g_{ij})$ y $\overline{g} = (\overline{g}_{ij})$ defined on M, they are hollomorphic proyective equivalences if

$$\overline{\Gamma}^k_{ki} = \Gamma^k_{ji} + F_i\delta^k_i + F_j\delta^k_i - J^k_j\widetilde{F}_i - J^k_i\widetilde{F}_j,$$

where $\widetilde{F}_i = J^a_i F_a$.

Tensors for harmonic and scalar curvatures are defined on the manifold M by means of the following relations:

$$\nabla_a R_{kji}{}^{a} = \nabla_k S_{ji} - \nabla_j S_{ki},$$

$$R = g^{ba} S_{ba},$$

respectively where $S_{ji} = R_{aji}{}^{a}$ is the tensor Ricci. The Laplacian of f is defined by

$$\Delta f = \nabla^a \nabla_a f = \Delta f,$$

where $f = \frac{1}{n+2}\nabla_a X^a$ with $f \in C^\infty(M)$ and $F_j = \nabla_j f$.

The classic commutative relationship of L_X and de ∇ for a tensor Y of (1,2) type is given by

$$L_X \nabla_k Y_{ji}{}^{h} - \nabla_k L_X Y^h_{ji} = \left(L_X \Gamma^h_{ka}\right) Y^h_{ji} - \left(L_X \Gamma^h_{kj}\right) Y^h_{ai} - \left(L_X \Gamma^a_{kj}\right) Y^h_{ai} - \left(L_X \Gamma^a_{ki}\right) Y_{ji}{}^{h}.$$

Being X an holomorphically proyective transformation with an F associated vector then the following identities are satisfied, watch [1]

i $2S^a_i F_a = -\nabla_i(\Delta f)$

ii $\nabla_j F_i = J^a_i J^b_j \nabla_b F_a$

iii $\nabla_k \nabla_j F_i = -J_k{}^{b} J^a_j R^c_{iab} F_c$.

In [3] proof

$$S_{ij} = \overline{S}_{ij} + \tau(F_{ij} - \overline{F}_{ji}), \tag{1.3}$$

where $F_{ij} = \nabla_i \nabla_j f$, $\tau-$ parameter.

An $A_n = (M, \nabla)$ space is a Peterson Codazzi one if $\nabla_k S_{ji} = \nabla_j S_{ki}$. If $\nabla_l R^h_{ijk} = F_l R_{ijk}{}^{h}$ it is a recurrent space where $F_l \neq 0$ or it is an Einstein space if $S = \lambda g$ taking S as the Ricci and g an the metric tensors and λ as a parameter.

Lemma 1.1. *If M is a compact Kaehlerian manifold of dimension n with a scalar curvature R and it admits an holomorphically proyective transformation then the following equations are fulfilled,*
i) $\Delta f = -\frac{2R}{n} f$
ii) $S_i{}^a F_a = \frac{R}{n} F_i$.

Proof. i) Since A_n is a recurrent space and M admits an holomorphically proyective transformation then we obtain

$$(1.4) g^{hi}\nabla_b\nabla_j X_i + g^{hi}R_{abji}X^a - F_b\delta_j^h - F_j\delta_b^h + F_a J_b^a J_j^h + F_a J_j^a J_b^h = 0,$$

multiplying (1.4) by g_{hk} and applying ∇^b it results that

$$\nabla^b\left(\nabla_b\nabla_j X_i + R_{abji}X^a - F_b g_{ji} - F_j g_{bi} + F_a J_b^a J_{ji} + F_a J_j^a J_{bi}\right) = 0.$$

Now using Ricci′s and Bianchi′s identities we obtain

$$(R_{abji} - 2R_{bjia})\nabla^b X^a - R_{ai}\nabla_j X^a + R_j^a\nabla_a X_i - (\nabla_a R_{ji})\, X^a = 0,$$

finally by applying ∇^j the result is

$$\begin{aligned} -2\nabla_i R_{ba}\nabla^b X^a &= 0 \quad \Rightarrow \nabla_i R_{ba} L_X g^{ba} = 0 \\ &\Rightarrow \quad -2RF_i = n\nabla_i(\Delta f) \\ &\Rightarrow \quad \nabla_i(\Delta f) = -\frac{2R}{n} f = \nabla_i\left(-\frac{2R}{n} f\right) \\ &\Rightarrow \quad \Delta f = -\frac{2R}{n} f, \end{aligned}$$

due to $(n\Delta f + 2Rf)$ is constant for being

$$\int_M \Delta f d\sigma = \int_M f d\sigma = 0,$$

a compact M and $d\sigma$ is a volumetric element of M. Finally we conclude that

$$\Delta f = -\frac{2R}{n} f.$$

ii) The demonstration is obtained by using I and part (i) from this lema.

Lemma 1.2. *Let X be an holomorphically proyective transformation with an F associated vector then*

$$L_X S_{ji} = -(n+2)\nabla_j F_i.$$

Proof. Using the definition of $L_X R_{kji}{}^h$ we have

$$L_X S_{ji} = L_X R_{hji}{}^h = \nabla_h L_X \Gamma^h_{ji} - \nabla_j L_X \Gamma_{hi}{}^h,$$

since X is an holomorphically projective transformation then

$$\begin{aligned} L_X S_{ji} &= \nabla_j F_i + \nabla_i F_j - J^h_j J^a_i F_a - J^h_i J^a_j F_a - n\nabla_j F_i - \nabla_j F_i + \\ &\quad + in J^a_i \nabla_j F_a - \nabla_j F_i. \end{aligned}$$

By considering the real part we obtain the desired result

$$L_X S_{ji} = -n\nabla_j F_i - 2\nabla_j F_i = -(n+2)\nabla_j F_i.$$

2. Results

The following theorem allows a Kaehlerian space to become into a Peterson-Codazzi space under the hypothesis that the former is holomorphically projective.

Theorem 2.1. *Let M be a Kaehlerian manifold and X be an holomorphically projective killing field with an associated vectorial field F then*

$$L_X\left(\nabla_j S_{ki} - \nabla_k S_{ji}\right)$$
$$= \left\{(n+2) R^a_{jki} - S_{ki}\delta^a_j + S_{ji}\delta^a_k - J^a_i H_{ki} + J^a_k H_{ji} + 2J^a_i H_{jk}\right\} F_a. \tag{2.1}$$

Proof. Using the classic relation of commutation for a (0,2) type tensor we obtain that

$$(L_X\nabla_j S_{ki} - L_X\nabla_k S_{ji}) - (\nabla_j L_X S_{ki} - \nabla_k L_X S_{ji}) = (L_X \Gamma^a_{ki}) S_{ja} - \left(L_X \Gamma^a_{ji}\right) S_{ka}. \tag{2.2}$$

If by hypothesis we consider X as an holomorphically projective transformation by using (1.2) then we have that

$$L_X \Gamma_{ji}{}^a = \delta^a_j F_i + \delta^a_i F_j - J^a_j J^h_i F_h - J^a_i J^h_j F_h. \tag{2.3}$$

Furthermore according to lemma (1.2),

$$L_X S_{ji} = -(n+2)\nabla_j F_i \tag{2.4}$$

and analogically we obtain $L_X \Gamma_{ki}^{\ a}$ and $L_X S_{ki}$.

By Substituting (2.3) and (2.4) in (2.2) it results

$$\begin{aligned}(L_X \nabla_j S_{ki} - L_X \nabla_k S_{ji}) - (\nabla_j [-(n+2)\nabla_k F_i] - \nabla_k [-(n+2)\nabla_j F_i]) \\ = (\delta_k^a F_i + \delta_i^a F_k - J_k^a J_i^h F_h - J_i^a J_k^h F_h) S_{ja} \\ - \left(\delta_j^a F_i + \delta_i^a F_j - J_j^a J_i^h F_h - J_i^a J_j^h F_h\right) S_{ka}.\end{aligned}$$

By doing certain manipulations and using simplification we conclude that

$$\begin{aligned}\left\{(n+2) R_{jki}^{\quad a} - S_{ki}\delta_j^a + S_{ji}\delta_k^a - J_i^a H_{ki} + J_k^a H_{ji} + 2J_i^a H_{jk}\right\} F_a \\ = L_X (\nabla_j S_{ki} - \nabla_k S_{ji}).\end{aligned}$$

From here on some applications of the previous results will be given.

1) If $\nabla_j S_{ki} = \nabla_k S_{ji}$ then M is Kaehler-Peterson-Codazzi space and

$$\left\{(n+2) R_{jki}^{\quad a} - S_{ki}\delta_j^a + S_{ji}\delta_k^a - J_i^a H_{ki} + J_k^a H_{ji} + 2J_i^a H_{jk}\right\} F_a = 0.$$

(2.5)

Consequence i

A Kaehler-Peterson-Codazzi space has an harmonic curvature since,

$$\nabla_j S_{ki} = \nabla_k S_{ji} \Leftrightarrow \nabla_a R_{jki}^a = 0.$$

Consequence ii

A Kaehler-Peterson-Codazzi space is an Einstenian space if the former has a constant scalar curvature. Factually by applying g^{ki} into (2.5) results in

$$\left\{(n+2) g^{ki} R_{jki}^{\quad a} - R\delta_j^a + g^{ai} S_{ji} - J_i^a g^{ki} H_{ki} + J_k^a g^{ki} H_{ji} + 2J_i^a g^{ki} H_{jk}\right\} F_a = 0.$$

Since $F_a \neq 0$ and developing the three last terms we have,

$$(n+2) g^{ki} R_{jki}^{\quad a} - R\delta_j^a + g^{ai} S_{ji} - J_j^a J_k^b g^{ki} S_{bi} + 3J_k^a J_j^b g^{ki} S_{bi} = 0,$$

by making the contraction $a = j$ and adding up from 1 to n we obtain

$$g_{ki} (nR + 2R - nR + R) = 3S_{ki},$$

In this way we conclude that $S_{ki} = \frac{R}{n} g_{ki}$. In other words the Kaehler-Peterson-Codazzi space is an Einstenian space.
2) If M is a recurrent space then

$$(n+2)\, R^a_{jki} F_a - L_X \left(R^a_{jki} F_a \right) = S_{ki}\delta^a_j - S_{ji}\delta^a_k + J^a_j H_{ki} - J^a_k H_{ji} - 2J^a_i H_{jk} F_a.$$

Consequence
If M is an harmonic curvature and $W = \{F = (F^i) : \ F \neq 0\}$ with

$$F_j F^k = \begin{cases} \|F\|^2 & si \quad j = k \\ 0 & si \quad k \neq j \end{cases},$$

then M has a null scalar curvature.

As a matter of fact if M admits an harmonic curvature then making the contraction $l = a$ and summing up from 1 to n in the relation

$$\nabla_l R_{jki}{}^{a} = R_{jki}{}^{a} F_l,$$

we obtain,

$$\nabla_l R_{jki}{}^{a} = R_{jki}{}^{a} F_l \quad \Rightarrow R_{jki}{}^{a} = 0.$$

from (2.1) we obtain

$$S_{ki} F_j - S_{ji} F_k + H_{ki} \widetilde{F}_j - H_{ji} \widetilde{F}_k - 2H_{jk} \widetilde{F}_i = 0$$

and multiplying the previous relation by g^{ki} it results that

$$g^{ki} S_{ji} F_k - g^{ki} S_{ki} F_j + g^{ki} H_{ji} \widetilde{F}_k - g^{ki} H_{ki} \widetilde{F}_j + 2g^{ki} H_{kj} \widetilde{F}_i = 0.$$

Therefore

$$S^k_j F_k - r F_j = H^k_j \widetilde{F}_k,$$

wherein by applying F^j it results that $r = 0$. This way we conclude that the manifold is plain.

Example
Be Einstein compact Kaehlerian spaces $A_n = (M, \nabla)$ and $\overline{A}_n = (M, \overline{\nabla})$, with metric $g = (g_{ij})$ y $\overline{g} = (\overline{g}_{ij})$ hollomorphic proyective equivalences, get an expression that relates the scalar curvature R and $\overline{R}$.

Solution Using [4],

$$S_{ij} = \overline{S}_{ij} + \tau(F_{ij} - \overline{F}_{ji}),$$

and as spaces of Einstein:

$$S_{ij} = c_1 g_{ij}, \qquad \overline{S}_{ij} = c_2 \overline{g}_{ij}$$

o

$$S_{ij} = \frac{R}{n} g_{ij}, \qquad \overline{S}_{ij} = \frac{\overline{R}}{n} \overline{g}_{ij},$$

it must be

$$\frac{R}{n} gij = \frac{\overline{R}}{n} \overline{g}_{ij} + \tau (F_{ij} - \overline{F}_{ji}), \quad \tau \in C,$$

Applying g^{ij} result

$$R = \frac{\overline{R}}{n} g^{ij} \overline{g}_{ij} + \tau (g^{ij} F_{ij} - g^{ij} \overline{F}_{ji}),$$

$$R = \frac{\overline{R}}{n} g^{ij} \overline{g}_{ij} + \tau (\|F\| - g^{ij} \overline{F}_{ji})$$

o

$$R g_{ij} = \frac{\overline{R}}{n} \overline{g}_{ij} + \tau (\|F\| g_{ij} - \overline{F}_{ij})$$

Applying now $\overline{g}^{ij}$,

$$R g_{ij} = \overline{R} \overline{g}_{ij} + \tau (\|F\| g_{ij} - \|\overline{F}\| \overline{g}_{ij}),$$

from here

$$(R - \tau \|F\|) g_{ij} = (\overline{R} - \tau \|\overline{F}\|) \overline{g}_{ij},$$

then

$$\frac{(R - \tau \|F\|)}{(\overline{R} - \tau \|\overline{F}\|)} = \frac{\det(\overline{g}_{ij})}{\det(g_{ij})}$$

In [3] proof

$$\ln \sqrt{\frac{det(\overline{g}_{ij})}{det(g_{ij})}} = (n+2)h, \quad h \in C^{\infty}(M).$$

Then he concludes

$$(R - \tau \|F\|) = (\overline{R} - \tau \|\overline{F}\|) \exp[2h(n+2)].$$

Example

Get an expression that compute the tensor Ricci in a compact kahlerian manifolds admitting proyective hollomorphic transformations with associated vector F, if A_n this is recurrent.

Solution In this case

$$\nabla_k R^h_{lji} = R^h_{lji} F_k,$$

by making the contraction $a = j$ and adding up from 1 to n we obtain

$$\nabla_k S_{ji} = S_{ji} F_k. \tag{2.6}$$

But

$$\nabla_k S_{ji} = \partial_k (S_{ji}) - \Gamma^a_{kj} S_{ai} - \Gamma^a_{ki} S_{ja}.$$

Applying g^{ij} result

$$\nabla_k S_{ji} = \partial_k (S_{ji}). \tag{2.7}$$

From (2.6) and (2.7) results

$$\partial_k (S_{ji}) - S_{ji} \partial_k f = 0.$$

The solution of this partial differential equation is the tensor de Ricci, hence is obtained scalar curvature.

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