Proyecciones Journal of Mathematics
Vol. 34, N^o 4, pp. 391-399, December 2015.
Universidad Católica del Norte
Antofagasta - Chile
DOI: 10.4067/S0716-09172015000400007

The Banach-Steinhaus Theorem in Abstract Duality Pairs

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Received : October 2015. Accepted : November 2015

Abstract

Let E, F be sets and G a Hausdorff, abelian topological group with $b : E \times F \to G$; we refer to E, F, G as an abstract duality pair with respect to G or an abstract triple and denote this by $(E, F : G)$. Let $(E_i, F_i : G)$ be abstract triples for $i = 1, 2$. Let $\mathcal{F}_i$ be a family of subsets of F_i and let $\tau_{\mathcal{F}_i}(E_i) = \tau_i$ be the topology on E_i of uniform convergence on the members of $\mathcal{F}_i$. Let Γ be a family of mappings from E_1 to E_2. We consider conditions which guarantee that Γ is $\tau_1 - \tau_2$ equicontinuous. We then apply the results to obtain versions of the Banach-Steinhaus Theorem for both abstract triples and for linear operators between locally convex spaces.

In [CLS] we established versions of the Orlicz-Pettis Theorem for subseries convergent series in abstract triples or abstract duality pairs based on results which were initiated at New Mexico State University during Professor Li Ronglu's tenure as a visiting scholar. In this note we present some further results on an equicontinuity version of the Banach-Steinhaus Theorem for abstract triples which were also the result of Professor Li's visit. After establishing our abstract version of the Banach-Steinhaus Theorem we present several applications to continuous linear operators between locally convex spaces and establish versions of the Banach-Steinhaus Theorem for arbitrary locally convex spaces.

We first recall the definition of abstract triples. Let E, F be sets and G a Hausdorff, abelian topological group with $b : E \times F \to G$; if $x \in E$ and $y \in F$, we often write $b(x, y) = x \cdot y$ for convenience. We refer to E, F, G as an abstract duality pair with respect to G or an abstract triple and denote this by $(E, F : G)$. Note that $(F, E : G)$ is an abstract triple under the map $\overline{b}(y, x) = b(x, y)$. Examples of abstract triples are given in [CLS]; in particular a pair of vector spaces in duality is an example where G is the scalar field.

In what follows $(E_i, F_i : G)$ will denote abstract triples for $i = 1, 2$. Let $\mathcal{F}_i$ be a family of subsets of F_i and let $\tau_{\mathcal{F}_i}(E_i) = \tau_i$ be the topology on E_i of uniform convergence on the members of $\mathcal{F}_i$ so a net $\{x_\alpha\}$ converges to $x \in E_i$ iff $x_\alpha \cdot y \to x \cdot y$ uniformly for y belonging to a member of $\mathcal{F}_i$. Let Γ be a family of mappings $T : E_1 \to E_2$. We consider conditions which guarantee that Γ is $\tau_1 - \tau_2$ equicontinuous. We then establish several versions of the Banach-Steinhaus Theorem for abstract triples and give applications to continuous linear operators between locally convex spaces.

To motivate the condition which guarantees that Γ is $\tau_1 - \tau_2$ equicontinuous, we consider the case of continuous linear operators between locally convex spaces. Let $(E_1, F_1), (E_2, F_2)$ be dual pairs and let τ_i be the polar topology of uniform convergence on the members of $\mathcal{F}_i$ and let Γ be a family of weakly continuous linear operators $T : E_1 \to E_2$. Suppose

$$\begin{aligned} (*) \ for\ every\ B &\in \mathcal{F}_2\ there\ exists\ A \in \mathcal{F}_1\ such\ that \\ T'B &= BT \subset A\ for\ every\ T \in \Gamma. \end{aligned}$$

or, taking polars in E_1,

$$(**)\ (T'B)_0 = T^{-1}B_0 \supset A_0\ for\ T \in \Gamma.$$

Condition (**) implies that Γ is $\tau_1 - \tau_2$ equicontinuous.

We consider abstracting condition (*) to abstract triples. For this regard the elements y of F_i as functions from $E_i \to G$ defined by $y(x) = x \cdot y$ for $x \in E_i$. We say that the pair $(\mathcal{F}_\infty, \mathcal{F}_\in)$ satisfies the equicontinuity condition (E) if

$$\begin{aligned} (E)\ \textit{for every}\ B &\in \mathcal{F}_2\ \textit{thereexists}\ A \in \mathcal{F}_1\ \textit{such that} \\ B\Gamma &= \{y \circ T : y \in B, T \in \Gamma\} \subset A \end{aligned}$$

[note if $x \in E_1$, $(y \circ T)(x) = y(Tx) = y \cdot Tx$].

Theorem 1. *If $(\mathcal{F}_1, \mathcal{F}_2)$ satisfies condition (E), the Γ is $\tau_1 - \tau_2$ equicontinuous.*

Proof. Suppose the net $\{x_\delta\}$ in E_1 converges to $x \in E_1$ with respect to τ_1 so $x_\delta \cdot y \to x \cdot y$ uniformly when y belongs to a member of $\mathcal{F}_1$. Let $B \in \mathcal{F}_2, z \in B$ and let A be as in condition (E). Then $z \circ T \in A$ for every $T \in \Gamma, z \in B$ so $z \cdot Tx_\delta \to z \cdot Tx$ uniformly for $T \in \Gamma, z \in B$ by the definition of convergence in τ_1. Therefore, $Tx_\delta \to Tx$ in τ_2 uniformly for $T \in \Gamma$. □

The case of a single operator satisfying condition (E) is of interest.

Corollary 2. *Suppose $T : E_1 \to E_2$ is such that for every $B \in \mathcal{F}_2$ there exists $A \in \mathcal{F}_1$ such that $BT \subset A$. Then T is $\tau_1 - \tau_2$ continuous.*

Corollary 3. *(Banach-Steinhaus) Suppose $\{T_\alpha\}$ is a net of maps from E_1 to E_2 such that $\tau_2 - \lim_\alpha T_\alpha x = Tx$ exists for every $x \in E_1$. If $\Gamma = \{T_\alpha\}$ satisfies condition (E), then T is $\tau_1 - \tau_2$ continuous.*

Proof. Suppose the net $\{x_\delta\}$ is τ_1 convergent to $x \in E_1$. Then by hypothesis $\tau_2 - \lim_\alpha T_\alpha x_\delta = Tx_\delta$ for each δ. Also, by Theorem 1, $\tau_2 - \lim_\delta T_\alpha x_\delta = T_\alpha x$ uniformly with respect to α. Therefore,

$$\lim_\delta Tx_\delta = \lim_\delta \lim_\alpha T_\alpha x_\delta = \lim_\alpha \lim_\delta T_\alpha x_\delta = \lim_\alpha T_\alpha x$$

([DS]I.7.6) and T is $\tau_1 - \tau_2$ continuous. □

We next consider conditions for which (E) holds and establish versions of the Banach-Steinhaus Theorem for topological vector spaces.

In what follows G will be a Hausdorff topological vector space.

We first give a motivation for the conditions which appear in a version of the Banach-Steinhaus Theorem for abstract triples.

Suppose (E, F) is a dual pair and $\tau_{\mathcal{F}}$ is a polar topology on E of uniform convergence on the members of $\mathcal{F}$. Recall a subset $C \subset E$ is $\tau_{\mathcal{F}}$ bounded iff $BC = \{\langle y, x\rangle : y \in B, x \in C\}$ is bounded for every $B \in \mathcal{F}$. We abstract this condition to abstract triples.

Definition 4. *A subset $C \subset E_2$ is $\mathcal{F}_2$ bounded if $C \cdot B = \{x \cdot y : y \in B, x \in C\}$ is bounded in G for every $B \in \mathcal{F}_2$.*

We give an equicontinuity version of the Banach-Steinhaus Theorem for abstract triples.

Theorem 5. *Suppose Γ is pointwise $\mathcal{F}_2$ bounded on E_1 (i.e., for every $x \in E_1$ the set Γx is $\mathcal{F}_2$ bounded in E_2). Let $\mathcal{B}$ be the family of subsets of F_1 which are pointwise bounded on E_1. Then the pair $(\mathcal{B},\mathcal{F}_2)$ satisfies condition (E). Hence, Γ is $\tau_{\mathcal{B}} - \tau_2$ equicontinuous.*

Proof. Let $B \in \mathcal{F}_2$. We claim $B\Gamma \in \mathcal{B}$. Let $x \in E_1$. Since Γx is $\mathcal{F}_2$ bounded, $B(\Gamma x)$ is bounded in G so $B\Gamma \in \mathcal{B}$. Therefore, $(\mathcal{B}, \mathcal{F}_2)$ satisfies condition (E) and the result follows from Theorem 1. □

From Corollary 3 we have another version of the Banach-Steinhaus Theorem.

Corollary 6. *Let $\{T_\alpha\}$ be a net of maps from $E_1 \to E_2$ which is pointwise $\mathcal{F}_2$ bounded on E_1. If $\tau_2 - \lim_\alpha T_\alpha x = Tx$ exists for every $x \in E_1$, then T is $\tau_{\mathcal{B}} - \tau_2$ continuous.*

We also have the more familiar form of the Banach-Steinhaus Theorem for sequences.

Corollary 7. *Let $T_k : E_1 \to E_2$ and suppose $\tau_2 - \lim_k T_k x = Tx$ exists for each $x \in E_1$. Then T is $\tau_{\mathcal{B}} - \tau_2$ continuous.*

Proof. For each $x \in E_1$, $\{T_k x\}$ is $\mathcal{F}_2$ bounded so the corollary above applies. □

We can also give a generalization of Theorem 1. Let $\mathcal{A}_1$ be a family of subsets of E_1 and let $\mathcal{B}_1$ be a family of subsets of F_1 which is uniformly bounded on members of $\mathcal{A}_1$ (i.e., $\mathcal{B}_1$ is $\mathcal{A}_1$ bounded).

Theorem 8. *Suppose ΓA is $\mathcal{F}_2$ bounded for every $A \in \mathcal{A}_1$. Then Γ is $\tau_{\mathcal{B}_1} - \tau_2$ equicontinuous.*

Proof. As in the proof of Theorem 5 the pair $(\mathcal{B}_1, \mathcal{F}_2)$ satisfies condition (E). □

In the case of Theorem 5, the family $\mathcal{A}_1$ consists of singletons.

We now give applications of the results above for abstract triples to continuous linear operators between locally convex spaces and obtain versions of the Banach-Steinhaus Theorem for arbitrary locally convex spaces.

In what follows let $(E_1, F_1), (E_2, F_2)$ be dual pairs with polar topologies τ_i on E_i of uniform convergence on members of $\mathcal{F}_i$. Let Γ be a family of $\tau_1 - \tau_2$ continuous linear operators. Let $\beta(E_i, F_i)$ be the strong topology on E_i from the duality. From Theorem 5 we obtain an equicontinuity version of the Banach-Steinhaus Theorem.

Theorem 9. *If Γ is pointwise bounded on E_1, then Γ is $\beta(E_1, F_1) - \tau_2$ equicontinuous.*

Proof. In Theorem 5 the family $\mathcal{B}$ is the family of $\sigma(F_1, E_1)$ bounded sets so $\tau_{\mathcal{B}} = \beta(E_1, F_1)$ and the result follows from Theorem 5. □

Note that the family Γ may fail to be equicontinuous with respect to the original topology of E_1 but that the result above holds for arbitrary locally convex spaces with no assumptions on the domain space E_1. If E_1 is a barrelled space, then the original topology of E_i is $\beta(E_1, F_1)$ so Theorem 7 gives one of the usual forms of the Banach-Steinhaus Theorem or the Uniform Boundedness Principle (see [Sw1] 24.11,[Wi1] 9.3.4). This result was established in [Sw2]. As noted in [LC] there are non-barrelled spaces which carry the strong topology so Theorem 9 gives a proper extension of the usual form of the Banach-Steinhaus Theorem for barrelled spaces.

From Corollary 7 and Theorem 9 we also obtain the sequential version of the Banach-Steinhaus Theorem.

Theorem 10. *Let $T_k : E_1 \to E_2$ be a sequence of $\tau_1 - \tau_2$ continuous linear operators such that $\tau_2 - lim_k T_k x = Tx$ exists for every $x \in E_1$. Then T is $\beta(E_1, F_1) - \tau_2$ continuous and $\{T_k\}$ is $\beta(E_1, F_1) - \tau_2$ equicontinuous.*

Again note that T may fail to be continuous with respect to the original topology of E_1. This result was established in [LC].

We can obtain an improvement of Theorem 9 for Banach-Mackey spaces. Recall a locally convex space E_1 is a Banach-Mackey space if the bounded subsets of E_1 are strongly bounded ([Wi1]10.4.3). For example, any sequentially complete locally convex space is a Banach-Mackey space ([Wi1] 10.4.8). We denote the topology on E_1 of uniform convergence on the $\beta(F_1, E_1)$ bounded subsets of F_1 by $\beta^*(E_1, F_1)$ (see [Sw1]20,[Wi1]10.1).

Theorem 11. *Suppose E_1 is a Banach-Mackey space. If Γ is pointwise bounded on E_1, then Γ is $\beta^*(E_1, F_1) - \tau_2$ equicontinuous.*

Proof. By the Banach-Mackey property the family $\mathcal{B}$ of Theorem 5 is the family of all $\beta(F_1, E_1)$ bounded subsets of F_1 so $\tau_{\mathcal{B}} = \beta^*(E_1, F_1)$ and the result follows from Theorem 5

Note $\beta^*(E_1, F_1) \subset \beta(E_1, F_1)$ so Theorem 11 improves the conclusion of Theorem 9 for Banach-Mackey spaces. We can also obtain an improvement of Theorem 10 for Banach-Mackey spaces. □

Theorem 12. *Suppose E_1 is a Banach-Mackey space. Let $T_k : E_1 \to E_2$ be a sequence of $\tau_1 - \tau_2$ continuous linear operators such that $\lim_k T_k x = Tx$ exists for every $x \in E_1$. Then T is $\beta^*(E_1, F_1) - \tau_2$ continuous and $\{T_k\}$ is $\beta^*(E_1, F_1) - \tau_2$ equicontinuous.*

We can also obtain a corollary of Theorem 8.

Corollary 13. *Let $\mathcal{A}_1$ be the family of all $\sigma(E_1, F_1)$ bounded subsets of E_1 and $\mathcal{B}_1$ be the family of all $\beta(F_1, E_1)$ bounded subsets of F_1. If Γ is uniformly bounded on members of $\mathcal{A}_1$, then Γ is $\beta^*(E_1, F_1) - \tau_2$ equicontinuous.*

Proof. $\tau_{\mathcal{B}_1} = \beta^*(E_1, F_1)$ so the result follows from Theorem 8. □

Corollary 13 about a single mapping also has an interesting application to linear operators.

Corollary 14. *Suppose $T : E_1 \to E_2$ is a bounded linear operator. Then T is $\beta^*(E_1, F_1) - \tau_2$ continuous.*

Note that T may not be continuous with respect to the original topology of E_1. Consider the identity operator on an infinite dimensional normed space when the domain has the weak topology and the range the norm topology.

The result in Corollary 2 also has an application to a Hellinger-Toeplitz result for linear operators. Let X, Y be locally convex spaces with duals X', Y'. A property $\mathcal{P}$ of subsets B of a dual space Y' is said to be linearly invariant if for every continuous linear operator $T : X \to Y$ there exists $A \subset X'$ with property $\mathcal{P}$ such that $BT = T'B \subset A$. For example, the family of subsets with finite cardinal, the weak* compact sets, the weak* convex compact sets, the weak* bounded sets, etc.

If $\mathcal{P}$ is a linearly invariant property, let $P(X, X')$ be the locally convex topology of uniform convergence on the members of X' with property $\mathcal{P}$. From Corollary 2 we have a Hellinger-Toeplitz result in the spirit of Wilansky ([Wi1]11.2.6).

Corollary 15. *If $T : X \to Y$ is a continuous linear operator, then T is $P(X, X') - P(Y, Y')$ continuous.*

In particular, T is continuous with respect to the Mackey topologies and strong topologies ([Wi1]11.2.6).

Finally we indicate an application concerning automatic continuity of matrix transformations between sequence spaces. Let λ_1, λ_2 be scalar sequence spaces containing c_{00}, the space of sequences with finite range and if $a = \{a_j\} \in \lambda_1^\beta$, the β-dual of λ_1, $t = \{t_j\} \in \lambda_1$, we write $a \cdot t = \sum_{j=1}^\infty a_j t_j$. Assume that λ_i has a locally convex polar topology τ_i from the duality pair $\lambda_i, \lambda_i^\beta$ and that $A = [a_{ij}]$ is an infinite matrix which maps λ_1 into λ_2. Under assumptions on the sequence spaces, we use Theorem 10 to show that A is continuous with respect to appropriate topologies. First, we assume that the β-dual of λ_1 is contained in the topological dual λ_1' and then we assume that λ_2 is an AK-space under its topology (i.e., the canonical unit vectors $\{e^i\}$ form a Schauder basis for λ_2 ([Wi2] 4.2.13,[Sw3] B.2). Now let a^i be the i^{th} row of the matrix A so $a^i \in \lambda_1^\beta \subset \lambda_1'$ and define $A_k : \lambda_1 \to \lambda_2$ by $A_k t = \sum_{i=1}^k (a^i \cdot t) e^i$. Then A_k is $\tau_1 - \tau_2$ continuous and $\tau_2 - \lim_k A_k t = \sum_{i=1}^\infty (a^i \cdot t) e^i = At$ by the AK assumption. By the Banch-Steinhaus Theorem 10, $\{A_k\}$ is $\beta(\lambda_1, \lambda_1) - \tau_2$ equicontinuous and A is $\beta(\lambda_1, \lambda_1^\beta) - \tau_2$ continuous, an automatic continuity result. In particular, if $\lambda_1 = \lambda_2 = l^2$, then this result implies that any matrix mapping l^2 into itself is continuous; this is the classic theorem of Hellinger and Toeplitz ([K2] 34.7). Further automatic continuity theorems for matrix mappings can be found in [K2] 34.7 and [Sw4] 12.6.

This paper was supported by Research Fund, Kumoh National Institute of Technology

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