Proyecciones Journal of Mathematics
Vol. 34, N^o 4, pp. 379-390, December 2015.
Universidad Católica del Norte
Antofagasta - Chile
DOI: 10.4067/S0716-09172015000400006

Computing the maximal signless Laplacian index among graphs of prescribed order and diameter

Nair Abreu *
Universidad Federal de Río de Janeiro, Brasil
Eber Lenes †
Universidad del Sinú, Colombia
and
Oscar Rojo ‡
Universidad Católica del Norte, Chile.
Received : July 2015. Accepted : August 2015

Abstract

A bug Bug_{p,r_1,r_2} is a graph obtained from a complete graph K_p by deleting an edge uv and attaching the paths P_{r_1} and P_{r_2} by one of their end vertices at u and v, respectively. Let $Q(G)$ be the signless Laplacian matrix of a graph G and $q_1(G)$ be the spectral radius of $Q(G)$. It is known that the bug $B_0 = Bug_{n-d+2,\lfloor \frac{d}{2} \rfloor, \lceil \frac{d}{2} \rceil}$ maximizes $q_1(G)$ among all graphs G of order n and diameter d. For a bug B of order n and diameter d, $n-d$ is an eigenvalue of $Q(B)$ with multiplicity $n-d-1$. In this paper, we prove that remainder $d+1$ eigenvalues of $Q(B)$, among them $q_1(B)$, can be computed as the eigenvalues of a symmetric tridiagonal matrix of order $d+1$. Finally, we show that $q_1(B_0)$ can be computed as the largest eigenvalue of a symmetric tridiagonal matrix of order $\frac{d}{2}+1$ whenever d is even.

Keyword : *Signless Laplacian index, diameter, bug, H-join.*

2000 AMS classification : *05C50, 05C35, 15A18.*

---

*Thanks the support of Grant 305372/2009-2, CNPq, Brazil.

†Thanks the support of Projects Mecesup UCN 0711, Mecesup UCN 1102 and Fondecyt Regular 1130135

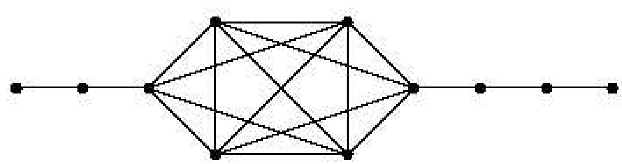
‡Thanks he support of Project Fondecyt Regular 1130135, Chile, and the hospitality of the Center For Mathematical Modeling, Universidad de Chile, Chile, in which this research was finished.

1. Introduction

Let $G = (V(G), E(G))$ be a simple undirected graph of order n with vertex set $V(G)$ and edge set $E(G)$. Let $D(G)$ be the diagonal matrix of order n whose $(i,i)-$ entry is the degree of the $i-th$ vertex of G and let $A(G)$ be the adjacency matrix of G. The matrices $L(G) = D(G) - A(G)$ and $Q(G) = D(G) + A(G)$ are the Laplacian matrix and signless Laplacian matrix of G, respectively. These matrices are both positive semidefinite matrices and they have the same characteristic polynomial if and only if G is a bipartite graph. The eigenvalues of $A(G)$, $L(G)$ and $Q(G)$ are called the eigenvalues, Laplacian eigenvalues and signless Laplacian eigenvalues of G, respectively. In particular, the spectral radius of $Q(G)$ is called the signless Laplacian index of G and it is usually denoted by $q_1(G)$. From the Perron - Frobenius Theory for nonnegative matrices, it follows that if G is a connected graph then $q_1(G)$ is a simple eigenvalue of $Q(G)$.

Let K_n and P_n be a complete graph and a path on n vertices, respectively. A bug Bug_{p,r_1,r_2} is a graph obtained from K_p by deleting an edge uv and attaching the paths P_{r_1} and P_{r_2} by one of their end vertices at u and v, respectively. Observe that Bug_{p,r_1,r_2} is a graph of order $p + r_1 + r_2 - 2$ and diameter $r_1 + r_2$.

Example 1. *For instance $Bug_{6,3,4}$ is the graph*



of 11 *vertices and diameter* 7.

Let $\mathcal{G}_n$ be the class of all connected graphs on n vertices and let $\mathcal{G}_{n,d}$ be the subclass of graphs in $\mathcal{G}_n$ with diameter d. Since $\mathcal{G}_{n,1}=\{K_n\}$ and $\mathcal{G}_{n,n-1} = \{P_n\}$, throughout this paper, we assume $2 \leq d \leq n-2$. Let

$$\mathcal{B}_{n,d} = \{Bug_{n-d+2,i,d-i} : 1 \leq i \leq d-1\}.$$

Clearly $\mathcal{B}_{n,d}$ is a subclass of $\mathcal{G}_{n,d}$.

Some results such that $q_1(G)$ is maximal among graphs with fixed invariants are known. For instance, in [4] the graph having the largest $q_1(G)$

among the graphs with fixed numbers of vertices and edges is found, in [6] the graphs with the largest $q_1(G)$ and the largest adjacency index among all graphs with a fixed vertex connectivity or a fixed edge connectivity are characterized and in [7] the author characterizes the graphs having the largest $q_1(G)$ among all the graphs on n vertices and a given matching number. of trees in $\mathcal{T}_{n,d}$ are characterized.

For a bug B of order n and diameter d, $n-d$ is an eigenvalue of $Q(B)$ with multiplicity $n-d-1$. In this paper, we prove that remainder $d+1$ eigenvalues of $Q(B)$ can be computed as the eigenvalues of a symmetric tridiagonal matrix of order $d+1$.

A conjecture proposed by Hansen and Lucas [5] states that, for a given $n \geq 9$, the bug $B_{\lfloor \frac{n}{2} \rfloor +1, \lfloor \frac{D}{2} \rfloor, \lceil \frac{D}{2} \rceil}$, where $D = \lceil \frac{n+1}{2} \rceil$, is the unique connected graph of order n that maximizes the product $q_1(G) diam(G)$ over all connected graphs G of order n. This conjecture was studied by H. Liu and M. Lu [3]. They proved that the bug $B_0 = Bug_{n-d+2, \lfloor \frac{d}{2} \rfloor, \lceil \frac{d}{2} \rceil}$ maximizes $q_1(G)$ among all graphs G of order n and diameter d and that, for a given n, the bug $B_{\lfloor \frac{n}{2} \rfloor +2, \lfloor \frac{D}{2} \rfloor, \lceil \frac{D}{2} \rceil}$, where $D = \lceil \frac{n+1}{2} \rceil$, is the unique connected graph of order n that maximizes the product $q_1(G) diam(G)$ over all connected graphs of order n.

Moreover, in this paper, we prove that $q_1(B_0)$ can be computed as the largest eigenvalue of a symmetric tridiagonal matrix of order $\frac{d}{2}+1$ whenever d is even.

We recall the notion of the join operation of graphs. Given two vertex disjoint graphs G_1 and G_2, the join of G_1 and G_2 is the graph $G = G_1 \vee G_2$ such that $\mathrm{V}(G) = V(G_1) \cup V(G_2)$ and $\mathrm{E}(G) = E(G_1) \cup E(G_2) \cup \{xy : x \in V(G_1), y \in V(G_2)\}$.

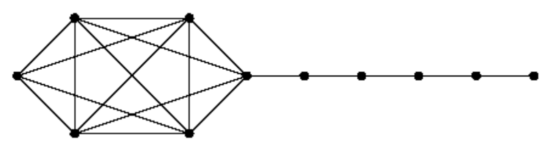
The join operation of two vertex disjoint graphs can be generalize as follows [1, 2]. Let H be a graph of order k. Let $V(H) = \{1, \dots, k\}$ be the vertex set of H. Let $\{G_1, G_2, \dots, G_k\}$ be a set of pairwise vertex disjoint graphs. For $1 \leq j \leq k$, the vertex $j \in V(H)$ is assigned to the graph G_j. Let G be the graph obtained from the graphs $G_1, G_2, \dots, G_k$ and the edges connecting each vertex of G_i with all the vertices of G_j if and only if $ij \in E(H)$. That is, G is the graph with vertex set $\mathrm{V(G)}=\bigcup_{i=1}^{k} V(G_i)$ and edge set $\mathrm{E(G)}=\left(\bigcup_{i=1}^{k} E(G_i)\right) \cup \left(\bigcup_{ij \in E(H)} \{uv : u \in V(G_i), v \in V(G_j)\}\right)$. This graph is called the $H-join$ of the graphs $G_1, \dots, G_k$ and it is denoted by $\mathrm{G}=\bigvee_H \{G_j : 1 \leq j \leq k\}$.

We see that if n_i is the order of G_i, $i = 1, 2, \dots, k$, then $H-join$ of $G_1, \dots, G_k$ is a graph of order $n_1 + n_2 + \ldots + n_k$.

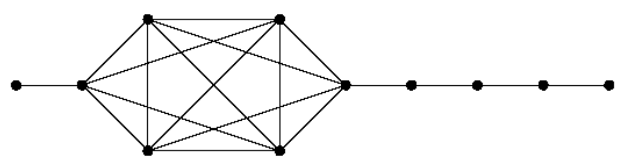
Important examples of this graph operation are the bugs $Bug_{n-d+2,i,d-i}$. In fact, the bug $Bug_{n-d+2,i,d-i}$ is the $P_{d+1}-join$ of the regular graphs $G_1 = \ldots = G_i = K_1, G_{i+1} = K_{n-d}, G_{i+2} = \ldots = G_{d+1} = K_1$.

Since $Bug_{n-d+2,i,d-i}$ and $Bug_{n-d+2,d-i,i}$, are isomorphic graphs, we may take $1 \leq i \leq \lfloor \frac{d}{2} \rfloor$.

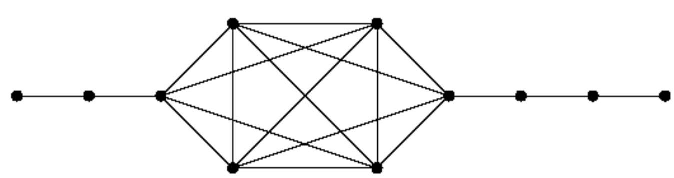
Example 2. *Below are the non-isomorphic bugs of order* 11 *and diameter* 7.
$Bug_{6,1,6}$ *is the* P_8-join *of* $G_1 = K_1$, $G_2 = K_4$ *and* $G_i = K_1$ *for* $i = 3, \ldots, 8$*:*



$Bug_{6,2,5}$ *is the* P_8-join *of* $G_1 = G_2 = K_1$, $G_3 = K_4$ *and* $G_i = K_1$ *for* $i = 4, \ldots, 8$*:*



$Bug_{6,3,4}$ *is the* P_8-join *of* $G_1 = G_2 = G_3 = K_1$, $G_4 = K_4$ *and* $G_i = K_1$ *for* $i = 5, \ldots, 8$*:*



2. The signless Laplacian eigenvalues of bugs

In [1], Theorem 5, the spectrum of the adjacency matrix of the H- join of regular graphs is obtained. The version of this result for the signless Laplacian matrix is given below and its proof is similar.

Theorem 1. *Let H be a graph with k vertices. Let $G = \bigvee_H \{G_j : 1 \leq j \leq k\}$. For $j = 1, \ldots, k$, let G_j be a r_j-regular graph of order n_j. Then*

(2.1) $\sigma\left(Q\left(G\right)\right) = \cup_{G_j \neq K_1} \left\{s_j + \lambda : \lambda \in \sigma\left(Q\left(G_j\right)\right) \setminus \{2r_j\}\right\} \cup \sigma\left(M(G)\right)$

where $M(G)$ is a matrix of order $k \times k$ given by

$$M(G) = \begin{bmatrix} s_1 + 2r_1 & \delta_{12}\sqrt{n_1 n_2} & \cdots & \delta_{1k}\sqrt{n_1 n_k} \\ \delta_{12}\sqrt{n_1 n_2} & s_2 + 2r_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \delta_{(k-1)k}\sqrt{n_{k-1} n_k} \\ \delta_{1k}\sqrt{n_1 n_k} & \cdots & \delta_{(k-1)k}\sqrt{n_{k-1} n_k} & s_k + 2r_k \end{bmatrix}$$

(2.2)

with

$$\delta_{ij} = \begin{cases} 1 \text{ if } ij \in E\left(H\right) \\ 0 \text{ otherwise} \end{cases}$$

and, for $j = 1, 2, \ldots, k$,

(2.3) $$s_j = \sum_{jl \in E(H)} n_l.$$

For brevity, let $B(i) = Bug_{n-d+2,i,d-i}$. Remember that we may take $1 \leq i \leq \lfloor \frac{d}{2} \rfloor$.

We already observed $B(i) = Bug_{n-d+2,i,d-i}$ is the $P_{d+1} - join$ of the regular graphs $G_1 = \ldots = G_i = K_1, G_{i+1} = K_{n-d}, G_{i+2} = \ldots = G_{d+1} = K_1$. Hence Theorem 1 can be applied to determine its signless Laplacian eigenvalues. For all the bugs $B(i)$, the graph H in Theorem 1 is the path P_{d+1}. Hence the matrix $M(B(i))$ in (2.2) becomes a symmetric tridiagonal matrix of order $(d+1)\times(d+1)$:

$$M(B(i)) = \begin{bmatrix} s_1+2r_1 & \sqrt{n_1n_2} & & & & \\ \sqrt{n_1n_2} & s_2+2r_2 & \sqrt{n_2n_3} & & & \\ & \sqrt{n_2n_3} & s_3+2r_3 & \ddots & & \\ & & \ddots & \ddots & \ddots & \\ & & & \ddots & s_d+2r_d & \sqrt{n_dn_{d+1}} \\ & & & & \sqrt{n_dn_{d+1}} & s_{d+1}+2r_{d+1} \end{bmatrix}. \tag{2.4}$$

For convenience, we define the matrices

Definition 1. *Let*

$$R(n-d) = \begin{bmatrix} n-d+1 & \sqrt{n-d} & 0 \\ \sqrt{n-d} & 2(n-d) & \sqrt{n-d} \\ 0 & \sqrt{n-d} & n-d+1 \end{bmatrix},$$

$$T_1 = [1]$$

and, for $s \geq 2$, let

$$T_s = \begin{bmatrix} 1 & 1 & & & \\ 1 & 2 & 1 & & \\ & 1 & \ddots & \ddots & \\ & & \ddots & 2 & 1 \\ & & & 1 & 2 \end{bmatrix}. \tag{2.5}$$

of order $s\times s$.

Moreover, we introduce the following matrices: I is the identity matrix, 0 is the zero matrix, J is the exchange matrix (the matrix with ones in the secondary diagonal and zeros elsewhere) and F is the matrix whose entries are zeros except for the entry in the last row and first column which is equal to 1. The orders of these matrices will be clear from the context in which they are used.

Theorem 2. *The eigenvalues of $B(i) = Bug_{n-d+2,i,d-i}$ are $n-d$ with multiplicity $n-d-1$ and the eigenvalues of the $(d+1)\times(d+1)$ symmetric tridiagonal matrix*

$$M(B(i)) = \begin{bmatrix} X_i & F \\ F^T & JT_{d-i-1}J \end{bmatrix} \tag{2.6}$$

where

$$X_1 = \begin{bmatrix} n-d & \sqrt{n-d} & 0 \\ \sqrt{n-d} & 2(n-d) & \sqrt{n-d} \\ 0 & \sqrt{n-d} & n-d+1 \end{bmatrix} \tag{2.7}$$

whenever $i = 1$,

$$X_i = \begin{bmatrix} T_{i-1} & F \\ F^T & R\,(n-d) \end{bmatrix} \tag{2.8}$$

whenever $2 \leq i \leq \left\lfloor \frac{d}{2} \right\rfloor$ *and* F *is the matrix defined above.*

Proof. We know that $B(i) = Bug_{n-d+2,i,d-i}$ is the $P_{d+1} - join$ of the regular graphs $G_1 = \ldots = G_i = K_1, G_{i+1} = K_{n-d}, G_{i+2} = \ldots = G_{d+1} = K_1$. Thus $G_j = K_1$ for all j except for $j = i+1$. For $j = i+1$, we have $G_{i+1} = K_{n-d}$ which is a $(n-d-1)$ - regular graph. From (2.3), $s_{i+1} = n_i + n_{i+2} = 1 + 1 = 2$. Then, from (2.1), we have

$$\sigma(Q(B(i))) = \{2 + \lambda : \lambda \in Q(K_{n-d}) \backslash \{2(n-d-1)\}\} \cup \sigma(M(B(i)). \tag{2.9}$$

At this point, we recall that the signless Laplacian eigenvalues of K_{n-d} are $2(n-d-1)$ and $n-d-2$ with multiplicity $n-d-1$. Using this fact in (2.9), we obtain $\sigma(Q(B(i))) = \{(n-d)^{[n-d-1]}\} \cup \sigma(M(B(i))$ where $(n-d)^{[n-d-1]}$ means that $n-d$ is an eigenvalue of multiplicity $n-d-1$.

We now search for the entries of $M(B(i))$ in (2.4). We begin with $M(B(1))$. The bug $B(1)$ is the $P_{d+1} - join$ of $G_1 = K_1$, $G_2 = K_{n-d}$ and $G_3 = G_4 = \ldots = G_{d+1} = K_1$. For this bug

$$\begin{array}{cc} n_1 = 1 & r_1 = 0 \\ n_2 = n-d & r_2 = n-d-1 \\ n_3 = 1 & r_3 = 0 \\ \vdots & \vdots \\ n_d = 1 & r_d = 0 \\ n_{d+1} = 1 & r_{d+1} = 0 \end{array}$$

Then $s_1 = n_2 = n-d, s_2 = n_1 + n_3 = 2, s_3 = n_2 + n_4 = n-d+1, s_4 = n_3 + n_5 = 2, \ldots, s_d = n_{d-1} + n_{d+1} = 2, s_{d+1} = n_d = 1$. Replacing these values in (2.4), we obtain

$$\mathrm{M(B(1))}=\begin{bmatrix} X_1 & F \\ F^T & JT_{d-2}J \end{bmatrix} \text{ with } \mathrm{X}_1 = \begin{bmatrix} n-d & \sqrt{n-d} & 0 \\ \sqrt{n-d} & 2(n-d) & \sqrt{n-d} \\ 0 & \sqrt{n-d} & n-d+1 \end{bmatrix}$$

and T_{d-2} as in (2.5). The theorem has been proved for $B(1)$. Let $2 \leq i \leq \lfloor \frac{d}{2} \rfloor$. The bug $B(2)$ is the $P_{d+1} - join$ of the regular graphs $G_1 = G_2 = K_1, G_3 = K_{n-d}, G_4 = \ldots = G_{d+1} = K_1$. For $B(2)$, we have

$$\begin{array}{cc} n_1 = 1 & r_1 = 0 \\ n_2 = 1 & r_2 = 0 \\ n_3 = n-d & r_3 = n-d-1 \\ n_4 = 1 & r_4 = 0 \\ \vdots & \vdots \\ n_d = 1 & r_d = 0 \\ n_{d+1} = 1 & r_{d+1} = 0 \end{array}$$

Then $s_1 = n_2 = 1, s_2 = n_1 + n_3 = n-d+1, s_3 = n_2 + n_4 = 2, s_4 = n_3 + n_5 = n-d+1, s_5 = n_4 + n_6 = 2, \ldots, s_d = n_{d-1} + n_{d+1} = 2, s_{d+1} = n_d = 1$. Replacing these values in (2.4), we get

$$\mathrm{M(B(2))}=\begin{bmatrix} X_2 & F \\ F^T & JT_{d-3}J \end{bmatrix}$$

where

$$\mathrm{X}_2 = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & n-d+1 & \sqrt{n-d} & 0 \\ 0 & \sqrt{n-d} & 2(n-d) & \sqrt{n-d} \\ 0 & 0 & \sqrt{n-d} & n-d+1 \end{bmatrix}$$

and T_{d-3} as in (2.5). The bug $B(3)$ is the $P_{d+1} - join$ of $G_1 = G_2 = G_3 = K_1$, $G_4 = K_{n-d}$, $G_5 = \ldots = G_{d+1} = K_1$. Similarly

$$\mathrm{M(B(3))}=\begin{bmatrix} X_3 & F \\ F^T & JT_{d-4}J \end{bmatrix}$$

where

$$\mathrm{X}_3 = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 2 & 1 & 0 & 0 \\ 0 & 1 & n-d+1 & \sqrt{n-d} & 0 \\ 0 & 0 & \sqrt{n-d} & 2(n-d) & \sqrt{n-d} \\ 0 & 0 & 0 & \sqrt{n-d} & n-d+1 \end{bmatrix}$$

and T_{d-4} as in (2.5). We continue in this fashion obtaining that the result also holds for $i = 4, \ldots, \lfloor \frac{d}{2} \rfloor$. $\square$

3. Computing the largest signless Laplacian index of graphs of prescribed order and diameter

We already mentioned that H. Liu and M. Lu, Theorem 3.2 in [3] characterized the largest signless Laplacian index among the graphs in $\mathcal{G}_{n,d}$.

Theorem 3. Among all the graphs G on n vertices and diameter d, $2 \leq d \leq n-2$, the largest $q_1(G)$ is attained by the bug $B(\lfloor \frac{d}{2} \rfloor) = Bug_{n-d+2,\lfloor \frac{d}{2} \rfloor,\lceil \frac{d}{2} \rceil}$.

Theorem 3 tell us that the largest signless Laplacian index among the graph in $\mathcal{G}_{n,d}$ is $q_1(B(\lfloor \frac{d}{2} \rfloor)$. From Theorem 2, $q_1(B(\lfloor \frac{d}{2} \rfloor)$ can be computed as the largest eigenvalues of the symmetric tridiagonal matrix $M(B(\lfloor \frac{d}{2} \rfloor))$ of order $d+1$. More precisely

Theorem 4. Let $G \in \mathcal{G}_{n,d}$.

(a) If $d = 3$ then the largest $q_1(G)$ can be computed as the largest eigenvalue of the symmetric tridiagonal matrix $M(B(1))$ of order 4 with diagonal entries

$$n-3, 2(n-3), n-2, 1$$

and codiagonal entries

$$\sqrt{n-3},\ \sqrt{n-3}, 1$$

(b) If $d \geq 4$ then the largest $q_1(G)$ can be computed as the largest eigenvalue of the symmetric tridiagonal matrix $M(B(\lfloor \frac{d}{2} \rfloor))$ of order $d+1$ with diagonal entries

$$1, \overbrace{1, 2, ..., 2}^{\lfloor \frac{d}{2} \rfloor - 2}, n-d+1, 2(n-d), n-d+1, \overbrace{2, ..., 2}^{\lceil \frac{d}{2} \rceil - 2}, 1$$

codiagonal entries

$$\overbrace{1, ..., 1}^{\lfloor \frac{d}{2} \rfloor - 1}, \sqrt{n-d}, \sqrt{n-d}, \overbrace{1, ..., 1}^{\lceil \frac{d}{2} \rceil - 1}.$$

We now prove that $q_1(B(\lfloor \frac{d}{2} \rfloor)$ can be computed as the largest eigenvalue of a symmetric tridiagonal matrix of order $\frac{d}{2} + 1$ whenever d is an even integer.

Theorem 5. If $d \geq 4$ is an even integer then among the graphs G on n vertices and diameter d, the largest $q_1(G)$ can be computed as the largest eigenvalue of a symmetric tridiagonal matrix of order $\frac{d}{2}+1$ with diagonal entries

$$1, 2, \ldots, 2, n-d+1, 2(n-d)$$

and codiagonal entries

$$1, \ldots, 1, \sqrt{2(n-d)}.$$

Proof. Let d be an even integer and $\alpha = n - d$. From Theorem 4, $q_1(M(B(\frac{d}{2}))$ can be computed as the largest eigenvalue of

$$M(B(\frac{d}{2})) = \begin{bmatrix} U & \mathbf{b} & 0 \\ \mathbf{b}^T & 2\alpha & \mathbf{b}^T J \\ 0 & J\mathbf{b} & JUJ \end{bmatrix}$$

of order $d+1$ where

$$U = \begin{bmatrix} 1 & 1 & & & \\ 1 & 2 & \ddots & & \\ & \ddots & \ddots & 1 & \\ & & 1 & 2 & 1 \\ & & & 1 & \alpha+1 \end{bmatrix}$$

of order $\frac{d}{2}$, $\mathbf{b}^T = \begin{bmatrix} 0 & \cdots & \cdots & 0 & \sqrt{\alpha} \end{bmatrix}$, J is the reverse matrix and 0 is the zero matrix, all of them of the appropriate sizes. Consider the orthogonal matrix

$$Q = \frac{1}{\sqrt{2}} \begin{bmatrix} I & \mathbf{0} & J \\ \mathbf{0}^T & \sqrt{2} & \mathbf{0}^T \\ -J & \mathbf{0} & I \end{bmatrix}.$$

An easy calculation shows that

$$QM(B(\frac{d}{2}))Q^T = \begin{bmatrix} U & \sqrt{2}\mathbf{b} & 0 \\ \sqrt{2}\mathbf{b}^T & 2\alpha & \mathbf{0}^T \\ 0 & \mathbf{0} & JUJ \end{bmatrix}.$$

Then the eigenvalues of $M(B(\frac{d}{2}))$ are the eigenvalues of $\begin{bmatrix} U & \sqrt{2}\mathbf{b} \\ \sqrt{2}\mathbf{b}^T & 2\alpha \end{bmatrix}$ and the eigenvalues of U. Since the eigenvalues of U strictly interlace the eigenvalues of $\begin{bmatrix} U & \sqrt{2}\mathbf{b} \\ \sqrt{2}\mathbf{b}^T & 2\alpha \end{bmatrix}$, the proof is complete.
□

References

[1] D. M. Cardoso, M. A. A. de Freitas, E. Martins., M. Robbiano, Spectra of graphs obtained by a generalization of the join graph operation, Discrete Mathematics 313, pp. 733-741, (2013).

[2] D. M. Cardoso, E. Martins., M. Robbiano, O. Rojo, Eigenvalues of a H-generalized operation constrained by vertex subsets, Linear Algebra Appl. 438, pp. 3278-3290, (2013).

[3] H. Liu, M. Lu, A conjecture on the diameter and signless Laplacian index of graphs, Linear Algebra Appl. 450, pp. 158-174, (2014).

[4] D. Cvetković, P. Rowlinson, S.K. Simić, Eigenvalue bounds for the signless Laplacian, Publications de L'Institute Mathématique, Nouvelle série, tome 81 (95), pp. 11-27, (2007).

[5] P. Hansen, C. Lucas, Bounds and conjectures for the signless Laplacian index of graphs, Linear Algebra Appl. 432, pp. 3319-3336, (2010).

[6] Miao-Lin Ye, Yi-Zheng Fan, Hai Feng Wang, Maximizing signless Laplacian or adjacency spectral radius of graphs subject to fixed connectivity, Linear Algebra Appl. 433, pp. 1180-1186, (2010).

[7] G. Yu, On the maximal signless Laplacian spectral radius of graphs with given matching number, Proc. Japan Acad. Ser. A 84, pp. 163-166, (2008).

Nair Abreu
Production Engineering Program,
PEP/COPPE
Universidade Federal do Rio de Janeiro
Rio de Janeiro,
Brazil
e-mail : nairabreunovoa@gmail.com

Eber Lenes
Departamento de Investigaciones
Universidad del Sinú. Elías Bechara Zainúm
Cartagena,
Colombia
e-mail: elenes@ucn.cl

and

Oscar Rojo
Department of Mathematics
Universidad Católica del Norte
Antofagasta,
Chile
e-mail: orojo@ucn.cl