

On the hyperstability of a Cauchy-Jensen type functional equation in Banach spaces

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Abstract

In this paper, we establish some hyperstability results of the following Cauchy-Jensen functional equation

$$f\left(\frac{x+y}{2} + z\right) + f\left(\frac{x-y}{2} + z\right) = f(x) + 2f(z)$$

in Banach spaces.

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1. Introduction and Preliminaries

The stability problem of functional equations was posed for the first time by Ulam [27] in the year 1940. Ulam stated the problem as follows:

Let (G, \cdot) be a group and let (H, \cdot, d) be a metric group with the metric d . Given $\delta > 0$, does there exist $\epsilon > 0$ such that if a mapping $h : G \rightarrow H$ satisfies the inequality

$$d(h(xy), h(x)h(y)) \leq \delta$$

for all $x, y \in G$, then there is a homomorphism $a : G \rightarrow H$ with

$$d(h(x), a(x)) \leq \epsilon$$

for all $x \in G$?

Ulam's problem was partially solved by Hyers in 1941 in the context of Banach spaces with $\epsilon = \delta$ as shown below [14].

Theorem 1.1. [D. H. Hyers (1941)] Let E be a normed vector space, F a Banach space and suppose that the mapping $f : E \rightarrow F$ satisfies the inequality

$$\|f(x+y) - f(x) - f(y)\| \leq \epsilon,$$

for all $x, y \in E$ where ϵ is a constant. Then the limit

$$T(x) = \lim_{n \rightarrow \infty} 2^{-n} f(2^n x)$$

exists for each $x \in E$ and T is the unique additive mapping satisfying

$$\|f(x) - T(x)\| \leq \epsilon,$$

for all $x \in E$. Also, if for each x the function $t \rightarrow f(tx)$ from \mathbf{R} to F is continuous for each fixed x , then T is linear. If f is continuous at a single point of E , then T is continuous in E .

Aoki [1], Z. Gajda [11] and Th. M. Rassias [23] provided a generalization of the Hyers theorem for additive and linear mappings, respectively, by allowing the Cauchy difference to be unbounded.

Theorem 1.2. Let $f : E \rightarrow F$ be a mapping from a real normed vector space E into a Banach space F satisfying the inequality

$$(1.1) \quad \|f(x+y) - f(x) - f(y)\| \leq \theta(\|x\|^p + \|y\|^p),$$

for all $x, y \in E \setminus \{0\}$, where θ and p are constants with $\theta > 0$ and $p \neq 1$. Then there exists a unique additive mapping $T : E \rightarrow F$ such that

$$(1.2) \quad \|f(x) - T(x)\| \leq \frac{\theta}{|1 - 2^{p-1}|} \|x\|^p,$$

for all $x \in E \setminus \{0\}$.

Theorem 1.2 is due to Aoki [1] for $0 < p < 1$ (see also [23]; Gajda [11] for $p > 1$; Hyers [14] for $p = 0$ and Th. M. Rassias [24] for $p < 0$ (see [[25], page 326] and [5]).

Beginning around the year 1980 the topic of approximate homomorphisms, or the stability of the equation of homomorphism, was studied by a number of mathematicians (see 10, 16, 17, 18, 26, 12, 15, 20, 21).

Recently, interesting results concerning Cauchy-Jensen functional equation

$$(1.3) \quad f\left(\frac{x+y}{2} + z\right) + f\left(\frac{x-y}{2} + z\right) = f(x) + 2f(z),$$

have been obtained in [2] and [20].

Lemma 1.1. Let X and Y be vector spaces. If a mapping $f : X \rightarrow Y$ satisfies (1.3), then the mapping $f : X \rightarrow Y$ is Cauchy additive and f is a solution of the Jensen equation $2f\left(\frac{x+y}{2}\right) = f(x) + f(y)$.

We say a functional equation D is *hyperstable* if any function f satisfying the equation D approximately is a true solution of D . The term hyperstability was used for first time probably in [19]. However, it seems that the first hyperstability result was published in [4] and concerned the ring homomorphisms. The hyperstability results for Cauchy equation were investigated by Brzdek in [7, 8, 9]. Gselmann in [13] studied the hyperstability of the parametric fundamental equation of information. In [3] Bahyrycz and Piszczek provided the hyperstability of the Jensen functional equation.

In this paper, we present the hyperstability results for the Cauchy-Jensen functional equation (1.3) in Banach spaces.

The method of the proof of the main results is motivated by an idea used by Brzdek in [7, 8, 9] and further by Piszczek in [22]. It is based on

a fixed point theorem for functional spaces obtained by Brzdek et al. (see [[6], Theorem 1]).

Throughout the paper, \mathbf{N} , \mathbf{R} , \mathbf{R}_+ and \mathbf{N}_{m_0} denote the set of all positive integers, the set of real numbers, the set of positive real numbers and the set of all integers greater than or equal to m_0 , respectively.

First, we take the following three hypotheses (all notations come from [6]).

(H1) X is a nonempty set, Y is a Banach space, $f_1, \dots, f_k : X \rightarrow X$ and $L_1, \dots, L_k : X \rightarrow \mathbf{R}_+$ are given.

(H2) $\mathcal{T} : Y^X \rightarrow Y^X$ is an operator satisfying the inequality

$$\|\mathcal{T}\xi(x) - \mathcal{T}\mu(x)\| \leq \sum_{i=1}^k L_i(x) \|\xi(f_i(x)) - \mu(f_i(x))\|$$

for all $\xi, \mu \in Y^X$, $x \in X$.

(H3) $\Lambda : \mathbf{R}_+^X \rightarrow \mathbf{R}_+^X$ is a linear operator defined by

$$\Lambda\delta(x) := \sum_{i=1}^k L_i(x)\delta(f_i(x))$$

for all $\delta \in \mathbf{R}_+^X$, $x \in X$.

The mentioned fixed point theorem is stated as follows.

Theorem 1.3. Let hypotheses **(H1)**-**(H3)** be valid and functions $\varepsilon : X \rightarrow \mathbf{R}_+$ and let $\varphi : X \rightarrow Y$ fulfil the following two conditions:

$$\|\mathcal{T}\varphi(x) - \varphi(x)\| \leq \varepsilon(x), \quad x \in X$$

$$\varepsilon^*(x) := \sum_{n=0}^{\infty} \Lambda^n \varepsilon(x) < \infty, \quad x \in X.$$

Then, there exists a unique fixed point ψ of \mathcal{T} with

$$\|\varphi(x) - \psi(x)\| \leq \varepsilon^*(x), \quad x \in X.$$

Moreover

$$\psi(x) = \lim_{n \rightarrow \infty} \mathcal{T}^n \varphi(x), \quad x \in X.$$

2. Hyperstability Results of eq (1.3)

The following theorems are the main results in this paper and concern the hyperstability of equation (1.3). Moreover the results that we will obtained in these theorems corresponds the results obtained in [3, 9].

Theorem 2.1. Let X be a normed space, Y be a Banach space, $c \geq 0$ and $p, q, r \in \mathbf{R}$ such that $p + q + r < 0$. If $f : X \rightarrow Y$ satisfies

$$\left\| f\left(\frac{x+y}{2} + z\right) + f\left(\frac{x-y}{2} + z\right) - 2f(z) - f(x) \right\| \leq c \|x\|^p \|y\|^q \|z\|^r \tag{2.1}$$

for all $x, y, z \in X \setminus \{0\}$, then f is a solution of (1.3) on $X \setminus \{0\}$.

Proof. It is sufficient to consider only the case $p + q < 0$. Replacing (x, y, z) by (mx, mx, m) , with $m \in \mathbf{N}_1$, in (2.1), we get

$$\|f((1+m)x) - f(mx) - f(x)\| \leq cm^{p+q} \|x\|^{p+q+r} \tag{2.2}$$

for all $x \in X \setminus \{0\}$. Further put

$$\begin{aligned} \mathcal{T}\xi(x) &:= \xi((1+m)x) - \xi(mx), \quad x \in X \setminus \{0\}, \quad \xi \in Y^{X \setminus \{0\}}, \\ \varepsilon(x) &:= cm^{p+q} \|x\|^{p+q+r}, \quad x \in X \setminus \{0\}. \end{aligned}$$

Then the inequality (2.2) takes the form

$$\|\mathcal{T}f(x) - f(x)\| \leq \varepsilon(x), \quad x \in X \setminus \{0\}.$$

The operator

$$\Lambda\delta(x) := \delta((1+m)x) + \delta(mx), \quad x \in X \setminus \{0\}, \quad \delta \in \mathbf{R}_+^{X \setminus \{0\}}$$

has the form described in **(H3)** with $k = 2$ and $f_1(x) = (1+m)x$, $f_2(x) = mx$, and $L_1(x) = L_2(x) = 1$ for $x \in X \setminus \{0\}$.

Moreover, for every $\xi, \mu \in Y^{X \setminus \{0\}}$ and $x \in X \setminus \{0\}$, we obtain

$$\begin{aligned} \|\mathcal{T}\xi(x) - \mathcal{T}\mu(x)\| &= \|(\xi - \mu)(f_1(x)) - (\xi - \mu)(f_2(x))\| \\ &\leq \|(\xi - \mu)(f_1(x))\| + \|(\xi - \mu)(f_2(x))\| \\ &= \sum_{i=1}^2 L_i(x) \|(\xi - \mu)(f_i(x))\|. \end{aligned}$$

So, **(H2)** is valid. Next, we can find $m_0 \in \mathbf{N}_1$ such that

$$(1 + m)^{p+q+r} + m^{p+q+r} < 1 \text{ for all } m \geq m_0.$$

Therefore, we obtain that

$$\begin{aligned} \varepsilon^*(x) &:= \sum_{n=0}^{\infty} \Lambda^n \varepsilon(x) \\ &= cm^{p+q} \|x\|^{p+q+r} \sum_{n=0}^{\infty} \left((1 + m)^{p+q+r} + m^{p+q+r} \right)^n \\ &= \frac{cm^{p+q} \|x\|^{p+q+r}}{1 - (1 + m)^{p+q+r} - m^{p+q+r}}, x \in X \setminus \{0\}, m \geq m_0 \end{aligned}$$

Thus, according to Theorem 1.3, for each $m \geq m_0$ there exists a unique solution $F_m : X \setminus \{0\} \rightarrow Y$ of the equation

$$F_m(x) = F_m((1 + m)x) - F_m(mx)$$

for all $x \in X \setminus \{0\}$, such that

$$\|f(x) - F_m(x)\| \leq \frac{cm^{p+q} \|x\|^{p+q+r}}{1 - (1 + m)^{p+q+r} - m^{p+q+r}},$$

for all $x \in X \setminus \{0\}$, $m \geq m_0$.

Moreover

$$F_m(x) = \lim_{n \rightarrow \infty} \mathcal{T}^n f(x).$$

We show that

$$\begin{aligned} &\left\| \mathcal{T}^n f\left(\frac{x+y}{2} + z\right) + \mathcal{T}^n f\left(\frac{x-y}{2} + z\right) - 2\mathcal{T}^n f(z) - \mathcal{T}^n f(x) \right\| \leq \\ (2.3) \quad &c \left((1 + m)^{p+q+r} + m^{p+q+r} \right)^n \|x\|^p \|y\|^q \|z\|^r \end{aligned}$$

for all $x, y, z \in X \setminus \{0\}$ and $n \in \mathbf{N}$.

Indeed, if $n = 0$, then (2.3) is simply (2.1). So, fix $n \in \mathbf{N}$ and suppose that (2.3) holds for n . Then

$$\begin{aligned} & \left\| \mathcal{T}^{n+1} f \left(\frac{x+y}{2} + z \right) + \mathcal{T}^{n+1} f \left(\frac{x-y}{2} + z \right) - 2\mathcal{T}^{n+1} f(z) - \mathcal{T}^{n+1} f(x) \right\| = \\ & \left\| \mathcal{T}^n f \left((1+m) \left(\frac{x+y}{2} + z \right) \right) - \mathcal{T}^n f \left(m \left(\frac{x+y}{2} + z \right) \right) \right. \\ & \left. + \mathcal{T}^n f \left((1+m) \left(\frac{x-y}{2} + z \right) \right) - \mathcal{T}^n f \left(m \left(\frac{x-y}{2} + z \right) \right) \right. \\ & \left. - 2\mathcal{T}^n f((m+1)z) + 2\mathcal{T}^n f(mz) - \mathcal{T}^n f((m+1)x) + \mathcal{T}^n f(mx) \right\| \\ & \leq \left\| \mathcal{T}^n f \left((m+1) \left(\frac{x+y}{2} + z \right) \right) + \mathcal{T}^n f \left((m+1) \left(\frac{x-y}{2} + z \right) \right) \right. \\ & \left. - 2\mathcal{T}^n f((1+m)z) - \mathcal{T}^n f(1+m)x \right\| \\ & + \left\| \mathcal{T}^n f \left(m \left(\frac{x+y}{2} + z \right) \right) + \mathcal{T}^n f \left(m \left(\frac{x-y}{2} + z \right) \right) - 2\mathcal{T}^n f(mz) - \mathcal{T}^n f(mx) \right\| \\ & \leq c \left[(m+1)^{p+q+r} + m^{p+q+r} \right]^n \| (1+m)x \|^p \| (1+m)y \|^q \| (1+m)z \|^r \\ & + c \left[(1+m)^{p+q+r} + m^{p+q+r} \right]^n \| mx \|^p \| my \|^q \| mz \|^r \\ & = c \left((1+m)^{p+q+r} + m^{p+q+r} \right)^{n+1} \| x \|^p \| y \|^q \| z \|^r \end{aligned}$$

for all $x, y, z \in X \setminus \{0\}$.

Thus, by induction, we have shown that (2.3) holds for all $x, y, z \in X \setminus \{0\}$ and for all $n \in \mathbf{N}$. Letting $n \rightarrow \infty$ in (2.3), we obtain

$$F_m \left(\frac{x+y}{2} + z \right) + F_m \left(\frac{x-y}{2} + z \right) = F_m(x) + 2F_m(z),$$

for all $x, y, z \in X \setminus \{0\}$. So, we find a sequence $(F_m)_{m \geq m_0}$ satisfies (1.3) on $X \setminus \{0\}$ such that

$$\| f(x) - F_m(x) \| \leq \frac{cm^{p+q} \| x \|^{p+q+r}}{1 - (1+m)^{p+q+r} - m^{p+q+r}}, \quad x \in X \setminus \{0\}, \quad m \geq m_0.$$

It follows, with $m \rightarrow \infty$, that f is a solution of (1.3) on $X \setminus \{0\}$. \square

In a similar way we can prove the following three theorems.

Theorem 2.2. Let X be a normed space, Y be a Banach space, $c \geq 0$ and $p, q, r \in \mathbf{R}$ such that $p + q + r > 1$. If $f : X \rightarrow Y$ satisfies (2.1) then f is a solution of (1.3) on $X \setminus \{0\}$.

Proof. Since $p + q + r > 1$, one of p , $r + q$ must be positive. It is sufficient to consider only the case $q + r > 0$ and $p \in \mathbf{R}$. Replacing (x, y, z) by $(x, \frac{x}{m}, \frac{x}{m})$, with $m \in \mathbf{N}_1$, in (2.1), we get

$$(2.4) \quad \left\| f\left(\frac{3+m}{2m}x\right) + f\left(\frac{1+m}{2m}x\right) - 2f\left(\frac{x}{m}\right) - f(x) \right\| \leq \frac{c}{m^{q+r}} \|x\|^{p+q+r}$$

for all $x \in X \setminus \{0\}$. Define operators $\mathcal{T} : Y^{X \setminus \{0\}} \rightarrow Y^{X \setminus \{0\}}$ and $\Lambda : \mathbf{R}_+^{X \setminus \{0\}} \rightarrow \mathbf{R}_+^{X \setminus \{0\}}$ by

$$\mathcal{T}\xi(x) := \xi\left(\frac{3+m}{2m}x\right) + \xi\left(\frac{1+m}{2m}x\right) - 2\xi\left(\frac{x}{m}\right), \quad x \in X \setminus \{0\}, \quad \xi \in Y^{X \setminus \{0\}},$$

$$\Lambda\delta(x) := \delta\left(\frac{3+m}{2m}x\right) + \delta\left(\frac{1+m}{2m}x\right) + 2\delta\left(\frac{x}{m}\right), \quad x \in X \setminus \{0\}, \quad \delta \in \mathbf{R}_+^{X \setminus \{0\}}.$$

Then it is easily seen that Λ has the form described in **(H3)** with $k = 3$ and

$$f_1(x) = \frac{3+m}{2m}x, \quad f_2(x) = \frac{1+m}{2m}x, \quad f_3(x) = \frac{x}{m}, \quad L_1(x) = L_2(x) = 1$$

$$\text{and } L_3(x) = 2, \quad x \in X \setminus \{0\}.$$

Then the inequality (2.4) take the form

$$\|\mathcal{T}f(x) - f(x)\| \leq \varepsilon(x), \quad x \in X \setminus \{0\},$$

with

$$\varepsilon(x) := \frac{c}{m^{q+r}} \|x\|^{p+q+r}.$$

Moreover, for every $\xi, \mu \in Y^{X \setminus \{0\}}$ and $x \in X \setminus \{0\}$, we have

$$\|\mathcal{T}\xi(x) - \mathcal{T}\mu(x)\| = \|(\xi - \mu)(f_1(x)) + (\xi - \mu)(f_2(x)) - 2(\xi - \mu)(f_3(x))\|$$

$$\leq \sum_{i=1}^3 L_i(x) \|(\xi - \mu)(f_i(x))\|.$$

So, **(H2)** is valid. We observe that there exists $m_0 \in \mathbf{N}_1$ such that

$$\left[\frac{3+m}{2m}\right]^{p+q+r} + \left[\frac{1+m}{2m}\right]^{p+q+r} + \frac{2}{m^{p+q+r}} < 1 \quad \text{for all } m \geq m_0.$$

We have

$$\begin{aligned} \varepsilon^*(x) &= \sum_{n=0}^{\infty} \Lambda^n \varepsilon(x) \\ &= \frac{c}{m^{q+r}} \|x\|^{p+q+r} \sum_{n=0}^{\infty} \left(\left[\frac{3+m}{2m} \right]^{p+q+r} + \left[\frac{1+m}{2m} \right]^{p+q+r} + \frac{2}{m^{p+q+r}} \right)^n \\ &= \frac{c \|x\|^{p+q+r}}{m^{q+r} \left(1 - \left[\frac{3+m}{2m} \right]^{p+q+r} - \left[\frac{1+m}{2m} \right]^{p+q+r} - \frac{2}{m^{p+q+r}} \right)} \end{aligned}$$

for all $x \in X \setminus \{0\}$ and $m \geq m_0$.

Thus, according to Theorem 1.3, for each $m \geq m_0$ there exists a unique solution $F_m : X \setminus \{0\} \rightarrow Y$ of the equation

$$F_m(x) = F_m\left(\frac{3+m}{2m}x\right) + F_m\left(\frac{1+m}{2m}x\right) - 2F_m\left(\frac{x}{m}\right),$$

for all $x \in X \setminus \{0\}$, such that

$$\|f(x) - F_m(x)\| \leq \frac{c \|x\|^{p+q+r}}{m^{q+r} \left(1 - \left[\frac{3+m}{2m} \right]^{p+q+r} - \left[\frac{1+m}{2m} \right]^{p+q+r} - \frac{2}{m^{p+q+r}} \right)},$$

for all $x \in X \setminus \{0\}$ and $m \geq m_0$.

Moreover

$$F_m(x) = \lim_{n \rightarrow \infty} \mathcal{T}^n f(x), \quad x \in X \setminus \{0\}.$$

We show that

$$\begin{aligned} &\left\| \mathcal{T}^n f\left(\frac{x+y}{2} + z\right) + \mathcal{T}^n f\left(\frac{x-y}{2} + z\right) - 2\mathcal{T}^n f(z) - \mathcal{T}^n f(x) \right\| \leq \\ (2.5) \quad &c \left(\left[\frac{3+m}{2m} \right]^{p+q+r} + \left[\frac{1+m}{2m} \right]^{p+q+r} + \frac{2}{m^{p+q+r}} \right)^n \|x\|^p \|y\|^q \|z\|^r \end{aligned}$$

for all $x, y, z \in X \setminus \{0\}$ and $n \in \mathbf{N}$.

Indeed, if $n = 0$, then (2.3) is simply (2.1). So, fix $n \in \mathbf{N}$ and suppose that (2.3) holds for n . Then

$$\begin{aligned}
& \left\| \mathcal{T}^{n+1} f\left(\frac{x+y}{2} + z\right) + \mathcal{T}^{n+1} f\left(\frac{x-y}{2} + z\right) - 2\mathcal{T}^{n+1} f(z) - \mathcal{T}^{n+1} f(x) \right\| \leq \\
& \left\| \mathcal{T}^n f\left(\frac{3+m}{2m}\left(\frac{x+y}{2} + z\right)\right) + \mathcal{T}^n f\left(\frac{3+m}{2m}\left(\frac{x-y}{2} + z\right)\right) - 2\mathcal{T}^n f\left(\frac{3+m}{2m}z\right) - \mathcal{T}^n f\left(\frac{3+m}{2m}x\right) \right\| \\
& + \left\| \mathcal{T}^n f\left(\frac{1+m}{2m}\left(\frac{x+y}{2} + z\right)\right) + \mathcal{T}^n f\left(\frac{1+m}{2m}\left(\frac{x-y}{2} + z\right)\right) - 2\mathcal{T}^n f\left(\frac{1+m}{2m}z\right) - \mathcal{T}^n f\left(\frac{1+m}{2m}x\right) \right\| \\
& + \left\| 2\mathcal{T}^n f\left(\frac{1}{m}\left(\frac{x+y}{2} + z\right)\right) + \mathcal{T}^n f\left(\frac{1}{m}\left(\frac{x-y}{2} + z\right)\right) - 2\mathcal{T}^n f\left(\frac{z}{m}\right) - \mathcal{T}^n f\left(\frac{x}{m}\right) \right\| \\
& \leq c\left[\left(\frac{3+m}{2m}\right)^{p+q+r} + \left(\frac{1+m}{2m}\right)^{p+q+r} + \frac{2}{m^{p+q+r}}\right]^n \left\| \frac{3+m}{2m}x \right\|^p \left\| \frac{3+m}{2m}y \right\|^q \left\| \frac{3+m}{2m}z \right\|^r \\
& + c\left[\left(\frac{3+m}{2m}\right)^{p+q+r} + \left(\frac{1+m}{2m}\right)^{p+q+r} + \frac{2}{m^{p+q+r}}\right]^n \left\| \frac{1+m}{2m}x \right\|^p \left\| \frac{1+m}{2m}y \right\|^q \left\| \frac{1+m}{2m}z \right\|^r \\
& + 2c\left[\left(\frac{3+m}{2m}\right)^{p+q+r} + \left(\frac{1+m}{2m}\right)^{p+q+r} + \frac{2}{m^{p+q+r}}\right]^n \left\| \frac{x}{m} \right\|^p \left\| \frac{y}{m} \right\|^q \left\| \frac{z}{m} \right\|^r \\
& = c\left(\left(\frac{3+m}{2m}\right)^{p+q+r} + \left(\frac{1+m}{2m}\right)^{p+q+r} + \frac{2}{m^{p+q+r}}\right)^{n+1} \|x\|^p \|y\|^q \|z\|^r
\end{aligned}$$

for all $x, y, z \in X \setminus \{0\}$.

Thus, by induction, we have shown that (2.5) holds for all $x, y, z \in X \setminus \{0\}$ and for all $n \in \mathbf{N}$. Letting $n \rightarrow \infty$ in (2.5), we obtain

$$F_m\left(\frac{x+y}{2} + z\right) + F_m\left(\frac{x-y}{2} + z\right) = F_m(x) + 2F_m(z),$$

for all $x, y, z \in X \setminus \{0\}$. So, we have a sequence $(F_m)_{m \geq m_0}$ satisfies (1.3) on $X \setminus \{0\}$ such that

$$\|f(x) - F_m(x)\| \leq \frac{c \|x\|^{p+q+r}}{m^{r+q} \left(1 - \left[\frac{3+m}{2m}\right]^{p+q+r} - \left[\frac{1+m}{2m}\right]^{p+q+r} - \frac{2}{m^{p+q+r}}\right)}$$

for all $x \in X \setminus \{0\}$ and $m \geq m_0$. It follows, with $m \rightarrow \infty$, that f is a solution of (1.3) on $X \setminus \{0\}$. \square

Theorem 2.3. Let X be a normed space, Y be a Banach space, $c \geq 0$ and $p, q, r \in \mathbf{R}$ such that $0 < p + q + r < 1$. If $f : X \rightarrow Y$ satisfies (2.1) then f is a solution of (1.3) on $X \setminus \{0\}$.

Proof. Since $p + q + r > 0$, one of $p, r + q$ must be positive, it is sufficient to consider only the case $p > 0$. Replacing (x, y, z) by $(\frac{2x}{m}, 2x, x)$, with $m \in \mathbf{N}_1$, in (2.1), we get

$$(2.6) \left\| \frac{1}{2}f\left(\frac{1+2m}{m}x\right) + \frac{1}{2}f\left(\frac{x}{m}\right) - \frac{1}{2}f\left(\frac{2x}{m}\right) - f(x) \right\| \leq \frac{c2^{p+q-1}}{m^p} \|x\|^{p+q+r}$$

for all $x \in X \setminus \{0\}$. Define operators $\mathcal{T} : Y^{X \setminus \{0\}} \rightarrow Y^{X \setminus \{0\}}$ and $\Lambda : \mathbf{R}_+^{X \setminus \{0\}} \rightarrow \mathbf{R}_+^{X \setminus \{0\}}$ by

$$\mathcal{T}\xi(x) := \frac{1}{2}\xi\left(\frac{1+2m}{m}x\right) + \frac{1}{2}\xi\left(\frac{x}{m}\right) - \frac{1}{2}\xi\left(\frac{2x}{m}\right), \quad x \in X \setminus \{0\}, \quad \xi \in Y^{X \setminus \{0\}},$$

$$\Lambda\delta(x) := \frac{1}{2}\delta\left(\frac{1+2m}{m}x\right) + \frac{1}{2}\delta\left(\frac{x}{m}\right) + \frac{1}{2}\delta\left(\frac{2x}{m}\right), \quad x \in X \setminus \{0\}, \quad \delta \in \mathbf{R}_+^{X \setminus \{0\}}.$$

Then it is easily seen that Λ has the form described in **(H3)** with $k = 3$ and

$$f_1(x) = \frac{1+2m}{m}x, \quad f_2(x) = \frac{x}{m}, \quad f_3(x) = \frac{2x}{m}, \quad \text{and}$$

$$L_1(x) = L_2(x) = L_3(x) = \frac{1}{2}, \quad x \in X \setminus \{0\}.$$

Then the inequality (2.6) take the form

$$\|\mathcal{T}f(x) - f(x)\| \leq \varepsilon(x), \quad x \in X \setminus \{0\},$$

with

$$\varepsilon(x) := \frac{c2^{p+q-1}}{m^p} \|x\|^{p+q+r}.$$

Moreover, for every $\xi, \mu \in Y^{X \setminus \{0\}}$ and $x \in X \setminus \{0\}$, we have

$$\begin{aligned} & \|\mathcal{T}\xi(x) - \mathcal{T}\mu(x)\| \\ &= \frac{1}{2} \|(\xi - \mu)(f_1(x)) + (\xi - \mu)(f_2(x)) - (\xi - \mu)(f_3(x))\| \\ &\leq \sum_{i=1}^3 L_i(x) \|(\xi - \mu)(f_i(x))\|. \end{aligned}$$

So, **(H2)** is valid. We observe that there exists $m_0 \in \mathbf{N}_1$ such that

$$\frac{1}{2} \left[\frac{1+2m}{m} \right]^{p+q+r} + \frac{1}{2} \left[\frac{1}{m} \right]^{p+q+r} + \frac{1}{2} \left(\frac{2}{m} \right)^{p+q+r} < 1 \quad \text{for all } m \geq m_0.$$

We have

$$\begin{aligned} \varepsilon^*(x) &:= \sum_{n=0}^{\infty} \Lambda^n \varepsilon(x) \\ &= \frac{c2^{p+q-1}}{m^p} \|x\|^{p+q+r} \sum_{n=0}^{\infty} \left(\frac{1}{2} \left[\frac{1+2m}{m} \right]^{p+q+r} + \frac{1}{2} \left[\frac{1}{m} \right]^{p+q+r} + \frac{1}{2} \left(\frac{2}{m} \right)^{p+q+r} \right)^n \\ &= \frac{c2^{p+q-1} \|x\|^{p+q+r}}{m^p \left(1 - \frac{1}{2} \left[\frac{1+2m}{m} \right]^{p+q+r} - \frac{1}{2} \left[\frac{1}{m} \right]^{p+q+r} - \frac{1}{2} \left(\frac{2}{m} \right)^{p+q+r} \right)} \end{aligned}$$

for all $x \in X \setminus \{0\}$ and $m \geq m_0$.

The rest of the proof is similar to the proof of Theorem 2.2. \square

Remark 1. Let X be a normed space, Y be a Banach space, $c \geq 0$ and $p, q, r > 0$ such that $p + q + r \neq 1$. If $f : X \rightarrow Y$ satisfies (2.1) on X , then f is a solution of (1.3) on X .

Theorem 2.4. Let X be a normed space, Y be a Banach space, $c \geq 0$ and $p < 0$. If $f : X \rightarrow Y$ satisfies

$$\left\| f\left(\frac{x+y}{2} + z\right) + f\left(\frac{x-y}{2} + z\right) - f(x) - 2f(z) \right\| \leq c(\|x\|^p + \|y\|^p + \|z\|^p) \quad (2.7)$$

for all $x, y, z \in X \setminus \{0\}$, then f is a solution of (1.3) on $X \setminus \{0\}$.

Proof. Replacing (x, y, z) by $(mx, mx, (1-m)x)$, where $m \in \mathbf{N}_1$, in (2.7), we get

$$(2.8) \quad \|f((1-m)x) + f(mx) - f(x)\| \leq c(2m^p + (m-1)^p) \|x\|^p$$

for all $x \in X \setminus \{0\}$. Further put

$$\begin{aligned} \mathcal{T}\xi(x) &= \xi((1 - m)x) + \xi(mx), \quad x \in X \setminus \{0\}, \quad \xi \in Y^{X \setminus \{0\}}, \\ \varepsilon(x) &:= (2m^p + (m - 1)^p) \|x\|^p. \end{aligned}$$

Inequality (2.8) takes the following form

$$\|\mathcal{T}f(x) - f(x)\| \leq \varepsilon(x), \quad x \in X \setminus \{0\}.$$

The operator

$$\Lambda\delta(x) = \delta((1 - m)x) + \delta(mx), \quad x \in X \setminus \{0\}, \quad \delta \in \mathbf{R}_+^{X \setminus \{0\}}$$

has the form described in **(H3)** with $k = 2$ and

$$f_1(x) = (1 - m)x, \quad f_2(x) = mx, \quad \text{and} \quad L_1(x) = L_2(x) = 1, \quad x \in X \setminus \{0\}$$

Moreover, for every $\xi, \mu \in Y^{X \setminus \{0\}}$ and $x \in X \setminus \{0\}$, we have

$$\begin{aligned} &\|\mathcal{T}\xi(x) - \mathcal{T}\mu(x)\| \\ &= \|(\xi - \mu)((1 - m)x) - (\xi - \mu)(mx)\| \\ &\leq \sum_{i=1}^2 L_i(x) \|(\xi - \mu)(f_i(x))\|. \end{aligned}$$

So, **(H2)** is valid. Now, we can find $m_0 \in \mathbf{N}_1$ such that

$$(m - 1)^p + m^p < 1 \quad \text{for all } m \geq m_0.$$

Therefore, we obtain that

$$\begin{aligned} \varepsilon^*(x) &:= \sum_{n=0}^{\infty} \Lambda^n \varepsilon(x) \\ &= \frac{c(2m^p + (m-1)^p) \|x\|^p}{1 - (m-1)^p - m^p} \end{aligned}$$

for all $x \in X \setminus \{0\}$ and $m \geq m_0$. The rest of the proof is similar to the proof of Theorem 2.1. \square

Corollary 2.4.1. Let X be a normed space and Y be a Banach space. Let $H : X^3 \rightarrow Y$ be a mapping such that $H(x_0, y_0, z_0) \neq 0$ for some $x_0, y_0, z_0 \in X$ and

$$(2.9) \quad \|H(x, y, z)\| \leq c \|x\|^p \|y\|^q \|z\|^r,$$

or

$$(2.10) \quad \|H(x, y, z)\| \leq c(\|x\|^p + \|y\|^p + \|z\|^p),$$

for all $x, y, z \in X \setminus \{0\}$, where $c \geq 0$ and $p, q, r \in \mathbf{R}$. Assume that the numbers p, q, r satisfy one of the following conditions:

$$(i) \ p + q + r < 0 \quad (ii) \ 0 < p + q + r < 1 \quad (iii) \ p + q + r > 1$$

in the case (2.9) and

$$(iv) \ p < 0 \text{ in the case (2.10)}$$

Then the functional equation

$$h\left(\frac{x+y}{2} + z\right) + h\left(\frac{x-y}{2} + z\right) = H(x, y, z) + h(x) + 2h(z), \quad x, y, z \in X \setminus \{0\}$$

$$(2.11)$$

has no solution in the class of functions $h : X \rightarrow Y$.

Proof. Suppose that $h : X \rightarrow Y$ is a solution to (2.11). Then (2.1) or (2.7) holds, and consequently, according to the above theorems, h is Jensen on $X \setminus \{0\}$, which means that $H(x_0, y_0, z_0) = 0$. This is a contradiction. \square

Remark 2. Let X be a normed space and Y be a Banach space. Let $H : X^3 \rightarrow Y$ be a mapping such that $H(x_0, y_0, z_0) \neq 0$ for some $x_0, y_0, z_0 \in X$ and

$$\|H(x, y, z)\| \leq c \|x\|^p \|y\|^q \|z\|^r,$$

for all $x, y, z \in X$, where $c \geq 0$ and $p, q, r > 0$ with $p + q + r \neq 1$.

Then the functional equation

$$h\left(\frac{x+y}{2} + z\right) + h\left(\frac{x-y}{2} + z\right) = H(x, y, z) + h(x) + 2h(z), \quad x, y, z \in X$$

has no solution in the class of functions $h : X \rightarrow Y$.

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