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# Hypo-$k$-Totally Magic Cordial Labeling of Graphs

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**Abstract**

*A graph $G$ is said to be hypo-k-totally magic cordial if $G-\{v\}$ is $k$-totally magic cordial for each vertex $v$ in $V(G)$. In this paper, we establish that cycle, complete graph, complete bipartite graph and wheel graph admit hypo-k-totally magic cordial labeling and some families of graphs do not admit hypo-k-totally magic cordial labeling.*

**Keywords :** *$k$-totally magic cordial labeling, hypo-k-totally magic cordial labeling, hypo-k-totally magic cordial graph,complete graph, complete bipartite graph,wheel graph.*

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## 1. Introduction

Let $G = (V(G), E(G))$ (or simply $G = (V, E)$) be a simple, finite and undirected graph of order $|V| = p$ and size $|E| = q$. For graph theoretic notations and terminology we refer [3]. The notion of cordial labeling was due to Cahit [1]. A binary vertex labeling $f : V(G) \rightarrow \{0, 1\}$ induces an edge labeling $f^* : E(G) \rightarrow \{0, 1\}$ defined by $f^*(uv) = |f(u) - f(v)|$. Such labeling is called cordial if the conditions $|v_f(0) - v_f(1)| \leq 1$ and $|e_{f^*}(0) - e_{f^*}(1)| \leq 1$ are satisfied, where $v_f(i)$ and $e_{f^*}(i)(i = 0, 1)$ are the number of vertices and edges with label $i$ respectively. A graph is called cordial if it admits cordial labeling. In [2] Cahit introduced totally magic cordial labeling (TMC) based on cordial labeling and generalized it into $k$-totally magic cordial labeling.

A graph $G$ is said to have totally magic cordial (TMC or 2-TMC) labeling with constant $C$ if there exists a mapping $f : V(G) \cup E(G) \rightarrow \{0, 1\}$ such that $f(a) + f(b) + f(ab) \equiv C(\text{mod } 2)$ for all $ab \in E(G)$ and $|n_f(0) - n_f(1)| \leq 1$, where $n_f(i)(i = 0, 1)$ is the sum of the number of vertices and edges with label $i$.

A graph $G$ is said to have a $k$-totally magic cordial ($k$-TMC) labeling with constant $C$ if there exists a mapping $f : V(G) \cup E(G) \rightarrow Z_k$ such that $f(a) + f(b) + f(ab) \equiv C \pmod{\text{k}}$ for all $ab \in E(G)$ provided for $i \neq j$, $|n_f(i) - n_f(j)| \leq 1$, where $n_f(i)$ $(i = 0, 1, 2, ..., k-1)$ is the sum of the number of vertices and edges with label $i$. A graph is called $k$-totally magic cordial if it admits $k$-totally magic cordial labeling. Further results on totally magic cordial labelings and $k$-totally magic cordial labelings were discussed in [4]-[8].

Let $G$ be a graph with vertex set $V$ and edge set $E$. Let $v \in V$. The subgraph of $G$ obtained by removing the vertex $v$ and all the edges incident with $v$ is called the subgraph obtained by the removal of the vertex $v$ and is denoted by $G - \{v\}$ .

Motivated by the concept of $k$-TMC labeling in [2], we define a new labeling called hypo-$k$-TMC labeling as follows : A graph $G$ is said to be hypo-$k$-TMC if $G - \{v\}$ is $k$-TMC for each vertex $v$ in $V(G)$. In this paper we establish that cycle, complete graph, complete bipartite graph and wheel graph admit hypo-$k$-totally magic cordial labeling and some families of graphs do not admit hypo-$k$-totally magic cordial labeling.

We use the following theorems and definitions in the subsequent section:

**Theorem 1.1.** *[8] Let $G$ be an odd graph with $p + q \equiv 2$ (mod 4). Then $G$ is not TMC.*

**Theorem 1.2.** *[8] The fan graph $F_n$ is TMC for $n \geq 2$.*

**Theorem 1.3.** *[7] Let $G$ be an odd graph with $p + q \equiv k \pmod{2k}$ and $k \equiv 2 \pmod 4$. Then $G$ is not $k$- TMC.*

**Theorem 1.4.** *[7] The complete graph $K_n (n \geq 3)$ is $n$-TMC.*

**Theorem 1.5.** *[7] The complete bipartite graph $K_{m,n}$ $(m \geq n \geq 2)$ is both $m$-TMC and $n$-TMC.*

**Theorem 1.6.** *[7] The star graph $S_n$ is $n$-TMC for all $n \geq 1$.*

**Definition 1.** *The helm graph $H_n$ is obtained from a wheel by attaching a pendant edge at each vertex of the $n$-cycle.*

**Definition 2.** *The closed helm graph $CH_n$ is obtained from a helm $H_n$ by joining each pendant vertex to form a cycle.*

**Definition 3.** *The web graph $Wb_n$ is obtained from a closed helm $CH_n$ by adding a pendant edge to each vertex of outer cycle.*

**Definition 4.** *The friendship graph $T_n (n \geq 2)$ is the one-point union of $n$ cycles of length 3.*

**Definition 5.** *The graph $S_{m,n}$ denotes a star with $m$ spokes in which each spoke is a path of length $n$.*

**Definition 6.** *The bistar graph $B_{m,n}$ is obtained from $K_2$ by joining $m$ pendant edges to one end of $K_2$ and $n$ pendant edges to the other end of $K_2$. The edge of $K_2$ is called central edge of $B_{m,n}$ and the vertices of $K_2$ are called central vertices of $B_{m,n}$.*

**Definition 7.** *The subdivision graph $S(G)$ is obtained from $G$ by subdividing each edge of $G$.*

**Definition 8.** *Let $K_{1,n_1}, K_{1,n_2}, ..., K_{1,n_k}$ be a family of disjoint stars with the vertex-sets $V(K_{1,n_i}) = \{c_i, a_{i1}, ..., a_{in_i}\}$ and $deg(c_i) = n_i$, $1 \leq i \leq k$. A banana tree $BT(n_1, n_2, ..., n_k)$ is a tree obtained by adding a new vertex $a$ and joining it to $a_{11}, a_{21}, ..., a_{k1}$.*

## 2. Main Results

**Theorem 2.1.** *The cycle $C_n (n \geq 3)$ is hypo-$(n-1)$-TMC.*

**Proof.** Suppose we remove any vertex from the cycle $C_n$ we get a path of length $n-1$. Let $v_1, v_2, ..., v_{n-1}$ be the successive vertices of $P_{n-1}$.

Define $f : V(P_{n-1}) \cup E(P_{n-1}) \to Z_{n-1}$ as follows: when $i$ is odd, $f(v_i) = \frac{i-1}{2}$,

and when $i$ is even, $f(v_i) = \begin{cases} \frac{i+n-1}{2} - 1 & \text{if } n \text{ is odd },\\ \frac{i+n}{2} - 1 & \text{if } n \text{ is even },\end{cases}$

Also $f(v_iv_{i+1}) = \begin{cases} \frac{n-2i+1}{2} \pmod{(n-1)} & \text{if } n \text{ is odd },\\ \frac{n-2i}{2} \pmod{(n-1)} & \text{if } n \text{ is even }.\end{cases}$

Clearly, $f(a) + f(b) + f(ab) \equiv 0 \pmod{(n-1)}$. Moreover, if $n$ is even, $n_f(i) = \begin{cases} 1 & \text{if } i = \frac{n}{2},\\ 2 & \text{if } i \neq \frac{n}{2},\end{cases}$ and if $n$ is odd, $n_f(i) = \begin{cases} 1 & \text{if } i = \frac{n+1}{2},\\ 2 & \text{if } i \neq \frac{n+1}{2}.\end{cases}$ Thus for $i \neq j$ and $0 \leq i, j \leq n-2$, $|n_f(i) - n_f(j)| \leq 1$ . Hence $P_{n-1}$ is $(n-1)$-TMC. Therefore, the cycle $C_n (n \geq 3)$ is hypo-$(n-1)$-TMC. □

**Theorem 2.2.** *The complete graph $K_n (n \geq 3)$ is hypo-$(n-1)$-TMC.*

**Proof.** The subgraph $K_{n-1}$ is obtained by removing any vertex from $K_n$. According to Theorem 1.4, $K_{n-1}$ is $(n-1)$-TMC. Hence the complete graph $K_n (n \geq 3)$ is hypo-$(n-1)$-TMC. □

**Theorem 2.3.** *The wheel graph $W_n$ is hypo-$n$-TMC for all odd $n \geq 3$.*

**Proof.** Let $v$ be the central vertex and $\{v_1, v_2, ..., v_n\}$ be the set of degree 3 vertices. Assume that $n$ is odd. Clearly, $W_n - \{v\} = C_n$. Define $f : V(C_n) \cup E(C_n) \to Z_n$ as follows: $f(v_i) = i - 1$ and for $1 \leq i \leq n-1$, $f(v_iv_{i+1}) = 1 - 2i \pmod{n}$ and $f(v_nv_1) = 1 - 2n \pmod{n}$. Thus, $f(v_i) + f(v_{i+1}) + f(v_iv_{i+1}) \equiv 0 \pmod{n}$ and $n_f(i) = 2$ for all $1 \leq i \leq n$. Therefore, $C_n$ is $n$-TMC.

The fan graph $F_{n-1}$ is obtained by removing any vertex from the cycle $C_n$. Let $u_1, u_2, ..., u_{n-1}$ be the successive vertices of $F_{n-1}$. Define $g : V(F_{n-1}) \cup E(F_{n-1}) \to Z_n$ as follows:

$g(v) = 0$, $g(u_i) = 2i \pmod{n}$, $g(vu_i) = n - i$ and for $1 \leq i \leq n-1$, $g(u_iu_{i+1}) = n - 4i - 2 \pmod{n}$. Clearly, $g(u_i) + g(u_{i+1}) + g(u_iu_{i+1}) \equiv 0 \pmod{n}$ and

$$n_g(i) = \begin{cases} 2 & \text{if } i = 0, 2, n-2,\\ 3 & \text{if } i = 1, 3, ..., n-3, n-1.\end{cases}$$

Thus $F_{n-1}$ is $n$-TMC. Hence, the wheel graph $W_n$ is hypo-$n$-TMC for all odd $n \geq 3$. □

**Theorem 2.4.** *If* $n \equiv 2 \pmod 4$, *then the closed helm graph* $CH_n$ *is not hypo-*$n$*-TMC.*

**Proof.** Assume that $n \equiv 2 \pmod 4$. Let $u$ be the central vertex of the closed helm $CH_n$. Let $G = CH_n - \{u\}$. Clearly, $|V(G)| + |E(G)| = 5n$. Thus by Theorem 1.3, $G$ is not $n$-TMC. Hence, the closed helm graph $CH_n$ is not hypo-$n$-TMC. □

**Theorem 2.5.** *If* $n \equiv 2 \pmod 4$, *then the web graph* $Wb_n$ *is not hypo-*$n$*-TMC.*

**Proof.** Let $u$ be the central vertex of the web graph $Wb_n$. Assume that $n \equiv 2 \pmod 4$. Let $G = Wb_n - \{u\}$. Clearly, $|V(G)| + |E(G)| = 7n$. Thus by Theorem 1.3, $G$ is not $n$-TMC. Hence, the web graph $Wb_n$ is not hypo-$n$-TMC. □

**Theorem 2.6.** *The friendship graph* $T_n (n \geq 2)$ *is hypo-2-TMC if and only if* $n \not\equiv 2 \pmod 4$.

**Proof.** Assume that $n \equiv 2 \pmod 4$. Let $V = \{u, u_i^1, u_i^2 | 1 \leq i \leq n\}$ be the vertex set and $E = \{uu_i^1, u_i^1 u_i^2, u_i^2 u | 1 \leq i \leq n\}$ be the edge set of $T_n$. The subgraph $nP_2$ obtained by removing the central vertex $u$ from the graph $T_n$ is an odd graph with $p + q = 3n$. Clearly, $3n \equiv 2 \pmod 4$ for $n \equiv 2 \pmod 4$. Thus by Theorem 1.1, the graph $nP_2$ is not 2-TMC. Hence, the friendship graph $T_n (n \geq 2)$ is not hypo-2-TMC when $n \equiv 2 \pmod 4$.

Suppose $n \not\equiv 2 \pmod 4$, label the vertices of $n - \lceil \frac{n}{4} \rceil$ copies of $P_n$ with 0 and the edges with 1 and the vertices and the edges of the remaining $\lceil \frac{n}{4} \rceil$ copies of $P_n$ with 1. Clearly, $C = 1$ and the difference between the sum of the number of vertices and edges labeled with 0 and the sum of the number of vertices and edges labeled with 1 is atmost 1. Thus $T_n - \{u\} = nP_2$ is 2-TMC. Again, let $T_n - \left\{u_k^j\right\} = G$. Choose $k$ and $j$ arbitrarily as $k = n$ and $j = 2$. Define $g : V(G) \cup E(G) \to Z_2$ as follows: $g(u_k^1) = 1$, $g(uu_k^1) = 0$ and $g(u) = 0$, $g(u_i^1) = g(u_i^2) = 0$, $g(uu_i^1) = g(uu_i^2) = g(u_i^1 u_i^2) = 1$ for $1 \leq i \leq \left\lceil \frac{n-1}{2} \right\rceil$ and $g(u_i^1) = 1$, $g(u_i^2) = 0$, $g(uu_i^1) = 0$, $g(uu_i^2) = 1$ and $g(u_i^1 u_i^2) = 0$ for $\left\lceil \frac{n-1}{2} \right\rceil < i \leq n - 1$ . If $n$ is even, $n_f(0) = n_f(1)$ and if $n$

is odd, $n_f(0) = n_f(1) + 1$ with $C = 1$. Thus $G$ is 2-TMC and hence the friendship graph $T_n (n \geq 2)$ is hypo-2-TMC. □

**Theorem 2.7.** *The complete bipartite graph $K_{m,n}$ is hypo-$(m-1)$-TMC as well as hypo-$(n-1)$-TMC.*

**Proof.** Proof follows from Theorem 1.5. □

**Theorem 2.8.** *The graph $S_{2n,2}$ is hypo-2-TMC if and only if $n$ is even.*

**Proof.** Let $V = \left\{u, u_j^1, u_j^2 | 1 \leq j \leq 2n\right\}$ and $E = \left\{uu_j^1, u_j^1 u_j^2 | 1 \leq j \leq 2n\right\}$ be the vertex set and the edge set of the graph $S_{2n,2}$ respectively. Assume that $n$ is odd. The subgraph $2nP_2$ obtained by removing the apex $u$ from the graph $S_{2n,2}$ is an odd graph with $p + q = 6n$. We can easily verify that $6n \equiv 2 \pmod 4$. Thus by Theorem 1.1, the graph $2nP_2$ is not 2-TMC. Hence, the graph $S_{2n,2}$ is not hypo-2-TMC when $n$ is odd.

Assume that $n$ is even. Define

$$f : 2nP_2 \to \{0, 1\} \text{ by } f(u_j^1) = f(u_j^2) =$$

$$\begin{cases} 0 & \text{if } \ j \not\equiv 0 \ (mod\ 4), \\ 1 & \text{if } \ j \equiv 0 \ (mod\ 4) \end{cases}$$

and $f(u_j^1 u_j^2) = 1$ for all $i \leq j \leq 2n$. Clearly, $n_f(0) = n_f(1)$ and $C = 1$. Thus $S_{2n,2} - \{u\}$ is 2-TMC. Again for any $j = k$, $S_{2n,2} - \{u_k^1\} = S_{2n-1,2} \cup \{u_k^2\}$. We label the vertices of $S_{2n-1,2}$ with 0 and the edges with 1 and also label the vertex $u_k^2$ with 1. We find that $n_f(0) = n_f(1)$ and $C = 1$. Thus $S_{2n,2} - \{u_k^1\}$ is also 2-TMC. Also, label the vertices and the edges of $S_{2n,2} - \{u_k^2\}$ with 0 and 1 respectively, we find that $n_f(0) = n_f(1) + 1$ and $C = 1$. Thus $S_{2n,2} - \{u_k^2\}$ is also 2-TMC. Hence, the graph $S_{2n,2}$ is hypo-2-TMC. □

**Theorem 2.9.** *If $m$ and $n$ are odd, then the graph $\langle B_{m,n} : u \rangle$ obtained by the subdivision of the central edge of $B_{m,n}$ with a vertex $u$, is not hypo-2-TMC.*

**Proof.** Let $G = \langle B_{m,n} : u \rangle$. Assume that $m$ and $n$ are odd. Clearly, $G - \{u\} = K_{1,m} \cup K_{1,n}$ with $p + q = 2m + 2n + 2$. We can easily verify that $p + q \equiv 2 \pmod 4$. Thus by Theorem 1.3, $G - \{u\}$ is not 2-TMC. Hence, the graph $\langle B_{m,n} : u \rangle$ is not hypo-2-TMC. □

**Corollary 2.10.** *If $m$ and $n$ are even, then the graph $\langle B_{m,n} : u\rangle$ obtained by the subdivision of the central edge of $B_{m,n}$ by three vertices, is not hypo-2-TMC.*

**Theorem 2.11.** *If $k \equiv 2 \pmod 4$, and $n_i$ is odd for $1 \le i \le k$ such that $n_1 + n_2 + \cdots + n_k \equiv 0 \pmod k$, then the banana tree $BT(n_1, n_2, ..., n_k)$ is not hypo-k-TMC.*

**Proof.** Let $k \equiv 2 \pmod 4$. Assume that $n_i$ is odd for $1 \le i \le k$. Let $G = BT(n_1, n_2, ..., n_k)$. Now, $G - \{a\} = K_{1,n_1} \cup K_{1,n_2} \cup ... \cup K_{1,n_k}$ with $p + q = 2(n_1 + n_2 + ... + n_k) + k$. Since $n_1, n_2, ..., n_k$ are odd and $n_1 + n_2 + \cdots + n_k \equiv 0 \pmod k$, degree of the vertices of $G - \{a\}$ are odd. We can easily verify that $p + q \equiv k \pmod{2k}$. Thus, $G - \{a\}$ is not $k$-TMC. Hence, the banana graph $BT(n_1, n_2, ..., n_k)$ is not hypo-$k$-TMC. □

**Theorem 2.12.** *The graph $S_n + K_1$ is hypo-n-TMC for all $n \ge 1$.*

**Proof.** Let $V(S_n) = \{v, v_1, v_2, ..., v_n\}$, $E(S_n) = \{vv_i | 1 \le i \le n\}$ and $u$ be the vertex of $K_1$. We remove the vertex $u$ or $v$ from $S_n + K_1$, then the resultant graph is the star graph $S_n$. By Theorem 1.6, $S_n$ is $n$-TMC for all $n \ge 1$. Let $G = S_n + K_1 - \{v_i\}$ for any $i$, $1 \le i \le n$. Let $u_1, u_2, ..., u_{n-1}$ be the successive vertices $v_i$ of $G$. Define $g : V(G) \cup E(G) \to Z_{n-1}$ as follows: $g(u) = 1$, $g(v) = 0$, $g(u_i) = i - 1$, $g(vu_i) = n - i + 1 \pmod n$, $g(u_i u) = n - i \pmod n$ and $g(uv) = n - 1$. Clearly, $n_g(i) = n_g(j) = 3$ for all $i \ne j$ and $0 \le i, j \le n - 1$. Thus, $G$ is $n$-TMC. Hence, the graph $S_n + K_1$ is hypo-$n$-TMC for all $n \ge 1$. □

**Theorem 2.13.** *If a graph $G$ is not k-TMC then the graph $G + K_1$ is not hypo-k-TMC.*

**Proof.** Suppose $G + K_1$ is hypo-$k$-TMC. Then $G$ must be $k$-TMC, which is a contradiction to $G$ is not $k$-TMC. Hence, the graph $G + K_1$ is not hypo-$k$-TMC. □

**Proposition 1.** *If $G$ is a hypo-k-TMC graph such that $n_f(i)$ is constant for all $i = 0, 1, 2, \cdots, k-1$ and $e \in E(G)$, then $G - \{e\}$ is also hypo-k-TMC.*

**Proposition 2.** *If $G_1(p_1, q_1)$ and $G_2(p_2, q_2)$ are two hypo-k-TMC graphs with $n(i)$ is a constant in k-TMC labeling of $G_1 - \{u\}$ or of $G_2 - \{v\}$ then $G_1 \cup G_2$ is also hypo-k-TMC.*

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