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# Comment on "Edge Geodetic Covers in Graphs"

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**Abstract**

*In this paper we show by counter example that one of the main results in the paper "Edge Geodetic Covers in Graphs"' by Mariano and Canoy (International Mathematical Forum, 4, 2009, no. 46, 2301 - 2310) does not hold. Further, we partially characterize connected graphs $G$ of order $n$ for which its edge geodetic number $g_e(G) = n-1$.*

**Key Words :** *Geodetic cover, geodetic number, edge geodetic cover, edge geodetic number.*

**AMS Subject Classification :** *05C12.*

## 1. Introduction

For the sake of clarity we start with some definitions and notations. By a graph $G = (V(G), E(G))$, we mean a finite undirected connected graph without loops or multiple edges. The order and size of $G$ are denoted by $n$ and $m$, respectively. For basic graph theoretic terminology we refer to [9]. For vertices $x$ and $y$ in a connected graph $G$, the *distance* $d(x, y)$ is the length of a shortest $x$-$y$ path in $G$. It is known that the distance is a metric on the vertex set of $G$. An $x$-$y$ path of length $d(x, y)$ is called an *$x$-$y$ geodesic*. A vertex $v$ is said to *lie on* an $x$-$y$ geodesic $P$ if $v$ is a vertex of $P$ including the vertices $x$ and $y$. A vertex $v$ is an *extreme vertex* of $G$ if the subgraph induced by its neighbors is complete.

The set $I_G[u, v]$ consists of all vertices lying in any $u$-$v$ geodesic in $G$, inclusive of $u$ and $v$. If $S \subseteq V(G)$, then the set $I_G[S] = \bigcup\limits_{x,y \in S} I_G[x, y]$. A subset $S$ of $V(G)$ is a *geodetic set* or *geodetic cover* of $G$ if $I_G[S] = V(G)$. The *geodetic number* of $G$, denoted by, $g(G)$ is the minimum cardinality of a geodetic cover of $G$. Any geodetic cover of $G$ of cardinality $g(G)$ is called a *geodetic basis* of $G$. Results regarding geodetic number of a graph have been extensively studied in [1, 2, 3, 4, 6, 7, 8, 10, 11, 12, 13]. The edge geodetic number of a graph was introduced and studied in [14].

The set $I^e_G[u, v]$ consists of all edges of $G$ lying in any $u$-$v$ geodesic in $G$. If $S \subseteq V(G)$, then the set $I^e_G[S]$ denotes the union of all $I^e_G[u, v]$, where $u, v \in S$. A subset $S$ of $V(G)$ is an *edge geodetic set* or an *edge geodetic cover* of G if every edge of G is contained in $I^e_G[S]$, that is, $I^e_G[S] = E(G)$. The *edge geodetic number* of $G$, denoted by $g_e(G)$, is the minimum cardinality of an edge geodetic cover of $G$. Any edge geodetic cover of $G$ of cardinality $g_e(G)$ is called an *edge geodetic basis* of $G$. Many more results on edge geodetic sets are studied in [14, 15, 16, 17, 18]. In [5], the authors quote Atici [1], however, they failed to note that some of the results proved by Atici in [1] are wrong. To cite an instant, the statement and proof of the second part of Corollary 3.2 in [1] are blunders.

**Corollary 3.2** [1] Let $G$ be a connected graph of order $n$. If $g_e(G) = n$, then $G$ has a vertex of degree $n - 1$.

For the graph $G$ in Figure 1.1, $\Delta(< N(v_i) >) = |N(v_i)| - 1$ for $i = 1, 2, \ldots, 6$. Hence it follows from Theorem 2.3(see Theorem 2.3) that $g_e(G) = n$, whereas $G$ has no vertex of degree $n - 1$. Thus, Corollary 3.2 in [1] goes wrong.

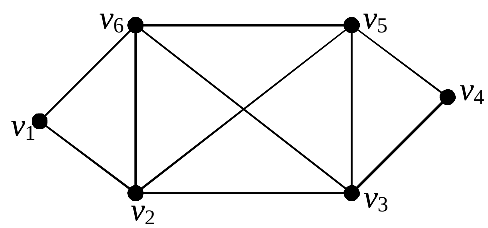


**Figure 1.1 :** G

We use the following theorem in the sequel.

**Theorem 1.1.** [14] Each extreme vertex of a connected graph $G$ belongs to every edge geodetic cover of $G$.

## 2. Graphs $G$ of order $n$ with $g_e(G) = n - 1$

First, we observe that the statement of Theorem 2.4 in [5] is wrong. For the graph $G$ given in Figure 2.1, it follows from Theorem 1.1 that every edge geodetic set contains $v_1, v_3$ and $v_5$. It is easily seen that $\{v_1, v_2, v_3, v_4, v_5\}, \{v_1, v_2, v_3, v_5, v_6\}$ and $\{v_1, v_3, v_4, v_5, v_6\}$ are the only three edge geodetic basis for $G$ and so $g(G) = 5 = n - 1$. For the vertices $v_2, v_4$ and $v_6$, we have $\Delta(< N(v_i) >) = 2$ and $|N(v_i)| - 1 = 3$ for $i = 2, 4, 6$. Thus, Theorem 2.4 in [5] goes wrong.

Now, we point out the place where the authors of [5] have committed the mistake in their proof. At this point, let us include the "statement" and "proof" of the above mentioned theorem.

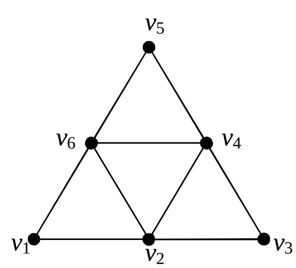


**Figure 2.1 :** G

**Theorem 2.4** Let $G$ be a connected graph of order $n$. Then $g_e(G) = n-1$ if and only if there exists a unique $v \in V(G)$ such that $\Delta(< N(v) >) \neq |N(v)| - 1$ In particular, $S = V(G) - \{v\}$ is the unique edge geodetic basis of $G$.

**Proof.** Suppose $g_e(G) = n - 1$. Then there exists a $v \in V(G)$ such that $S = V(G) - \{v\}$ is an edge geodetic basis of $G$. Suppose there exists a $u \in N(v)$ such that $deg_{<N(v)>}(u) = |N(v)| - 1$. Since $S$ is an edge geodetic cover of $G$, there exist $x, y \in S$ such that $vu$ is an edge in some $x - y$ geodesic $P(x, y) = [x_1, x_2, \ldots, x_k]$, where $x_1 = x, x_k = y, x_{i-1} = v$, and $x_i = u$ for some $3 \leq i \leq k$. This, however, is not possible because $x_i x_{i-2} \in E(G)$. Thus $\Delta(< N(v) >) \neq |N(v)| - 1$. Next, suppose that there exists $w \in V(G) - \{v\}$ such that $\Delta(< N(v) >) \neq |N(v)| - 1$. *Then it can easily be verified that* $S^* = V(G) - \{v, w\}$ *is an edge geodetic cover of* $G$. This contradicts our assumption that $g_e(G) = n - 1$. Therefore, $v$ is the only vertex of $G$ possessing the said property.

It is not difficult to realize that the italic statement in the above "proof" is not (necessarily) true. For example the vertices $v_2, v_4$ and $v_6$ satisfy the property $\Delta(< N(v_i) >) \neq |N(v_i)| - 1$ for $i = 2, 4, 6$. However, neither $S^* = V(G) - \{v_2, v_4\}$ nor $S^* = V(G) - \{v_2, v_6\}$ nor $S^* = V(G) - \{v_4, v_6\}$ is an edge geodetic cover of $G$.

Finally, we proceed to give a necessary condition of graphs $G$ of order $n$ for which $g_e(G) = n - 1$. Also, we develop a sufficent condition of graphs $G$ for which $g_e(G) = n - 1$. Thus, we prove not only Theorem 2.4.[5] is wrong, but also give a partial characterization.

**Definition 2.1.** For a connected graph $G$, let $S = \{x \in V(G) : \Delta(< N(x) >) = |N(x)| - 1\}$.

A vertex $v$ in $G$ is called a *semi-extreme vertex* of $G$ if $\Delta(< N(v) >) = |N(v)| - 1$. That is, the induced subgraph of $N(v)$ has a full degree vertex in $N(v)$.

The following two theorems are used to prove the required results.

**Theorem 2.2.** Let $G$ be a connected graph. Then $S$ is contained in every edge geodetic set of $G$.

**Proof.** Let $T$ be an edge geodetic set of $G$. Suppose that there exists $u \in S$ such that $u \notin T$. Since $\Delta(< N(u) >) = |N(u)| - 1$, there exists $v \in N(u)$ such that $deg_{<N(u)>}(v) = |N(u)| - 1$. Since $T$ is an edge geodetic set of $G$, the edge $e = uv$ lies on a $x - y$ geodesic $P : x = x_0, x_1, \ldots, x_i = u, x_{i+1} = v, \ldots, x_n = y$ with $x, y \in T$. Then $v \neq x, y$. Since $deg_{<N(u)>}(v) = |N(u)| - 1$, $u$ is adjacent to $x_{i+2}$, which is a contradiction to the fact that $P$ is a $x - y$ geodesic. Hence $S$ is contained in every edge geodetic set of $G$. $\square$

**Theorem 2.3.** Let $G$ be a connected graph. Then $g_e(G) = n$ if and only if $S = V(G)$.

**Proof.** If $S = V(G)$, then by Theorem 2.2, $g_e(G) = n$. Conversely, suppose that $g_e(G) = n$. If $S \neq V(G)$, then there exists $u \in V(G)$ such that $u \notin S$. Thus $\Delta(< N(u) >) \leq |N(u)| - 2$. It follows that $deg_{<N(u)>}(v) \leq |N(u)| - 2$ for all $v \in N(u)$. Let $T = V(G) - \{u\}$. We claim that $T$ is an edge geodetic set of $G$. Let $e = xy$ be an edge in $G$. If $x, y \in T$, then there is nothing to prove. So, assume that $y = u$. Then $x \in N(u) \cap T$. Let $x' \in N(u)$ be such that $x$ and $x'$ are non-adjacent. Then it is clear that the edge $e = xu$ lies on the geodesic $P : x, u, x'$ with $x, x' \in T$. Hence $T$ is an edge geodetic set of $G$ and so $g_e(G) \leq n - 1$, which is a contradiction to the fact that $g_e(G) = n$. Thus $S = V(G)$. $\square$

**Theorem 2.4.** Let $G$ be a connected graph of order $n$ and let $S$ denote the set of all semi-extreme vertices of $G$. If $g_e(G) = n - 1$, then $< V(G) - S >$ is a clique in $G$.

**Proof.** Suppose that $g_e(G) = n - 1$. Then it follows from Theorem 2.3 that $V(G) - S \neq \phi$. Suppose that there exist non-adjacent vertices $x$ and $y$ in $V(G) - S$. Then $|N(x)| \geq 2$ and $|N(y)| \geq 2$. Since $\Delta(< N(x) >) \leq |N(x)| - 2$ and $\Delta(< N(y) >) \leq |N(y)| - 2$, we have $deg_{<N(x)>}(u) \leq |N(x)| - 2$ for all $u \in N(x)$ and $deg_{<(N(y)>}(v) \leq |N(y)| - 2$ for all $v \in N(y)$. Now, let $T = V(G) - \{x, y\}$. We claim that $T$ is an edge geodetic set of $G$. Let $e = uv$ be an edge in $G$. If $u, v \in T$, then there is nothing to prove. So, assume that $u = x$. Then $v \in N(x)$ and $v \neq y$. Since $deg_{<N(x)>}(v) \leq |N(x)| - 2$, there is a vertex $v' \in N(x)$ such that $v$ and $v'$ are non-adjacent.Then it is clear that the edge $e = xv$ lies on the geodesic $P : v, x, v'$ with $v, v' \in T$. Hence $T$ is an edge geodetic set of $G$ and so $g_e(G) \leq n - 2$, which is a contradiction. Hence $< V(G) - S >$ is a clique in $G$. $\square$

**Remark 2.5.** The converse of the above theorem is not true. For the graph $G$ in Figure 2.2, we have $S = \{v_1, v_3, v_5, v_6\}$ and so $V(G) - S$ is a clique. It is clear that $S$ an edge geodetic set of $G$ so that $g_e(G) = 4 \neq 5 = n - 1$.

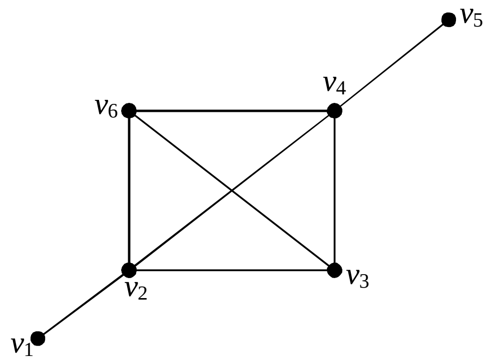


**Figure 2.2 :** G

**Theorem 2.6.** Let $G$ be a connected graph of order $n$ and let $S$ denote the set of all semi-extreme vertices of $G$. If $< V(G) - S >$ is a clique and $d(x, y) \leq 2$ for all $x, y \in N(V(G) - S))$, then $g_e(G) = n - 1$.

**Proof.** Suppose that $< V(G) - S >$ is a clique and $d(x, y) \leq 2$ for all $x, y \in N(V(G) - S))$. Since $< V(G) - S >$ is a clique, it follows that $S \neq V(G)$ and so by Theorem 2.3, $g_e(G) \leq n - 1$. If $|V(G) - S| = 1$, then $|S| = n - 1$ and it follows from Theorem 2.2 that $g_e(G) \geq n - 1$. Thus $g_e(G) = n - 1$. So, assume that $|V(G) - S| \geq 2$. Now, suppose that $g_e(G) \leq n - 2$. Let $T$ be an edge geodetic set of $G$ of cardinality $n - 2$. Let $u, v \in V(G)$ be such that $u, v \notin T$. Then by Theorem 2.2, $u, v \in (V(G) - S)$. Since $< V(G) - S >$ is a clique, $u$ and $v$ are adjacent in $G$. Since $T$ is an edge geodetic set of $G$, the edge $e = uv$ lies on a $x - y$ geodesic $P : x = x_0, x_1, \ldots, x_{i-1}, x_i = u, x_{i+1} = v, x_{i+2}, \ldots, x_n = y$ with $x, y \in T$ and $1 \leq i \leq n - 1$. Since $d(x, y) \leq 2$ for all $x, y \in N(V(G) - S))$, it follows that $d(x_{i-1}, x_{i+1}) \leq 2$, which is a contradiction to the fact that $P$ is a geodesic. Hence $g_e(G) = n - 1$. $\square$

We leave the following problem as open.

**Problem 2.7.** Characterize the class of graphs $G$ for which $g_e(G) = n - 1$.

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