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Sufficient conditions for the boundedness and square integrability of solutions of fourth-order differential equations

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Abstract

Sufficient conditions for the boundedness and square integrability of solutions and their derivatives of certain fourth order nonlinear differential equation are given by means of the Lyapunov's second method. Our results obtained in this work, generalize existing results on fourth order nonlinear differential equations in the literature. For illustration, an example is also given.

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1. Introduction

During the past few years there has been many excellent results concerning the boundedness of the solutions of nonlinear ordinary differential equations by the use of the Lyapunov's method (Yoshizawa [27]). Today, this method is widely recognized as an excellent tool not only in the study of differential equations but also in the theory of control systems, dynamical systems, systems with time lag, power system analysis, time varying nonlinear feedback systems, and so on. But, finding an appropriate Lyapunov function for higher order differential equations is in general a difficult task. Many works concerning have investigated the boundedness of solutions of certain differential equations of the fourth order. We mention, the works of Ezeilo [6], [7], Harrow [8], [9], Afuwape and Adesina [1], Tiryaki and Tunç [17], Tunç [18], [19], [20], Tunç and Tiryaki [21] where the Lyapunov's second method was used.

In [12], Omeike by using the Cauchy formula for the particular solution of nonlinear differential equations with constant coefficients, has proved that the solution of the equation

$$x'''' + ax''' + bx'' + cx' + h(x) = p(t), \tag{1.1}$$

and its derivatives up to order three are bounded.

In [22], and [25] Tunç established sufficient conditions for the asymptotic stability of the zero solution and the boundedness of the the following equations

$$\begin{aligned} x'''' + a_1x''' + \psi(x,x')x'' + a_4x' + h(x) &= 0, &(1.2)\\ x'''' + a_1x''' + \psi(x,x')x'' + g(x') + a_4x &= 0, &(1.3)\\ x'''' + ax''' + \psi(x,x',x'') + g(x,x') + h(x) &= p(t). &(1.4)\end{aligned}$$

The problem related to the study of square integrability of solutions for higher order nonlinear differential equations is also of great interest, but it should be noted that only a few results are related to the fourth order nonlinear differential equations. In 1989 Andres and Vlček [2], discussed the square integrable solutions of (1.1).

The purpose of this paper is to define a Lyapunov function and use it to study the boundedness of solutions and we also study the square integrability of solutions of the differential equation of the form

$$\Big(g\Big(x(t)\Big)x''(t)\Big)'' + a\,(t)\,\Big(p\Big(x(t)\Big)x''(t)\Big)' + b\,(t)\,\Big(q\Big(x(t)\Big)x'(t)\Big)'$$

$$+\mathrm{c}(t)\,f\Big(x(t)\Big)x'(t) + d\,(t)\,h\Big(x(t)\Big) = e(t),$$

(1.5)

where $a(t), b(t), c(t), d(t), e(t), f(x), g(x), p(x), q(x)$ and $h(x)$ are continuous functions depending only on the arguments shown and $p'(x)$, $q'(x)$, $f'(x)$ and $h'(x)$ exist and are continuous.

Hence, the aim of this paper consists mainly in further extension of the related existing results.

2. Assumptions and main results

We begin by presenting some sufficient assumptions which will be used in equation (1.5), and suppose that there are positive constants $a_0, b_0, c_0, d_0, f_0, g_0, p_0, q_0, a_1, b_1, c_1, d_1, f_1, g_1, p_1, q_1, m, M, \delta, \eta_1, h_0, \delta_0$, such that the following conditions are satisfied

i) $0 < a_0 \le a\,(t) \le a_1$; $0 < b_0 \le b\,(t) \le b_1$; $0 < c_0 \le c\,(t) \le c_1; 0 < d_0 \le d\,(t) \le d_1$ for $t \ge 0$.

ii) $0 < f_0 \le f\,(x) \le f_1$; $g_0 \le g\,(x) \le g_1$; $0 < p_0 \le p\,(x) \le p_1$; $0 < q_0 \le q\,(x) \le q_1$ for $x \in R$
and $0 < m < \min\Big\{f_0, p_0, g_0, 1\Big\}$, $M > \max\Big\{f_1, , g_1, p_1, 1\Big\}$.

iii) $\dfrac{h(x)}{x} \ge \delta > 0$ (for $x \ne 0$); $h\,(0) = 0$.

iv) $\dfrac{h_0}{m} - \dfrac{a_0 m \delta_0}{d_1} \le h'\,(x) \le \dfrac{h_0}{2M}$ for $x \in R$.

The following lemma will be needed in the proof of our results. [11] Let $h(0) = 0,\ xh(x) > 0\ (x \neq 0)$ and $\delta(t) - h'(x) \geq 0\ (\delta(t) > 0)$, then

$$2\delta(t)H(x) \geq h^2(x) \qquad \text{where} \quad H(x) = \int_0^x h(s)ds.$$

Before stating the theorem, we introduce the following notations:

$$\begin{cases} \kappa_1 = \dfrac{a_1 h_0 d_1 M^2}{c_0 m^3} + \dfrac{M^3(c_1+\delta_0)}{a_0 m^2} + a_0 a_1 m\,(M-1)\,, \\ \\ \kappa_2 = \dfrac{2d_1 h_0 a_0}{c_0\,(M-1)}\left(\dfrac{1}{m} - \dfrac{1}{M}\right)^2 + 2\dfrac{c_0 M}{a_0} + 2a_1\dfrac{d_1 h_0 M}{c_0 m^3} + \dfrac{c_0 c_1 (M^2+2)mM}{d_1 h_0}. \end{cases}$$

In addition to conditions (i) $\sim$ (iv) being satisfied, assume that there are positive constants η_2, η_3 and η_4 such that the following conditions hold

H_0) $\quad b_0 q_0 > \max\left\{\kappa_1, \kappa_2\right\}.$

H_1) $\quad \displaystyle\int_0^{+\infty} \left(|a'(t)| + |b'(t)| + |c'(t)| + |d'(t)|\right) dt < \eta_1.$

H_2) $\quad \displaystyle\int_{-\infty}^{+\infty} \left(|g'(s)| + |p'(s)| + |q'(s)| + |f'(s)|\right) ds < \eta_2.$

H_3) $\quad \displaystyle\int_0^{+\infty} |e(s)|\, ds < \eta_3.$

H_4) $\quad |g'(x)| < \eta_4,$ for all x.

Then any solution $x(t)$ of (**??**) and its derivatives $x'(t), x''(t)$ and $x'''(t)$ are bounded and satisfy

$\int_0^\infty \left(x^2(s) + x'^2(s) + x''^2(s) + x'''^2(s)\right) ds < \infty.$

Proof. The equation (1.5) can be expressed as the following system

$$\begin{cases} x' = y, \\ y' = \frac{1}{g(x)} z, \\ z' = w, \\ w' = -a(t)\frac{p(x)}{g(x)} w + \Big(a\left(t\right) p(x)\theta_1 - b(t)\frac{q(x)}{g(x)} - a(t)g(x)\theta_2\Big)\, z \\ \qquad -\Bigg(b(t)g^2(x)\theta_3 + c\left(t\right) f\left(x\right)\Bigg) y - d\left(t\right) h\left(x\right) + e(t), \end{cases}$$

(2.1)

where

$$\theta_1\left(t\right) = \frac{g'\left(x\left(t\right)\right)}{g^2\left(x\left(t\right)\right)} x'\left(t\right), \theta_2\left(t\right) = \frac{p'\left(x\left(t\right)\right)}{g^2\left(x\left(t\right)\right)} x'\left(t\right), \theta_3\left(t\right) = \frac{q'\left(x\left(t\right)\right)}{g^2\left(x\left(t\right)\right)} x'\left(t\right),$$

$$\theta_4\left(t\right) = \frac{f'\left(x\left(t\right)\right)}{g^2\left(x\left(t\right)\right)} x'\left(t\right).$$

Boundedness of solutions: First we proof the boundedness of solutions. The proof depend on the Lyapunov function $W = W(t, x, y, z, w)$ defined as

$$W = e^{-\frac{1}{\eta}\int_0^t (\gamma_1(s) + \gamma_2(s))ds} V, \tag{2.2}$$

where
$\gamma_1\left(t\right) = \left|a'\left(t\right)\right| + \left|b'\left(t\right)\right| + \left|c'\left(t\right)\right| + \left|d'\left(t\right)\right|$, $\gamma_2(t) = |\theta_1(t)| + |\theta_2(t)| + |\theta_3(t)| + |\theta_4(t)|$,
and

$$\begin{aligned} 2V \;\; &= \;\; 2V(t, x, y, z, w) = 2\beta d\left(t\right) H\left(x\right) + c\left(t\right) g\left(x\right) f\left(x\right) y^2 + \alpha b\left(t\right) \frac{q(x)}{g\left(x\right)} z^2 \\ &+ \mathrm{a}(t)\,\frac{p(x)}{g(x)} z^2 + 2\beta a\left(t\right) \frac{p(x)}{g(x)} yz + \left[\beta b(t)q(x) - \alpha h_0 d\left(t\right)\right] y^2 - \beta \frac{1}{g(x)} z^2 \\ &+ \alpha w^2 + 2d\left(t\right) g\left(x\right) h(x)y + 2\alpha d(t)h\left(x\right) z + 2\alpha c\left(t\right) f\left(x\right) yz + 2\beta yw + 2zw, \end{aligned}$$

with $\alpha = \dfrac{M}{a_0 m} + \epsilon$, $\beta = \dfrac{d_1 h_0}{c_0 m} + \epsilon$. ϵ, η are positive constants to be determined later in the proof. $2V$ can be rearranged in the form

$$2V \;\; = \;\; a\left(t\right) p(x) \left(\frac{w}{a\left(t\right) p(x)} + z + \beta \frac{1}{g\left(x\right)} y\right)^2 + c\left(t\right) f\left(x\right) \left(\frac{d\left(t\right) h\left(x\right)}{c\left(t\right) f\left(x\right)} + y + \alpha z\right)^2$$

$$+\,\mathrm{c}(t)\,f\,(x)\left[\left(g\,(x)-1\right)y+\tfrac{d(t)h(x)}{c(t)f(x)}\right]^2+2\epsilon d\,(t)\,H\,(x)+V_1+V_2+V_3,$$

such that

$$\mathrm{V}_1=2d\,(t)\textstyle\int_0^x h\,(s)\left(\frac{d_1h_0}{c_0m}-2\frac{d(t)}{c(t)f(x)}h'\,(s)\right)ds,$$

$$\mathrm{V}_2=\left(\alpha b\,(t)\,\tfrac{q(x)}{g(x)}-\beta\tfrac{1}{g(x)}-\alpha^2c\,(t)\,f\,(x)+a\,(t)\,p(x)\left(\tfrac{1}{g(x)}-1\right)\right)z^2$$

and

$$\mathrm{V}_3=\left(\beta b\,(t)\,q(x)-\alpha h_0d\,(t)-\beta^2a\,(t)\,\tfrac{p(x)}{g^2(x)}-c\,(t)\,f(x)\left(g^2\,(x)-3g\,(x)+2\right)\right)y^2$$

$$+\left(\alpha-\tfrac{1}{a(t)p(x)}\right)w^2+2\beta\left(1-\tfrac{1}{g(x)}\right)yw.$$

Now we will prove that V is positive definite. Take

$$\epsilon<\min\left\{\frac{M}{a_0m}\,,\,\frac{d_1h_0}{c_0m}\,,\,\frac{m^2(b_0q_0-\kappa_1)}{M^2(a_1+mMc_1)}\right\},\tag{2.3}$$

then

$$\frac{M}{a_0m}<\alpha<2\frac{M}{a_0m},\qquad\frac{d_1h_0}{c_0m}<\beta<2\frac{d_1h_0}{c_0m}.\tag{2.4}$$

Conditions (i) $\sim$ (iv) and (H_0) imply that

$$V_1\;\geq\;4d_0\frac{d_1}{c_0m}\int_0^x h\,(s)\left(\frac{h_0}{2M}-h'\,(s)\right)ds\geq0.$$

We can rewrite V_2 as

$$\mathrm{V}_2=\alpha\left(b\,(t)\,\frac{q(x)}{g\,(x)}-\beta\frac{a\,(t)}{g\,(x)}-\alpha c\,(t)\,f\,(x)-\frac{a\,(t)\,p(x)}{\alpha}\left(1-\frac{1}{g\,(x)}\right)\right)z^2$$
$$+\,\beta\left(\alpha\frac{a\,(t)}{g\,(x)}-\frac{1}{g\,(x)}\right)z^2.$$

From conditions (i) $\sim$(iii) and inequalities (2.3),(2.4), it follows that

$$\mathrm{V}_2\geq\alpha\left(\frac{b_0q_0}{M}-\Big(\frac{d_1h_0}{c_0m}+\epsilon\Big)\frac{a_1}{m}-\Big(\frac{M}{a_0m}+\epsilon\Big)c_1M-a_1\frac{a_0m}{M}\Big(M-1\Big)\right)z^2$$
$$+\,\beta\left(\alpha\frac{a_0}{M}-\frac{1}{m}\right)z^2$$

$$\geq \alpha\Bigg(\frac{b_0q_0}{M}-\frac{d_1h_0a_1}{c_0m^2}-\frac{c_1M^2}{a_0m}-a_1\frac{a_0m}{M}\Big(M-1\Big)-\frac{\epsilon}{m}\Big(a_1+c_1mM\Big)\Bigg)z^2$$
$$\geq \frac{\alpha}{Mm}\Bigg(m(b_0q_0-\kappa_1)-\epsilon M\left(a_1+c_1mM\right)\Bigg)z^2\geq 0,$$

and

$$\begin{aligned}V_3 \;\;\geq\;\; & \beta\Bigg(b_0q_0-\frac{\alpha}{\beta}h_0d_1-a_1\beta\frac{M}{g^2\left(x\right)}-\frac{c_1M(M^2+2)}{\beta}\Bigg)y^2+\Bigg(\frac{M-1}{a_0m}\Bigg)w^2\\ & +2\beta\Bigg(1-\frac{1}{g\left(x\right)}\Bigg)yw\\ & \geq\beta\Bigg(b_0q_0-2\tfrac{M}{a_0}c_0-2a_1\tfrac{d_1h_0M}{c_0m^3}-\tfrac{c_0c_1(M^2+2)mM}{d_1h_0}\Bigg)y^2+\Bigg(\tfrac{M-1}{a_0m}\Bigg)w^2\\ & +\;2\beta\Bigg(1-\tfrac{1}{g(x)}\Bigg)yw\\ & \geq\psi(y,\omega),\end{aligned}$$

such that

$$\psi(y,\omega)=\beta\frac{2d_1h_0a_0}{c_0\left(M-1\right)}\left(\frac{1}{m}-\frac{1}{M}\right)^2y^2+\Bigg(\frac{M-1}{a_0m}\Bigg)w^2+2\beta\Bigg(1-\frac{1}{g\left(x\right)}\Bigg)yw.$$

It is clear that $\psi(y,\omega)$ is positive definite. To show this we calculate the discriminant

$$\triangle=\beta^2\left[1-\frac{1}{g\left(x\right)}\right]^2-\frac{2\beta d_1h_0}{c_0m}\left[\frac{1}{M}-\frac{1}{m}\right]^2.$$

Using condition (ii) we have

$$\frac{1}{M}<\frac{1}{g(x)}<\frac{1}{m},\text{ and }\;\frac{1}{M}<1<\frac{1}{m},$$

it follows that

$$\left|1-\frac{1}{g\left(x\right)}\right|<\frac{1}{m}-\frac{1}{M}.$$

Using (2.4) we get

$$\triangle\leq\beta\left[\frac{2d_1h_0}{c_0m}\left(\frac{1}{M}-\frac{1}{m}\right)^2-\frac{2d_1h_0}{c_0m}\left(\frac{1}{M}-\frac{1}{m}\right)^2\right]=0.$$

Thus there exists positive number D_0 such that

$$2V \geq D_0 \left(y^2 + z^2 + w^2 + H(x) \right). \tag{2.5}$$

By Lemma 2 and conditions (iii) and (H_1) we get the existence of a positive number D_1 such that

$$2V \geq D_1 \left(x^2 + y^2 + z^2 + w^2 \right), \tag{2.6}$$

thus V is positive definite which implies that W is also positive definite.

Hence we can find positive definite functions $U_1(\|)$ and $U_2(\|X\|)$ ($X = (x, y, z, w)$) such that $U_1(\|X\|) \leq V \leq U_2(\|X\|)$. By (ii) and (H_2), we get

$\int_0^t \left(\gamma_1(s) + \gamma_2(s) \right) ds \leq \eta_1 + \int_{\alpha_1(t)}^{\alpha_2(t)} \frac{|g'(u)|+|p'(u)|+|q'(u)|+|f'(u)|}{g^2(u)} du$

$\leq \eta_1 + \frac{1}{m^2} \int_{-\infty}^{+\infty} \left(|g'(s)| + |p'(s)| + |q'(s)| + |f'(s)| \right) ds$

$\leq \eta_1 + \frac{\eta_2}{m^2}$,

(2.7)

with $\alpha_1(t) = \min\{x(0), x(t)\}$, and $\alpha_2(t) = \max\{x(0), x(t)\}$. By condition (H_1) and inequalities (2.2),(2.6) and (2) we have

$$W \geq D_2(x^2 + y^2 + z^2 + w^2), \tag{2.8}$$

where $D_2 = \dfrac{D_1}{2} e^{-\frac{1}{\eta}(\eta_1 + \frac{\eta_2}{m^2})}$. Therefore we can find positive definite functions $W_1(\|X\|)$ and $W_2(\|X\|)$ such that $W_1(\|X\|) \leq W \leq W_2(\|X\|)$.

Next we show that $\dot{W}$ is negative definite functional. The derivative of the function V, along any solution $(x(t), y(t), z(t), w(t))$ of system (2.1), with respect to t is after elementary calculation

$$2.V_{(2.1)} \quad = \quad -2\epsilon c\,(t)\, f(x) y^2 + V_4 + V_5 + V_6 + V_7 + 2(\beta y + z + \alpha w) e(t) + 2\frac{\partial V}{\partial t},$$

where

$$V_4 = -2\left(\frac{d_1 h_0}{c_0 m} c(t) f(x) - d(t) g(x) h'(x)\right) y^2 - 2\alpha d(t)\left(\frac{h_0}{g(x)} - h'(x)\right) yz,$$

$V_5 = -2\left(\frac{b(t)q(x)}{g(x)} - \alpha c(t)\frac{f(x)}{g(x)} - \beta a(t)\frac{p(x)}{g^2(x)}\right) z^2,$

$V_6 = -2\left(\alpha\frac{a(t)p(x)}{g(x)} - 1\right) w^2$

and

$$\begin{aligned} V_7 &= \theta_1\Big(a(t)p(x)z^2 - \alpha b(t)q(x)z^2 + c(t)f(x)g^2(x)y^2 + \beta z^2 + 2d(t)g^2(x)h(x)y \\ &\quad +2\alpha a(t)p(x)zw\Big) - b(t)\theta_3 g(x)\Big(\alpha z^2 + 2\alpha g(x)zw + \beta g(x)y^2 + 2g(x)yz\Big) \\ &\quad -a(t)\theta_2 g(x)\Big(z^2 + 2\alpha zw\Big) + \theta_4\Big(c(t)g^3(x)y^2 + 2\alpha c(t)g^2(x)yz\Big). \end{aligned}$$

By conditions (i), (ii), (iv), (H_0) and inequalities (2.3), (2.4) we get

$$\begin{aligned} V_4 &\le -2\Big(d(t)h_0 - d(t)g(x)h'(x)\Big)y^2 - 2\alpha d(t)\left(\frac{h_0}{g(x)} - h'(x)\right)yz \\ &\le -2d(t)m\left(\frac{h_0}{g(x)} - h'(x)\right)\left[\left(y + \frac{\alpha}{2m}z\right)^2 - \left(\frac{\alpha}{2m}z\right)^2\right] \\ &\le \frac{\alpha^2}{2m}d(t)\left(\frac{h_0}{m} - h'(x)\right)z^2. \end{aligned}$$

Therefore,

$$V_4+V_5 \le -2\left[\frac{b_0q_0}{M} - \left(\frac{M}{a_0 m} + \epsilon\right)\frac{c_1 M}{m} - \left(\frac{d_1 h_0}{c_0 m} + \epsilon\right)\frac{a_1 M}{m^2} - \frac{\alpha^2}{4m}\Big(a_0\delta_0\Big)\right] z^2$$

$$\le -2\left[\frac{b_0q_0}{M} - \frac{M^2}{a_0 m^2}c_1 - \frac{d_1 h_0 a_1 M}{c_0 m^3} - \frac{M^2\delta_0}{a_0 m^2} - \epsilon\frac{M}{m}\left(\frac{a_1}{m} + c_1\right)\right] z^2$$

$$\le -\frac{2}{Mm^2}\Big(m^2(b_0q_0 - \kappa_1) - \epsilon M^2(a_1 + c_1 m)\Big)z^2 \le 0.$$

We have also,

$$V_6 \le -2\left(\alpha\frac{a_0 m}{M} - 1\right)w^2 = -2\epsilon\frac{a_0 m}{M}w^2 \le 0.$$

Putting

$$D_3 = \min\left\{\epsilon c_0 m,\ \epsilon\frac{a_0 m}{M},\ \frac{1}{Mm^2}\Big(m^2(b_0 q_0 - \kappa_1) - \epsilon M^2\left(a_1 + c_1 mM\right)\Big)\right\}$$

we obtain

$$-2\epsilon c\,(t)\,f(x)y^2 + V_4 + V_5 + V_6 \leq -2D_3\left(y^2 + z^2 + w^2\right).$$

From Lemma (2.1) and inequalities (2.5), $2uv \leq u^2 + v^2$ we obtain the following

$V_7 \leq |\theta_1|\Big(a\,(t)\,p(x)z^2 + \alpha b\,(t)\,q(x)z^2 + c\,(t)\,f\,(x)\,g^2\,(x)\,y^2 + \beta z^2 + d\,(t)\,g^2\,(x)$
$(h^2\,(x) + y^2)$
$+\ \alpha a\,(t)\,p(x)(z^2 + w^2)\Big) + |\theta_4|\Big(c\,(t)\,g^3\,(x)\,y^2 + \alpha c\,(t)\,g^2\,(x)\,(y^2 + z^2)\Big)$
$+\ b(t)|\theta_3|g(x)\Big(\alpha z^2 + \alpha g(x)(z^2 + w^2) + \beta g(x)y^2 + g(x)(y^2 + z^2)\Big)$
$+\ a(t)|\theta_2|g(x)\Big(z^2 + \alpha(z^2 + w^2)\Big)$
$\leq \frac{2K_1}{D_0}\Big(|\theta_1| + |\theta_2| + |\theta_3| + |\theta_4|\Big)V, .$

with K_1 some positive constant. We get also,

$$\begin{aligned} 2\frac{\partial V}{\partial t} \quad &= \quad d'\,(t)\left[2\beta H\,(x) - \alpha h_0 y^2 + 2g\,(x)\,h\,(x)\,y + 2\alpha h\,(x)\,z\right] \\ &\quad +c'\,(t)\left[g\,(x)\,f(x)y^2 + 2\alpha f(x)yz\right] + b'\,(t)\left[\alpha\frac{q(x)}{g\,(x)}z^2 + \beta q(x)y^2\right] \\ &\quad +a'\,(t)\left[\frac{p(x)}{g\,(x)}z^2 + 2\beta\frac{p(x)}{g\,(x)}yz\right] \end{aligned}$$

$$\leq 2\frac{K_2}{D_0}\Bigg(\ |a'\,(t)| + |b'\,(t)| + |c'\,(t)| + |d'\,(t)|\ \Bigg)V,$$

with K_2 positive constant. Thus for $\dfrac{1}{\eta} = \dfrac{1}{D_0}\max\Big\{K_1, K_2\Big\}$ we have

$$.V_{(2.1)} \leq -D_3(y^2 + z^2 + w^2) + \frac{1}{\eta}(\gamma_1(t) + \gamma_2(t))V + \Big(\beta y + z + \alpha w\Big)e(t).$$

(2.9)

By condition (H_1) and using the inequalities (2.7), (2.8), (2.9) together with, $2uv \leq u^2 + v^2$, we obtain

$$.\mathrm{W}_{(2.1)} = \Big(.V_{(??)} - \tfrac{1}{\eta}(\gamma_1(t) + \gamma_2(t))V\Big)\, e^{-\dfrac{1}{\eta}\displaystyle\int_0^t (\gamma_1(s) + \gamma_2(s))ds}$$

$$\leq \Big(- D_3 \left(y^2 + z^2 + w^2\right) + \Big(\beta y + z + \alpha w\Big) e(t)\Big) e^{-\dfrac{1}{\eta}\displaystyle\int_0^t (\gamma_1(s) + \gamma_2(s))ds}$$

(2.10)

$$\leq D_4\left(|y| + |z| + |w|\right)|e(t)|$$
$$\leq D_4\left(3 + y^2 + z^2 + w^2\right)|e(t)|$$
$$\leq D_4\left(3 + \tfrac{1}{D_2}W\right)|e(t)|$$
$$\leq 3D_4|e(t)| + \tfrac{D_4}{D_2}W|e(t)|,$$

(2.11)

where $D_4 = \max\{\alpha, \beta, 1\}$. Integrating (2) from 0 to t, using the condition (H_3) and the Gronwall inequality, we have

$$\mathrm{W(t,x,y,z,w)} \leq W\Big(0, x(0), y(0), z(0), w(0)\Big) + 3D_4\eta_3$$
$$+ \tfrac{D_4}{D_2} \textstyle\int_0^t W\Big(s, x(s), y(s), z(s), w(s)\Big)|e(s)|ds$$
$$\leq \Big(W\Big(0, x(0), y(0), z(0), w(0)\Big) + 3D_4\eta_3\Big) e^{\dfrac{D_4}{D_2}\displaystyle\int_0^t |e(s)|ds}$$
$$\leq \Big(W\Big(0, x(0), y(0), z(0), w(0)\Big) + 3D_4\eta_3\Big) e^{\dfrac{D_4}{D_2}\eta_3} = \lambda_1 < \infty.$$

(2.12)

In view of inequalities (2.8) and (2.12), we get

$$(x^2 + y^2 + z^2 + w^2) \leq \frac{1}{D_2} W \leq \lambda_2$$

where $\lambda_2 = \frac{\lambda_1}{D_2}$. From the above inequality it follows that

$$|x(t)| \leq \sqrt{\lambda_2},\ |y(t)| \leq \sqrt{\lambda_2},\ |z(t)| \leq \sqrt{\lambda_2},\ |w(t)| \leq \sqrt{\lambda_2} \quad \text{for all} \ \ t \geq 0.$$

(2.13)

By using (2.13) and the condition (ii) and since $x' = y$ and $x'' = \dfrac{1}{g(x)}z$ we obtain

$$|x(t)| \leq \sqrt{\lambda_2},\ \ |x'(t)| \leq \sqrt{\lambda_2},\ \ |x''(t)| = \left|\frac{1}{g(x(t))}z(t)\right| \leq \frac{1}{m}\sqrt{\lambda_2} \quad \text{for all } t \geq 0. \tag{2.14}$$

On the other hand, conditions (ii), (H_4) and (2.14), show that

$$|\theta_1(t)| = \left|\frac{g'(x(t))}{g^2(x(t))}x'(t)\right| < \frac{\eta_4}{m^2}\sqrt{\lambda_2} \quad \text{for all } \ t \geq 0. \tag{2.15}$$

According to (2.13),(2.15) and since $x'''(t) = \dfrac{1}{g(x(t))}w(t) - \theta_1(t)z(t)$, we get

$$|x'''(t)| \leq \frac{1}{g(x(t))}|w(t)| + |\theta_1(t)||z(t)| \leq \frac{1}{m}\sqrt{\lambda_2} + \frac{\eta_4}{m^2}\lambda_2 \quad \text{for all } \ t \geq 0. \tag{2.16}$$

Square integrable solutions: Now we proof the square integrability of solutions and their derivatives. let $\rho > 0$, we define $F_t = F(t, x(t), y(t), z(t), w(t))$ as

$$F_t = W + \rho \int_0^t \Big(y^2(s) + z^2(s) + w^2(s)\Big)\, ds,$$

where $W = W(t, x, y, z)$ is is defined as (2.2). Note that F_t is positive definite since W is positive definite. From (2.10), and the estimate

$$e^{-\frac{1}{\eta}(\eta_1 + \frac{\eta_2}{m^2})} \leq e^{-\frac{1}{\eta}\int_0^t (\gamma_1(s) + \gamma_2(s))ds} \leq 1$$

we obtain

$$.F_{t(2.1)} \leq -D_3\Big(y^2(t) + z^2(t) + w^2(t)\Big)e^{-\frac{1}{\eta}(\eta_1 + \frac{\eta_2}{m^2})} \tag{2.17}$$

$$+D_4\Big(|y(t)|+|z(t)|+|w(t)|\Big)|e(t)|$$
$$+\rho\Big(y^2(t)+z^2(t)+w^2(t)\Big).$$

Choosing $\rho = D_3 e^{-\frac{1}{\eta}(\eta_1+\frac{\eta_2}{m^2})}$ we get

$$.F_{t(??)} \leq D_4\Big(3+\frac{1}{D_2}W\Big)|e(t)|$$
$$\leq 3D_4|e(t)| + \frac{D_4}{D_2}F_t|e(t)|. \tag{2.18}$$

Integrating (2.18) from 0 to t, using the condition (H_3) and the Gronwall inequality, we have

$\mathrm{F}_t \leq F_0 + 3D_4\eta_3 + \frac{D_4}{D_2}\int_0^t F_s|e(s)|ds$

$$\leq \Big(F_0+3D_4\eta_3\Big)e^{\dfrac{D_4}{D_2}\int_0^t |e(s)|ds}$$

$$\leq \Big(F_0+3D_4\eta_3\Big)e^{\dfrac{D_4}{D_2}\eta_3} = \lambda_3 < \infty.$$

(2.19)

from the above it follows that,

$$\lim_{t\to\infty} F_t \leq \lambda_3,$$

hence

$$\int_0^\infty y^2(s)ds < \lambda_3 \quad , \quad \int_0^\infty z^2(s) < \lambda_3 \text{ and } \int_0^\infty w^2(s)ds < \lambda_3. \tag{2.20}$$

By using (2.20) and the condition (ii) and since $x' = y$ and $x'' = \dfrac{1}{g(x)}z$ we obtain

$$\int_0^\infty x'^2(s)ds \leq \lambda_3 \quad , \quad \int_0^\infty x''^2(s)ds \leq \frac{1}{m}\int_0^\infty z^2(s)ds \leq \frac{\lambda_3}{m} = \lambda_4. \tag{2.21}$$

On the other hand, Combining the conditions (ii), (H_4) and (**??**), (2.14), gives

$$\theta_1^2(t) = \left[\frac{g'(x(t))}{g^2(x(t))}x'(t)\right]^2 \leq \frac{\eta_4^2}{m^4}\lambda_3, \quad \text{for all } \ t \geq 0.$$

Using again inequality $2uv \leq u^2 + v^2$ we get

$$\int_0^\infty x'''^2(s)ds = \int_0^\infty \frac{w^2(s)}{g^2(x(s))}ds + \int_0^\infty \theta_1^2(s)z^2(s)ds - 2\int_0^\infty \frac{\theta_1(s)}{g(x(s))}z(s)w(s)ds$$

$\leq 2\frac{w^2(s)}{g^2(x(s))}ds + 2\int_0^\infty \theta_1^2(s)z^2(s)ds$

$\leq 2N\int_0^\infty (z^2(s) + w^2(s))ds \leq 4N\lambda_3 = \lambda_5,$
(2.22)

where $N = \dfrac{1}{m^2}\max\left\{1, \dfrac{\eta_4^2}{m^2}\lambda_3\right\}$. Next, multiply (**??**) by $x(t)$ and integrate by parts from 0 to t all the terms on the LHS of (**??**) we obtain

$$(2.23)\int_0^t d(s)x(s)h(x(s))ds = I_1(t) + I_2(t) + I_3(t) + I_4(t) + I_5(t) + R_0,$$

where
$\mathrm{I}_1(t) = -\Big(g'(x(t))x'(t)x''(t) + g(x(t))x'''(t)\Big)x(t) + g(x(t))x''(t)x'(t) - \int_0^t g(x(s))x''^2(s)ds,$
$I_2(t) = -a(t)p(x(t))x(t)x''(t) + \int_0^t a'(s)p(x(s))x(s)x''(s)ds + \int_0^t a(s)p(x(s))x'(s)x''(s)ds,$
$I_3(t) = -b(t)q(x(t))x(t)x'(t) + \int_0^t b'(s)q(x(s))x(s)x'(s)ds + \int_0^t b(s)q(x(s))x'^2(s)ds,$
$I_4(t) = -\frac{1}{2}c(t)f(x(t))x^2(t) + \frac{1}{2}\int_0^t c'(s)f(x(s))x^2(s)ds + \frac{1}{2}\int_0^t c(s)f'(x(s))x'(s)x^2(s)ds,$
$I_5(t) = \int_0^t e(s)x(s)ds,$

and

$$\begin{aligned} R_0 \quad &= \quad \Big(g'(x(0))x'(0)x''(0) + g(x(0))x'''(0)\Big)x(0) \\ &\quad - g(x(0))x''(0)x'(0) + a(0)p(x(0))x(0)x''(0) \\ &\quad + b(0)q(x(0))x(0)x'(0) + \frac{1}{2}c(0)f(x(0))x^2(0). \end{aligned}$$

From (2.14), (2.16), (2.21) and the conditions (i), (ii), $(H_1)\sim(H_4)$, we have

$$\begin{aligned}
I_1(t) &\le \Big(\frac{\eta_4}{m}\lambda_2^{\frac{3}{2}} + M\Big(\frac{1}{m}\sqrt{\lambda_2} + \frac{\eta_4}{m^2}\lambda_2\Big)\Big)\sqrt{\lambda_2} + \frac{M}{m}\lambda_2 + M\int_0^t x''^2(s)ds,\\
I_2(t) &\le a_1\frac{M}{m}\lambda_2 + \frac{M}{m}\lambda_2\int_0^t |a'(s)|ds + a_1 M\int_0^t x'(s)x''(s)ds,\\
&\le a_1\frac{M}{m}\lambda_2 + \frac{1}{2}a_1 M\lambda_2 + \frac{M}{m}\lambda_2\int_0^t |a'(s)|ds,\\
I_3(t) &\le b_1q_1\lambda_2 + q_1\lambda_2\int_0^t |b'(s)|ds + b_1q_1\int_0^t x'^2(s)ds,\\
I_4(t) &\le \frac{1}{2}c_1M\lambda_2 + \frac{1}{2}M\lambda_2\int_0^t |c'(s)|ds, + \frac{1}{2}c_1\lambda_2^{\frac{3}{2}}\int_0^t |f'(s)|ds,
\end{aligned}$$

and

$$I_5(t) \le \sqrt{\lambda_2}\int_0^t |e(s)|ds.$$

Hence

$$\lim_{t\to+\infty} I_1(t) \le \Big(\frac{\eta_4}{m}\lambda_2^{\frac{3}{2}} + M\Big(\frac{1}{m}\sqrt{\lambda_2} + \frac{\eta_4}{m^2}\lambda_2\Big)\Big)\sqrt{\lambda_2} + \frac{M}{m}\lambda_2 + M\lambda_4 = R_1,$$
$$\lim_{t\to+\infty} I_2(t) \le a_1\frac{M}{m}\lambda_2 + \frac{1}{2}a_1M\lambda_2 + \frac{M}{m}\lambda_2\eta_1 = R_2,$$
$$\lim_{t\to+\infty} I_3(t) \le b_1q_1\lambda_2 + q_1\lambda_2\eta_1 + b_1q_1\lambda_3 = R_3,$$
$$\lim_{t\to+\infty} I_4(t) \le \frac{1}{2}c_1M\lambda_2 + \frac{1}{2}M\lambda_2\eta_1 + \frac{1}{2}c_1\lambda_2^{\frac{3}{2}}\eta_2 = R_4, \quad \text{and} \quad \lim_{t\to+\infty} I_5(t) \le \sqrt{\lambda_2}\eta_3 = R_5.$$

Thus

$$\lim_{t\to+\infty}\Big(I_1(t) + I_2(t) + I_3(t) + I_4(t) + I_5(t)\Big) \le \sum_{i=1}^{5} R_i < \infty. \tag{2.24}$$

Consequently, (2.23), (2.24) and condition (iii) gives

$$\int_0^\infty x^2(s)ds \le \frac{1}{d_0\delta}\int_0^\infty d(s)x(s)h(x(s))ds \le \frac{1}{d_0\delta}\sum_{i=0}^{5} R_i < \infty.$$

The proof of the theorem is now completed. □

If $e(t) = 0$, similarly to above proof, the inequality (2.10) becomes

$$.W_{(??)} \;=\; \Big(.V_{(??)} - \frac{1}{\eta}(\gamma_1(t) + \gamma_2(t))V\Big)\, e^{-\frac{1}{\eta}\int_0^t(\gamma_1(s)+\gamma_2(s))ds}$$

$$\begin{aligned}&\leq -D_3\left(y^2+z^2+w^2\right)e^{-\frac{1}{\eta}\int_0^t(\gamma_1(s)+\gamma_2(s))ds}\\&\leq -\mu\left(y^2+z^2+w^2\right),\end{aligned}$$

where $\mu = D_3 e^{-\frac{1}{\eta}(\eta_1+\frac{\eta_2}{m^2})}$. It is easy to see that the only solution of system (2.1) for which $.W_{(??)}(t,x,y,z,w)=0$ is the solution $x=y=z=w=0$. Hence the trivial solution of equation (1.5) is uniformly asymptotically stable, and the same conclusion as in the proof of Theorem 3.2 can be drawn for the square integrability of solutions of equation (1.5).

3. Example

We consider the following fourth order non-autonomous differential equation

$$\left(\left(\frac{x^2\sin x+5x^4+5}{5(1+x^4)}\right)x''\right)''+\left(e^{-t}\sin t+2\right)\left(\left(\frac{x+4e^x+4e^{-x}}{4(e^x+e^{-x})}\right)x''\right)'$$
$$+\left(\tfrac{\cos t+7t^2+7}{1+t^2}\right)\left(\left(\tfrac{\sin x+6e^x+6e^{-x}}{e^x+e^{-x}}\right)x'\right)'$$
$$+(e^{-2t}\sin^3 t+2)\left(\tfrac{x\cos x+5x^4+5}{5(1+x^4)}\right)x'+\left(\tfrac{\cos^2 t+t^2+1}{10(1+t^2)}\right)\left(\tfrac{x}{x^2+1}\right)=\tfrac{2\sin t}{t^2+1}.(3.1)$$

(3.2)

Taking

$g(x)=\dfrac{x^2\sin x+5x^4+5}{5(1+x^4)}$, $p(x)=\dfrac{x+4e^x+4e^{-x}}{4(e^x+e^{-x})}$, $q(x)=\dfrac{\sin x+6e^x+6e^{-x}}{e^x+e^{-x}}$,
$f(x)=\dfrac{x\cos x+5x^4+5}{5(1+x^4)}$, $h(x)=\dfrac{x}{x^2+1}$, $a(t)=e^{-t}\sin t+2$, $b(t)=\dfrac{\cos t+7t^2+7}{1+t^2}$,
$c(t)=e^{-2t}\sin^3 t+2$, $d(t)=\dfrac{\cos^2 t+t^2+1}{10(1+t^2)}$ and $e(t)=\dfrac{2\sin t}{t^2+1}$.

we have

$m=\dfrac{9}{10}$, $M=\dfrac{11}{10}$, $q_0=\dfrac{11}{2}$, $q_1=\dfrac{13}{2}$, $h_0=\dfrac{5}{2}$, $\delta_0=\dfrac{5}{3}$, $a_0=$

1 , $a_1 = 3$, $b_0 = 6$,
$b_1 = 8$, $c_0 = 1$, $c_1 = 3$, $d_0 = \frac{1}{10}$, $d_1 = \frac{1}{5}$,

we find

$$\frac{h_0}{m} - \frac{a_0 m \delta_0}{d_1} \quad = \quad -4.55 \leq h'(x) \leq 1.1 \leq \frac{h_0}{2M},$$

and

$$\begin{aligned}
\kappa_1 &= \frac{a_1 h_0 d_1 M^2}{c_0 m^3} + \frac{M^3(c_1 + \delta_0)}{a_0 m^2} + a_0 a_1 m (M-1) \leq 10 \\
\kappa_2 &= \frac{2 d_1 h_0 a_0}{c_0 (M-1)} \left(\frac{1}{m} - \frac{1}{M} \right)^2 + 2 \frac{c_0 M}{a_0} + 2 a_1 \frac{d_1 h_0 M}{c_0 m^3} + \frac{c_0 c_1 (M^2+2) m M}{d_1 h_0} \leq 27
\end{aligned}$$

Also

$$\begin{aligned}
\int_{-\infty}^{+\infty} |g'(x)|\, dx &= \frac{1}{5} \int_{-\infty}^{+\infty} \left| \frac{-4x^5 \sin x + (2x \sin x + x^2 \cos x)(x^4+1)}{(x^4+1)^2} \right| dx \\
&\leq \frac{1}{5} \int_{-\infty}^{+\infty} \left(\frac{x^2}{x^4+1} + \frac{4x^6}{(x^4+1)^2} + \frac{2x^2}{x^4+1} \right) dx = \frac{3}{5}\sqrt{2}\pi, \\
\int_{-\infty}^{+\infty} |p'(x)|\, dx &= \frac{1}{4} \int_{-\infty}^{+\infty} \left| \frac{1}{e^x + e^{-x}} + x \frac{e^{-x} - e^x}{(e^x + e^{-x})^2} \right| dx \\
&\leq \frac{1}{4} \int_{-\infty}^{0} \left(\frac{1}{e^x + e^{-x}} - x \frac{e^{-x} - e^x}{(e^x + e^{-x})^2} \right) dx \\
&\quad + \frac{1}{4} \int_{0}^{+\infty} \left(\frac{1}{e^x + e^{-x}} - x \frac{e^{-x} - e^x}{(e^x + e^{-x})^2} \right) dx = \frac{\pi}{4},
\end{aligned}$$

$$\begin{aligned}
\int_{-\infty}^{+\infty} |q'(x)|\, dx &= \frac{1}{5} \int_{-\infty}^{+\infty} \left| \frac{(e^x + e^{-x}) \cos x - (e^x - e^{-x}) \sin x}{(e^x + e^{-x})^2} \right| dx \\
&\leq \frac{1}{5} \int_{-\infty}^{+\infty} \left(\frac{1}{e^x + e^{-x}} + \frac{x}{(e^x + e^{-x})^2} (e^x - e^{-x}) \right) dx = \frac{1}{5}\pi, \\
\int_{-\infty}^{+\infty} |f'(x)|\, dx &= \frac{1}{5} \int_{-\infty}^{+\infty} \left| \frac{(\cos x - x \sin x)(x^4+1) - 4x^4 \cos x}{(x^4+1)^2} \right| dx
\end{aligned}$$

$$\begin{aligned}
&= \frac{1}{5}\int_{-\infty}^{+\infty}\left|\frac{\cos x}{x^4+1} - 4x^4\frac{\cos x}{(x^4+1)^2} - x\frac{\sin x}{x^4+1}\right| dx \\
&\leq \frac{1}{5}\int_{-\infty}^{+\infty}\left(\frac{5}{x^4+1} + \frac{x^2}{x^4+1}\right) dx = \frac{9}{10}\sqrt{2}\pi.
\end{aligned}$$

Then

$$\int_{-\infty}^{+\infty}\left(\left|g'(s)\right| + \left|p'(s)\right| + \left|q'(s)\right| + \left|f'(s)\right|\right) ds < \infty, \quad \left|g'(t)\right| < 3,$$

and

$$\int_0^{+\infty}\left|e(t)\right| dt = \int_0^{+\infty}\left|\frac{2\sin t}{t^2+1}\right| dt \leq \int_0^{+\infty}\frac{2}{t^2+1} dt = \pi,$$

$$\begin{aligned}
\int_0^{+\infty}\left|a'(t)\right| dt &= \int_0^{+\infty}\left|(\cos t)e^{-t} - (\sin t)e^{-t}\right| dx \leq \int_0^{+\infty} 2e^{-t} dx = 2, \\
\int_0^{+\infty}\left|b'(t)\right| dt &= \int_0^{+\infty}\left|-\frac{\sin t}{t^2+1} - 2t\frac{\cos t}{(t^2+1)^2}\right| dx \leq \int_0^{+\infty}\left(\frac{1}{t^2+1} + \frac{2|t|}{(t^2+1)^2}\right) dx
\end{aligned}$$

$\leq \int_0^{+\infty}\left(\frac{1}{t^2+1} + \frac{t^2+1}{(t^2+1)^2}\right) dx = \int_0^{+\infty}\frac{2}{t^2+1} dx = \pi,$

$\int_0^{+\infty}\left|c'(t)\right| dt = \int_0^{+\infty}\left|c'(t)\right| dx = \int_0^{+\infty}\left|3\left(\cos t\sin^2 t\right)e^{-2t} - 2\left(\sin^3 t\right)e^{-2t}\right| dx$

$\leq \int_0^{+\infty} 5e^{-2t} dx = \frac{5}{2},$

$\int_0^{+\infty}\left|d'(t)\right| dt = \int_0^{+\infty}\left|-2(\cos t)\frac{\sin t}{t^2+1} - 2t\frac{\cos^2 t}{(t^2+1)^2}\right| dx$

$\leq \int_0^{+\infty}\left(\frac{2}{t^2+1} + \frac{2|t|}{(t^2+1)^2}\right) dx \leq \int_0^{+\infty}\frac{3}{t^2+1} dx = \frac{3\pi}{2}.$

Then

$$\int_0^{+\infty}\left(\left|a'(t)\right| + \left|b'(t)\right| + \left|c'(t)\right| + \left|d'(t)\right|\right) dt < +\infty.$$

Thus all the assumptions of Theorem (2) hold, this shows that every solution $x(t)$ of (**??**) and their derivatives $x'(t), x''(t)$ and $x'''(t)$ are bounded and square integrable.

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