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On the classification of hypersurfaces in Euclidean spaces satisfying $L_r\overrightarrow{\mathbf{H}}_{r+1} = \lambda\overrightarrow{\mathbf{H}}_{r+1}$

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Abstract

In this paper, we study isometrically immersed hypersurfaces of the Euclidean space E^{n+1} satisfying the condition $L_r\overrightarrow{H}_{r+1} = \lambda\overrightarrow{H}_{r+1}$ for an integer r ($0 \leq r \leq n-1$), where $\overrightarrow{H}_{r+1}$ is the $(r+1)$th mean curvature vector field on the hypersurface, L_r is the linearized operator of the first variation of the $(r+1)$th mean curvature of hypersurface arising from its normal variations. Having assumed that on a hypersurface $x : M^n \rightarrow E^{n+1}$, the vector field $\overrightarrow{H}_{r+1}$ be an eigenvector of the operator L_r with a constant real eigenvalue λ, we show that, M^n has to be an L_r-biharmonic, L_r-1-type, or L_r-null-2-type hypersurface. Furthermore, we study the above condition on a well-known family of hypersurfaces, named the weakly convex hypersurfaces (i.e. on which principal curvatures are nonnegative). We prove that, any weakly convex Euclidean hypersurface satisfying the condition $L_r\overrightarrow{H}_{r+1} = \lambda\overrightarrow{H}_{r+1}$ for an integer r ($0 \leq r \leq n-1$), has constant mean curvature of order $(r+1)$. As an interesting result, we have that, the L_r-biharmonicity condition on the weakly convex Euclidean hypersurfaces implies the r-minimality.

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1. Introduction

The biharmonic functions as the solution of some well-known partial differential equations frequently appear in mathematical physics. Especially, when it becomes very difficult to find harmonic maps, sometimes the biharmonic ones are helpful. From geometric points of view, the role of biharmonic surfaces in elasticity and fluid mechanics can be considered as a physical motivation for the theory of biharmonicity. From the differential geometric points of view, B.Y. Chen (in the eighties) has started to investigate the properties of biharmonic submanifolds in the Euclidean spaces (whose position vector filed $x : M^n \to E^{n+k}$ satisfies the condition $\Delta^2 x = 0$, where Δ is the Laplace operator). He introduced some open problems and conjectures in [5], among them, a longstanding conjecture says that a biharmonic submanifold in a Euclidean space is a minimal one. Chen himself has proved the conjecture for surfaces in E^3. Later on, I. Dimitrić has verified Chen's conjecture in several different cases such as special curves, submanifolds of constant mean curvature and also, hypersurfaces of the Euclidean spaces with at most two distinct principal curvatures. T. Hasanis and T. Vlachos in [10] proved the conjecture for hypersurfaces in E^4. Having assumed the completeness, Akutagawa and Maeta ([1]) gave an affirmative answer to the global version of Chen's conjecture for biharmonic submanifolds in Euclidean spaces. Recently, in [8], it is proved that only biharmonic hypersurfaces in space forms with three distinct principal curvatures are minimal ones. An equivalent condition for the biharmonicity of an Euclidean hypersurfaces can be expressed as $\Delta\overrightarrow{\mathbf{H}} = 0$, where $\overrightarrow{\mathbf{H}}$ is the mean curvature vector field on the hypersurface. In 1988, Chen has started the study of a natural extension of this condition by assuming $\overrightarrow{\mathbf{H}}$ to be an eigenvector of Δ associated to an arbitrary constant real eigenvalue. In [6], Defever has proved that the hypersurfaces of E^4 satisfying the condition $\Delta\overrightarrow{\mathbf{H}} = \lambda\overrightarrow{\mathbf{H}}$ have constant mean curvature.

On the other hand, the Laplacian operator Δ can be seen as the first one of a sequence of n operators $L_0 = \Delta, L_1, \ldots, L_{n-1}$, where L_r stands for the linearized operator of the first variation of the $(r+1)$th mean curvature arising from normal variations of the hypersurface (see, for instance, [2]). These operators are given by $L_r(f) = tr(P_r \circ \nabla^2 f)$ for any $f \in C^\infty(M)$, where P_r denotes the rth Newton transformation associated to the second fundamental from of the hypersurface and $\nabla^2 f$ is the hessian of f. In this paper we consider the Euclidean hypersurfaces satisfying $L_r\overrightarrow{\mathbf{H}}_{r+1} = \lambda\overrightarrow{\mathbf{H}}_{r+1}$, where

$\overrightarrow{\mathbf{H}}_{r+1}$ is the $(r+1)$th mean curvature vector field the Euclidean hypersurface M. From this point of view, as an extension of finite type theory, S.M.B. Kashani ([11]) has introduced the notion of L_r-finite type hypersurface in the Euclidean space, which can be found in the second edition of Chen's book [4]. Furthermore, In [3], it is proved that every L_r-biharmonic hypersurface in $\mathbf{E}^m$ (for arbitrary integer $m > 2$) with at most two distinct principal curvatures is r-minimal, $0 < r < m$.

In this paper, we try to classify the Euclidean hypersurfaces satisfying $L_r\overrightarrow{\mathbf{H}}_{r+1} = \lambda\overrightarrow{\mathbf{H}}_{r+1}$. Also, we study this condition together with the weak convexity. Here are our main results:

Theorem 1.1. *If $x : M^n \rightarrow E^{n+1}$ is an isometric immersion of a hypersurface into Euclidean space, then the $(r+1)$th mean curvature vector field $\overrightarrow{\mathbf{H}}_{r+1}$ is an eigenvector of L_r if and only if it satisfies one of the following families:*
(a) L_r-biharmonic hypersurfaces,
(b) L_r-1-type hypersurfaces,
(c) L_r-null-2-type hypersurfaces.

Theorem 1.2. *Let $x : M^n \rightarrow E^{n+1}$ be an isometrically immersed Euclidean hypersurface satisfying $L_{n-1}\overrightarrow{\mathbf{H}}_n = \lambda\overrightarrow{\mathbf{H}}_n$, then H_n is constant. Moreover, if $\lambda = 0$ then M^n is n-minimal or ordinary minimal.*

Theorem 1.3. *Let $x : M^n \rightarrow E^{n+1}$ be a weakly convex hypersurface satisfying $L_r\overrightarrow{\mathbf{H}}_{r+1} = \lambda\overrightarrow{\mathbf{H}}_{r+1}$. Then the $(r+1)$th mean curvature is constant.*

Theorem 1.4. *Assume that $x : M^n \rightarrow E^{n+1}$ is a weakly convex L_r-biharmonic hypersurface in E^{n+1}, i.e. $L_r^2 x = 0$. Then $H_{r+1} = 0$*

2. Preliminaries

In this section we recall some prerequisites about Newton transformations P_r and their associated second order differential operators L_r from [2].

Let $x : M^n \rightarrow \mathbf{E}^{n+1}$ be an isometrically immersed hypersurface in the Euclidean space, with the Gauss map N. We denote by ∇^0 and ∇ the Levi-Civita connections on $\mathbf{E}^{n+1}$ and M, respectively, then, the basic Gauss and Weingarten formulae of the hypersurface are written as

$$\nabla^0_X Y = \nabla_X Y + < SX, Y > N$$

and

$$SX = -\nabla^0_X N$$

for all tangent vector fields $X, Y \in \chi(M)$, where $S : \chi(M) \to \chi(M)$ is the shape operator (or Weingarten endomorphism) of M with respect to the Gauss map N. As is well known, S defines a self-adjoint linear operator on each tangent plane T_pM, and its eigenvalues $\lambda_1(p), \ldots, \lambda_n(p)$ are the principal curvatures of the hypersurface. Associated to the shape operator there are n algebraic invariants given by

$$s_r(p) = \sigma_r(\lambda_1(p), \ldots, \lambda_n(p)), \qquad 1 \leq r \leq n,$$

where $\sigma_r : \mathbf{R}^n \to \mathbf{R}$ is the elementary symmetric function in $\mathbf{R}^n$ given by

$$\sigma_r(x_1, \ldots, x_n) = \sum_{i_1 < \cdots < i_r} x_{i_1} \ldots x_{i_r}.$$

Observe that the characteristic polynomial of S can be written in terms of the s_r as

$$Q_s(t) = det(tI - S) = \sum_{r=0}^{n} (-1)^r s_r t^{n-r}, \tag{2.1}$$

where $s_0 = 1$ by definition. Then for any integer $r \in \{0, 1, \ldots, n-1\}$, we introduce rth mean curvature function H_r and $(r+1)$th mean curvature vector field $\overrightarrow{\mathbf{H}}_{r+1}$ as follows:

$$\binom{n}{r} H_r = s_r, \qquad \overrightarrow{\mathbf{H}}_{r+1} = H_{r+1} N.$$

In particular, when $r = 1$

$$H_1 = \frac{1}{n} \sum_{i=1}^{n} \lambda_i = \frac{1}{n} tr(S) = H$$

is nothing but the mean curvature of M, which is the main extrinsic curvature of the hypersurface. On the other hand, $H_n = \lambda_1 \cdots \lambda_n$ is called the Gauss-Kronecker curvature of M. A hypersurface with zero $(r+1)$th mean curvature in $\mathbf{E}^{n+1}$ is called r-minimal (see [14]).

The classical Newton transformations $P_r : \chi(M) \to \chi(M)$ are defined inductively by

$$P_0 = I \;\; and \;\; P_r = s_r I - S \circ P_{r-1} = \binom{n}{r} H_r I - S \circ P_{r-1}$$

for every $r = 1, \dots, n$ where I denotes the identity in $\chi(M)$.

Equivalently,

$$P_r = \sum_{j=0}^{r} (-1)^j s_{r-j} S^j = \sum_{j=0}^{r} (-1)^j \binom{n}{r-j} H_{r-j} S^j. \tag{2.2}$$

Note that by the Cayley-Hamilton theorem stating that any operator T is annihilated by its characteristic polynomial, we have $P_n = 0$ from (2.1).

Each $P_r(p)$ is also a self-adjoint linear operator on the tangent space T_pM which commutes with $S(p)$. Indeed, $S(p)$ and $P_r(p)$ can be simultaneously diagonalized: if $\{E_1, \dots, E_n\}$ are the eigenvectors of $S(p)$ corresponding to the eigenvalues $\lambda_1(p), \dots, \lambda_n(p)$, respectively, then they are also the eigenvectors of $P_r(p)$ with corresponding eigenvalues given by

$$\mu_{i,r}(p) = \sum_{i_1 < \dots < i_r, i_j \neq i} \lambda_{i_1}(p) \cdots \lambda_{i_r}(p), \tag{2.3}$$

for every $1 \leq i \leq n$.

Associated to each Newton transformation P_r, we consider the second-order linear differential operator $L_r : C^\infty(M) \to C^\infty(M)$ given by

$$L_r(f) = tr(P_r \circ \nabla^2 f).$$

Here, $\nabla^2 f : \chi(M) \to \chi(M)$ denotes the self-adjoint linear operator metrically equivalent to the Hessian of f and is given by

$$< \nabla^2 f(X), Y > = < \nabla_X(\nabla f), Y >, \qquad X, Y \in \chi(M).$$

3. Hypersurfaces in Euclidean spaces satisfying $L_r \mathbf{H}_{r+1} = \lambda \mathbf{H}_{r+1}$

First, we recall the definition of an L_r-finite type hypersurface from [11], which is the basic notion of the paper.

Definition 3.1. *An isometrically immersed hypersurface $x : M^n \to \mathbf{E}^{n+1}$ is said to be of L_r-finite type if x has a finite decomposition $x = \sum_{i=0}^m x_i$, for some positive integer m satisfying the condition that $L_r x_i = \kappa_i x_i, \kappa_i \in \mathbf{R}, 1 \leq i \leq m$, where $x_i : M^n \to \mathbf{E}^{n+1}$ are smooth maps, $1 \leq i \leq m$, and x_0 is constant. If all κ_i's are mutually different, M^n is said to be of L_r-m-type. An L_r-m-type hypersurface is said to be null if some $\kappa_i; 1 \leq i \leq m$, is zero.*

Let $x : M^n \to E^{n+1}$ be a connected orientable hypersurface immersed into Euclidean space, with Gauss map N. Then, as is well known (see [2]),

$$L_r x = c_r \overrightarrow{\mathbf{H}}_{r+1}, \tag{3.1}$$

where $c_r = (n-r)\binom{n}{r}$. This shows, in particular, that M^n is a r-minimal hypersurface of $\mathbf{E}^{n+1}$ if and only if its coordinate functions are L_r-harmonic (i.e., if they are eigenfunctions with eigenvalue 0):

$$\overrightarrow{\mathbf{H}}_{r+1} = 0 \Longleftrightarrow L_r x = 0. \tag{3.2}$$

Condition (3.2) can be generalized in several directions. In [13] and inspired by Takahashi theorem, the first author jointly with Kashani studied and classified hypersurfaces in Euclidean spaces for which

$$L_r x = \lambda x; \quad \lambda \in \mathbf{R}, \tag{3.3}$$

that is, hypersurfaces for which all coordinate functions are eigenfunctions of L_r with the same eigenvalue λ. In terms of L_r-finite type theory, condition (3.3) characterizes the L_r-1-type hypersurfaces of $\mathbf{E}^{n+1}$. In [13], the authors showed that r-minimal hypersurfaces and open parts of hyperspheres are the only L_r-1-type Euclidean hypersurfaces.

Most recently, condition (3.2) generalized in another direction by Aminian and Kashani([3]), they studied the hypersurfaces of $\mathbf{E}^{n+1}$ satisfying

$$L_r \overrightarrow{\mathbf{H}}_{r+1} = 0 \Longleftrightarrow L_r^2 x = 0. \tag{3.4}$$

Hypersurfaces of $\mathbf{E}^{n+1}$ satisfying (3.4) called L_r-biharmonic hypersurfaces. Conditions (3.3) and (3.4) may be generalized and combined into the

$$L_r \overrightarrow{\mathbf{H}}_{r+1} = \lambda \overrightarrow{\mathbf{H}}_{r+1}, \quad \lambda \in \mathbf{R}. \tag{3.5}$$

Theorem 1.1 determines hypersurfaces of $\mathbf{E}^{n+1}$ which satisfy $L_r \overrightarrow{\mathbf{H}}_{r+1} = \lambda \overrightarrow{\mathbf{H}}_{r+1}$ for some $\lambda \in \mathbf{R}$.

Proof of Theorem 1.1. Under the hypothesis, assume that $L_r \overrightarrow{\mathbf{H}}_{r+1} = \lambda \overrightarrow{\mathbf{H}}_{r+1}$ holds for some real number λ. If $\lambda = 0$, then M^n is a L_r-biharmonic hypersurface, which gives (a). Now, assume that $L_r \overrightarrow{\mathbf{H}}_{r+1} = \lambda \overrightarrow{\mathbf{H}}_{r+1}$ with $\lambda \neq 0$. Taking

$$x_p = \frac{1}{\lambda} L_r x \quad and \quad x_0 = x - x_p,$$

we find

$$L_r x_p = \frac{1}{\lambda} L_r^2 x = \frac{c_r}{\lambda} L_r \overrightarrow{\mathbf{H}}_{r+1} = c_r \overrightarrow{\mathbf{H}}_{r+1} = L_r x.$$

Hence, M is either of L_r-1-type or of L_r-null-2-type, depending on x_0 is a constant or non-constant. Conversely, if M is L_r-biharmonic or L_r-null-2-type hypersurface, then $L_r^2 x = \lambda x$; $\lambda \in R$, so formula (3.1) gives the result. If M is L_r-1-type hypersurface, then $L_r x = \lambda x$; $\lambda \in R$, so by taking L_r of this equation and using (3.1) we get the result. □

By formulae in [2] page 122, we have

$$L_r^2 x = -c_r \binom{n}{r+1} H_{r+1} \nabla H_{r+1} - 2(S \circ P_r)(\nabla H_{r+1})$$

$$\text{-c}_r \left(\tbinom{n}{r+1} H_{r+1}(nH_1 H_{r+1} - (n-r-1)H_{r+2}) - L_r H_{r+1} \right) N. \tag{3.6}$$

By identifying normal and tangent parts of (3.6), one obtains necessary and sufficient conditions for the $(r+1)$th mean curvature vector field $\overrightarrow{\mathbf{H}}_{r+1}$ be an eigenvector of L_r, namely

$$L_r H_{r+1} - \binom{n}{r+1} H_{r+1}(nH_1 H_{r+1} - (n-r-1)H_{r+2}) = \lambda H_{r+1} \tag{3.7}$$

and

$$(S \circ P_r)(\nabla H_{r+1}) = -\frac{1}{2} \binom{n}{r+1} H_{r+1} \nabla H_{r+1}. \tag{3.8}$$

Since $P_n = 0$, $(n = \dim M)$; $S \circ P_{n-1} = H_n I$, by equations (3.7) and (3.8), hence one leads to consider the case $r = n-1$, at first. Here we prove Theorem 1.2.

Proof of Theorem 1.2. By (3.8) we have

$$(S \circ P_{n-1})(\nabla H_n) = -\frac{1}{2} H_n \nabla H_n.$$

We know that $P_n = 0$, hence $S \circ P_{n-1} = H_n I$. So $\frac{3}{2}\nabla H_n^2 = 0$. Therefore H_n is constant.
If $\lambda = 0$ and $H_n \neq 0$, by using (3.7) we obtain that $H = 0$. □

4. Weakly convex hypersurfaces in Euclidean spaces

Recently, in [12], the ordinary biharmonicity condition is verified on the hypersurfaces of space forms of nonpositive sectional curvature with an additional condition named weak convexity. A hypersurfaces of a space form is said to be weakly convex if all of its principal curvatures be non-negative. Here, we study the L_r-biharmonicity condition and in general $L_r\overrightarrow{\mathbf{H}}_{r+1} = \lambda\overrightarrow{\mathbf{H}}_{r+1}$ on weakly convex Euclidean hypersurfaces. We prove the Theorem 1.3.

Proof of Theorem 1.3.

Define

$$B := \{p \in M : \nabla H_{r+1}^2(p) \neq 0\}.$$

We will prove that B is an empty set by a contradiction argument, and so $(r+1)$th mean curvature is constant and we are done. We choose a local orthonormal frame $\{E_1, \ldots, E_n\}$ such that $S(E_i) = \lambda_i E_i$ and $P_r(E_i) = \mu_{i,r} E_i$, where λ_i, s and $\mu_{i,r}, s$ are eigenvalues of S and P_r, respectively, $1 \leq i \leq n$, which are nonnegative by the assumption that M^n is weakly convex.

We have $\nabla H_{r+1} = \sum_{i=1}^{n} < \nabla H_{r+1},\ E_i > E_i$, so (3.8) is equivalent to

$$< \nabla H_{r+1},\ E_i > \left(\lambda_i \mu_{i,r} + \frac{1}{2}\binom{n}{r+1} H_{r+1}\right) = 0, \qquad on\ B, \tag{4.1}$$

for every $i = 1, \ldots, n$. Therefore, for every i such that $< \nabla H_{r+1},\ E_i > \neq 0$ on B we get

$$(\lambda_i \mu_{i,r} + \frac{1}{2}\binom{n}{r+1} H_{r+1}) = 0, \qquad on\ B.$$

So by the assumption that M^n is weakly convex, we obtain that $H_{r+1} = 0$ locally on B, which is a contradiction with the definition of B. This finishes the proof. □

Using the idea of the last proof, we prove Theorem 1.4 as follows.

Proof of Theorem 1.4. By Theorem 1.3, the $(r + 1)$th mean curvature H_{r+1} is constant. It is always true that

$$H_{i-1} H_{i+1} \leq H_i^2$$

and
$$H_1 \geq H_2^{1/2} \geq H_3^{1/3} \geq \cdots \geq H_i^{1/i} \qquad (1 \leq i < n),$$
provided $H_1, H_2, \ldots, H_i$ are nonnegative, [page 52 of [9]].
Then, from these above inequalities, we obtain

$$HH_{r+1} - H_{r+2} \geq \frac{H_{r+1}}{H_r}(HH_r - H_{r+1}) \geq \frac{H_{r+1}}{H_r}(HH_r - H_r^{\frac{r+1}{r}}) \geq H_{r+1}(H - H_r^{\frac{1}{r}}) \geq 0. \tag{4.2}$$

And the other hand, since H_{r+1} is a constant and M^n is L_r-biharmonic, by using formula (3.7) we get

$$nHH_{r+1} = (n - r - 1)H_{r+2},$$

so, when $r = n - 1$, we have $H = 0$ therefore from the above inequalities, we get $H_n = 0$. When $r < n - 1$, formula (4.2) and this above equation we get $H_{r+1} = 0$. $\square$

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