Proyecciones Journal of Mathematics
Vol. 35, N^o 2, pp. 213-223, June 2016.
Universidad Católica del Norte
Antofagasta - Chile
DOI: 10.4067/S0716-09172016000200006

On Jensen's and the quadratic functional equations with involutions

B. Fadli
A. Chahbi
Iz. El-Fassi
and
S. Kabbaj
IBN Tofail University, Morocco
Received : March 2016. Accepted : May 2016

Abstract

We determine the solutions $f : S \to H$ of the generalized Jensen's functional equation

$$f(x + \sigma(y)) + f(x + \tau(y)) = 2f(x), \quad x, y \in S,$$

and the solutions $f : S \to H$ of the generalized quadratic functional equation

$$f(x + \sigma(y)) + f(x + \tau(y)) = 2f(x) + 2f(y), \quad x, y \in S,$$

where S is a commutative semigroup, H is an abelian group (2-torsion free in the first equation and uniquely 2-divisible in the second) and σ, τ are two involutions of S.

Subjclass [2010] : *Primary 39B52.*

Keywords : *Functional equation, Jensen, quadratic, additive function, semigroup.*

1. Set up, notation and terminology

Throughout the paper we work in the following framework and with the following notation and terminology. We use it without explicit mentioning. S is a commutative semigroup [a set equipped with an associative composition rule $(x, y) \mapsto x + y$], $\sigma, \tau : S \to S$ are two homomorphisms satisfying $\sigma \circ \sigma = \tau \circ \tau = id$, and $(H, +)$ denotes an abelian group with neutral element 0. We say that H is 2-torsion free if $[h \in H \text{ and } 2h = 0] \Rightarrow h = 0$.
H is said to be uniquely 2-divisible if for any $h \in H$ the equation $2x = h$ has exactly one solution $x \in H$.

A function $A : S \to H$ is said to be additive if $A(x + y) = A(x) + A(y)$ for all $x, y \in S$.

We recall that the Cauchy difference Cf of a function $f : S \to H$ is defined by

$$Cf(x, y) := f(x + y) - f(x) - f(y), \quad x, y \in S.$$

2. Introduction

In [16], Sinopoulos determined the general solution $f : S \to H$, where H is 2-torsion free, of Jensen's functional equation

$$f(x + y) + f(x + \tau(y)) = 2f(x), \quad x, y \in S, \tag{2.1}$$

and the general solution $f : S \to H$, where H is uniquely 2-divisible, of the quadratic functional equation

$$f(x + y) + f(x + \tau(y)) = 2f(x) + 2f(y), \quad x, y \in S. \tag{2.2}$$

Some information, applications and numerous references concerning (2.1) and (2.2) and their further generalizations can be found, e.g., in [3-8, 10-14, 17, 18]. For more details, we refer to the monographs [9, 15, 19].

The purpose of the present paper is to solve the following functional equations

$$f(x + \sigma(y)) + f(x + \tau(y)) = 2f(x), \quad x, y \in S, \tag{2.3}$$

$$f(x + \sigma(y)) + f(x + \tau(y)) = 2f(x) + 2f(y), \quad x, y \in S. \tag{2.4}$$

Thus the contribution by our paper of new knowledge consists in introducing an involution σ and in solving the corresponding extensions (2.3) and

(2.4) of the functional equations (2.1) and (2.2). Our solution formulas contain the previous ones as special cases.

A similar functional equation that has been studied is

$$f(x+\sigma(y))+f(x+\tau(y))=2f(x)f(y), \quad x,y\in S, \tag{2.5}$$

where $f: S\to \mathbf{C}$ is the function to determine. Eq. (2.5) was solved in a more general framework (see [2]).

3. On Jensen's functional equation

In this section, we solve the functional equation (2.3) by expressing its solutions in terms of additive functions.

Lemma 3.1. *Let $f: S\to H$ be a solution of the functional equation (2.3). Then*

$$f(x+(\tau\circ\sigma)^2(y))=f(x+y) \quad \text{for all } x,y\in S. \tag{3.1}$$

Proof. Making the substitutions $(x,\sigma(y))$ and $(x,\tau(y))$ in (2.3), we get respectively

$$f(x+y)+f(x+\tau(\sigma(y)))=2f(x),$$
$$f(x+\sigma(\tau(y)))+f(x+y)=2f(x).$$

So

$$f(x+\tau(\sigma(y)))=f(x+\sigma(\tau(y))) \quad \text{for all } x,y\in S.$$

Replacing here y by $\tau(\sigma(y))$, we obtain (3.1). □

Theorem 3.2. *Suppose that H is 2-torsion free. The general solution $f: S\to H$ of the functional equation (2.3) is $f=A+c$, where $A: S\to H$ is an additive map such that $A\circ\tau=-A\circ\sigma$, and where $c\in H$ is a constant.*

Proof. The method used here is closely related to and inspired by the one in [16, Proof of Theorem 2]. Assume that $f: S\to H$ is a solution of (2.3). Then

$$f(x+y)+f(x+\tau(\sigma(y)))=2f(x), \quad x,y\in S. \tag{3.2}$$

Making the substitutions $(x, y + \tau(\sigma(y)))$ and $(x + z, y)$ in (3.2) and using Lemma 3.1, we get respectively

$$f(x + y + \tau(\sigma(y))) = f(x), \tag{3.3}$$
$$f(x + z + y) + f(x + z + \tau(\sigma(y))) = 2f(x + z).$$

Interchanging y and z in the last equation we have

$$f(x + y + z) + f(x + y + \tau(\sigma(z))) = 2f(x + y).$$

Adding the last two equations we obtain

$$\begin{aligned} 2f(x + y + z) + f(x + y + \tau(\sigma(z))) + f(x + z + \tau(\sigma(y))) \\ = 2f(x + z) + 2f(x + y). \end{aligned} \tag{3.4}$$

Using Lemma 3.1, we get that

$$f(x + z + \tau(\sigma(y))) = f(x + (\tau \circ \sigma)[y + (\tau \circ \sigma)(z)]).$$

So, using (3.2), we can reformulate (3.4) to

$$2f(x + y + z) + 2f(x) = 2f(x + z) + 2f(x + y).$$

Setting here $z = \tau(\sigma(x))$ and using (3.3) and the fact that H is 2-torsion free, we get

$$f(y) + f(x) = f(x + \tau(\sigma(x))) + f(x + y). \tag{3.5}$$

Interchanging x and y in (3.5), we get that

$$f(x + \tau(\sigma(x))) = f(y + \tau(\sigma(y)))$$

for all $x, y \in S$. So $f(x + \tau(\sigma(x)))$ is a constant, say c. By using (3.5), we infer that the function $A(x) := f(x) - c$ is additive. Substituting f into (2.3) we see that $A \circ \tau = -A \circ \sigma$.
The other direction of the proof is trivial to verify. $\square$

As a immediate consequence of Theorem 3.2, we have the following result.

Corollary 3.3. *[16, Theorem 2] Suppose that H is 2-torsion free. The general solution $f : S \to H$ of the functional equation (2.1) is $f = A + c$, where $A : S \to H$ is an additive map such that $A \circ \tau = -A$, and where $c \in H$ is a constant.*

4. On the quadratic functional equation

In this section, we generalize Sinopoulos's result [16, Theorem 3] on semigroups by solving the functional equation (2.4). The following lemma lists pertinent basic properties of any solution $f : S \to H$ of (2.4).

Lemma 4.1. *Suppose that H is 2-torsion free and let $f : S \to H$ be a solution of the functional equation (2.4).*

a) $f \circ \sigma + f \circ \tau = 2f$.

b) Let $A : S \to H$ be $A := f \circ \sigma - f \circ \tau$. Then A is additive and $A \circ \sigma = A \circ \tau = -A$.

c) For all $x, y, z \in S$, we have

$$f(x+y+z) = f(x+y) + f(x+z) + f(y+z) - f(x) - f(y) - f(z). \tag{4.1}$$

d) $Cf : S \times S \to H$ is a symmetric, bi-additive map satisfying $Cf(x, \tau(y)) = -Cf(x, \sigma(y))$ for all $x, y \in S$.

e) Let $\varphi : S \to H$ be $\varphi(x) := A(x) + 2f(x + \tau(x))$, $x \in S$. Then $\varphi \circ \sigma + \varphi \circ \tau = 2\varphi$ and φ satisfies that

$$\varphi(x+y) = \varphi(x) + \varphi(y) + 4\{Cf(x,y) - Cf(x, \sigma(y))\}, \quad x, y \in S.$$

f) φ is a solution of (2.4).

g) $4f(x) = 2Cf(x, \sigma(x)) + \varphi(x)$ for all $x \in S$.

Proof. (a) Let us first observe that $f \circ \sigma + f \circ \tau$ is a solution of (2.4). We next replace x, first by $\sigma(x)$ and then by $\tau(x)$, in (2.4) we find that

$$f(\sigma(x) + \sigma(y)) + f(\sigma(x) + \tau(y)) = 2f(\sigma(x)) + 2f(y), \tag{4.2}$$

$$f(\tau(x) + \sigma(y)) + f(\tau(x) + \tau(y)) = 2f(\tau(x)) + 2f(y). \tag{4.3}$$

Summing these two equations and using (2.4) and the fact that H is 2-torsion free, we obtain

$$[f(x) + f(\sigma(y))] + [f(x) + f(\tau(y))] = f(\sigma(x)) + f(\tau(x)) + 2f(y),$$

i.e.

$$2f(x) - f(\sigma(x)) - f(\tau(x)) = 2f(y) - f(\sigma(y) - f(\tau(y)),$$

for all $x, y \in S$. From this last equation we infer that $2f - f \circ \sigma - f \circ \tau$ is a constant in H, say c. Using the fact that $2f - (f \circ \sigma + f \circ \tau)$ is a solution of (2.4) and that H is 2-torsion free, we see that $c = 0$.
(b) We subtract (4.3) from (4.2) and get that

$$(f \circ \sigma - f \circ \tau)(x+y) + [f(\sigma(x)+\tau(y)) - f(\tau(x)+\sigma(y))] = 2(f \circ \sigma - f \circ \tau)(x), \tag{4.4}$$

for all $x, y \in S$. By using (2.4) and (a), we have

$$\begin{aligned} & f(\sigma(x)+\tau(y)) - f(\tau(x)+\sigma(y)) \\ & \quad = [f(\sigma(x)+\tau(y)) + f(\tau(x)+\tau(y))] - 2f \circ \tau(x) - 2f(y) \\ & \quad = 2f(x) + 2f \circ \tau(y) - 2f \circ \tau(x) - 2f(y) \\ & \quad = (f \circ \sigma - f \circ \tau)(x) - (f \circ \sigma - f \circ \tau)(y), \end{aligned}$$

which turns the identity (4.4) into

$$(f \circ \sigma - f \circ \tau)(x+y) = (f \circ \sigma - f \circ \tau)(x) + (f \circ \sigma - f \circ \tau)(y),$$

for all $x, y \in S$. This show that the function $A = f \circ \sigma - f \circ \tau$ is additive. Using (a), we see that

$$A = 2f - 2f \circ \tau = 2f \circ \sigma - 2f.$$

So, $A \circ \sigma = A \circ \tau = -A$.
(c) Making the substitutions $(x+y, \sigma(z))$, $(x+\tau(\sigma(z)), \sigma(y))$, and $(x, \sigma(y+z))$ in (2.4), we get respectively

$$\begin{aligned} f(x+y+z) + f(x+y+\tau(\sigma(z))) &= 2f(x+y) + 2f(\sigma(z)), \\ f(x+\tau(\sigma(z))+y) + f(x+\tau(\sigma(y+z))) &= 2f(x+\tau(\sigma(z)) + 2f(\sigma(y)) \\ &= 2\,[2f(x) + 2f(\sigma(z)) - f(x+z)] + 2f(\sigma(y)), \\ f(x+y+z) + f(x+\tau(\sigma(y+z))) &= 2f(x) + 2f(\sigma(y+z)). \end{aligned}$$

Subtracting the middle identity from the sum of the other two we get that

$$\begin{aligned} 2f(x+y+z) \quad = \quad & 2f(x+y) + 2f(x+z) + 2f(\sigma(y+z)) \\ & -2f(x) - 2f(\sigma(y)) - 2f(\sigma(z)). \end{aligned}$$

Replacing here $2f \circ \sigma$ by $2f + A$ and using the fact that H is 2-torsion free, we get (4.1).

(d) That Cf is symmetric and bi-additive follows immediately from the very definition of Cf and (4.1). Let $x, y \in S$ be arbitrary. By help of (4.1) and (a), we get that

$$\begin{aligned} Cf(x, \tau(y)) &= f(x + \tau(y)) - f(x) - f(\tau(y)) \\ &= 2f(x) + 2f(y) - f(x + \sigma(y)) - f(x) - f(\tau(y)) \\ &= f(x) + f(\sigma(y)) - f(x + \sigma(y)) \\ &= -Cf(x, \sigma(y)). \end{aligned}$$

(e) For all $x \in S$, we have

$$\begin{aligned} &(\varphi \circ \sigma + \varphi \circ \tau)(x) \\ &\quad = A(\sigma(x)) + 2f(\sigma(x) + \tau(\sigma(x))) + A(\tau(x)) + 2f(\tau(x) + x) \\ &\quad = -2A(x) + 2f(x + \tau(x)) + 8f(\sigma(x)) - 2f(x + \sigma(x)) \\ &\quad = -2A(x) + 2f(x + \tau(x)) + 8f(x) + 4A(x) - 2f(x + \sigma(x)) \\ &\quad = 2A(x) + 4f(x + \tau(x)) \\ &\quad = 2\varphi(x). \end{aligned}$$

So, $\varphi \circ \sigma + \varphi \circ \tau = 2\varphi$.

Next, let $x, y \in S$ be arbitrary. Using (4.1) repeatedly and the fact that $2f \circ \tau = 2f - A$ and that $A \circ \tau = -A$ we find

$$\begin{aligned} &\varphi(x + y) = A(x + y) + 2f((x + \tau(x)) + y + \tau(y)) \\ &\quad = A(x) + A(y) + 2f(x + \tau(x) + y) + 2f(x + \tau(x) + \tau(y)) \\ &\qquad + 2f(y + \tau(y)) - 2f(x + \tau(x)) - 2f(y) - 2f(\tau(y)) \\ &\quad = A(x) + A(y) + [2f(x + \tau(x)) + 2f(x + y) + 2f(\tau(x) + y) \\ &\qquad - 2f(x) - 2f(\tau(x)) - 2f(y)] + [2f(x + \tau(x)) + 2f(x + \tau(y)) \\ &\qquad + 2f \circ \tau(x + y) - 2f(x) - 2f(\tau(x)) - 2f(\tau(y))] \\ &\qquad + 2f(y + \tau(y)) - 2f(x + \tau(x)) - 2f(y) - 2f(\tau(y)) \\ &\quad = \varphi(x) + \varphi(y) + [2f(x + y) + 2f \circ \tau(x + y)] + [2f(x + \tau(y)) \\ &\qquad + 2f \circ \tau(x + \tau(y))] - 4f(x) - 4f \circ \tau(x) - 4f(y) - 4f \circ \tau(y) \\ &\quad = \varphi(x) + \varphi(y) + 4f(x + y) + 4f(x + \tau(y)) - A(x + y) \\ &\qquad - A(x + \tau(y)) - 8f(x) + 2A(x) - 4f(y) - 4f \circ \tau(y) \\ &\quad = \varphi(x) + \varphi(y) + 4Cf(x, y) + 4Cf(x, \tau(y)) \\ &\quad = \varphi(x) + \varphi(y) + 4\left\{Cf(x, y) - Cf(x, \sigma(y))\right\}. \end{aligned}$$

(f) Using (e) and (d), we get

$$\begin{aligned}
&\varphi(x+\sigma(y))+\varphi(x+\tau(y))\\
&\quad=2\varphi(x)+\varphi(\sigma(y))+\varphi(\tau(y))+4\{Cf(x,\sigma(y))-Cf(x,y)\}\\
&\qquad+4\{Cf(x,\tau(y))-Cf(x,\sigma(\tau(y)))\}\\
&\quad=2\varphi(x)+2\varphi(y)+4\{Cf(x,\sigma(y))-Cf(x,y)\}\\
&\qquad+4\{-Cf(x,\sigma(y))+Cf(x,y)\}\\
&\quad=2\varphi(x)+2\varphi(y).
\end{aligned}$$

So, φ is a solution of (2.4).
(g) Using the equality $A=2f\circ\sigma-2f$ and (2.4), we obtain

$$\begin{aligned}
&2Cf(x,\sigma(x))+\varphi(x)\\
&\quad=2f(x+\sigma(x))-2f(x)-2f(\sigma(x))+A(x)+2f(x+\tau(x))\\
&\quad=[2f(x+\sigma(x))+2f(x+\tau(x))]-2f(x)-2f(\sigma(x))+2f(\sigma(x))\\
&\qquad-2f(x)\\
&\quad=8f(x)-4f(x)\\
&\quad=4f(x)\quad\text{for all } x\in S.
\end{aligned}$$

□

The second main theorem of the present paper reads as follows.
Theorem 4.2. Suppose that H is uniquely 2-divisible. The general solution $f:S\to H$ of the functional equation (2.4) is

$$f(x)=Q(x,\sigma(x))+\psi(x),\quad x\in S,$$

where $Q:S\times S\to H$ is an arbitrary symmetric, bi-additive map such that $Q(x,\tau(y))=-Q(x,\sigma(y))$ for all $x,y\in S$, and where $\psi:S\to H$ is an arbitrary solution of

$$\psi(x+y)=\psi(x)+\psi(y)+2\{Q(x,y)-Q(x,\sigma(y))\},\quad x,y\in S,$$

such that $\psi\circ\sigma+\psi\circ\tau=2\psi$.

Proof. That all solutions of (2.4) have this form is a consequence of Lemma 4.1 and the fact that H is uniquely 2-divisible. Conversely, simple computations based on the properties of Q and ψ, show that the indicated functions are solutions. □

As a immediate consequence of Theorem 4.2, we have the following result.

Corollary 4.2. *[16, Theorem 3] Suppose that H is uniquely 2-divisible. The general solution $f : S \to H$ of the functional equation (2.2) is*

$$f(x) = Q(x, x) + \psi(x),$$

where $Q : S \times S \to H$ is an arbitrary symmetric, bi-additive map such that $Q(x, \tau(y)) = -Q(x, y)$ for all $x, y \in S$, and where $\psi : S \to H$ is an arbitrary additive map such that $\psi \circ \tau = \psi$.

References

[1] J. Aczél and J. Dhombres, Functional equations in several variables, Cambridge University Press, New York (1989).

[2] A. Chahbi, B. Fadli, S. Kabbaj, A generalization of the symmetrized multiplicative Cauchy equation, Acta Math. Hungar., pp. 1-7, (2016).

[3] J. K. Chung, B. R. Ebanks, C. T. Ng and P. K. Sahoo, On a quadratic trigonometric functional equation and some applications, Trans. Amer. Math. Soc., 347, pp. 1131-1161, (1995).

[4] B. Fadli, D. Zeglami and S. Kabbaj, On a Gajda's type quadratic equation on a locally compact abelian group, Indagationes Math., 26, pp. 660-668, (2015).

[5] B. Fadli, D. Zeglami and S. Kabbaj, A variant of Jensen's functional equation on semigroups, Demonstratio Math., to appear.

[6] P. de Place Friis and H. Stetkær, On the quadratic functional equation on groups, Publ. Math. Debrecen 69, pp. 65-93, (2006).

[7] S-M. Jung, Quadratic functional equations of Pexider type, J. Math. & Math. Sci., 24, pp. 351-359, (2000).

[8] P. Kannappan, Quadratic functional equation and inner product spaces, Results Math., 27, pp. 368-372, (1995).

[9] P. Kannappan, Functional equations and inequalities with applications, Springer, New York, (2009).

[10] C. T. Ng, Jensen's functional equation on groups, Aequationes Math. 39, pp. 85-99, (1990).

[11] C. T. Ng, Jensen's functional equation on groups, II, Aequationes Math. 58, pp. 311-320, (1999).

[12] C. T. Ng, Jensen's functional equation on groups, III, Aequationes Math. 62, pp. 143-159, (2001).

[13] C. T. Ng, A Pexider-Jensen functional equation on groups, Aequationes Math. 70, pp. 131-153, (2005).

[14] J. C. Parnami and H.L. Vasudeva, On Jensen's functional equation, Aequationes Math. 43, pp. 211-218, (1992).

[15] Th. M. Rassias, Inner Product Spaces and Applications, Pitman Research Notes in Mathematics Series, Addison Wesley Longman Ltd, 376, (1997).

[16] P. Sinopoulos, Functional equations on semigroups, Aequationes Math., 59, pp. 255-261, (2000).

[17] H. Stetkær, Functional equations on abelian groups with involution, Aequationes Math. 54, pp. 144-172, (1997).

[18] H. Stetkær, On Jensen's functional equation on groups, Aequationes Math. 66, pp. 100-118, (2003).

[19] H. Stetkær, Functional Equations on Groups, World Scientific Publishing Co, Singapore, (2013).

B. Fadli
Department of Mathematics,
Faculty of Sciences,
Ibn Tofail University,
B. P. 14000. Kenitra,
Morocco
e-mail : himfadli@gmail.com

A. Chahbi
Department of Mathematics,
Faculty of Sciences,
Ibn Tofail University,
B. P. 14000. Kenitra,
Morocco
e-mail : abdellatifchahbi@gmail.com

Iz. EL-Fassi
Department of Mathematics,
Faculty of Sciences,
Ibn Tofail University,
B. P. 14000. Kenitra,
Morocco
e-mail : izidd-math@hotmail.fr

and

S. Kabbaj
Department of Mathematics,
Faculty of Sciences,
Ibn Tofail University,
B. P. 14000. Kenitra,
Morocco
e-mail : samkabbaj@yahoo.fr