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COUNTABLE COMPACTNESS AND THE LINDELOF PROPERTY IN L-FUZZY TOPOLOGICAL SPACES *

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Abstract

In this paper, the concepts of L-fuzzy countable compactness and the L-fuzzy Lindelöf property are introduced in L-fuzzy topological spaces, where L is a completely distributive DeMorgan algebra. An L-fuzzy compact L-fuzzy set is L-fuzzy countably compact and has the L-fuzzy Lindelöf property. An L-fuzzy set having the L-fuzzy Lindelöf property is L-fuzzy countably compact if and only if it is L-fuzzy compact. Many characterizations of L-fuzzy countable compactness and the L-fuzzy Lindelöf property are presented.

Keywords : *L-fuzzy topology, L-fuzzy countable compactness, the L-fuzzy Lindelöf property.*

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1. Introduction

In 1976, the concept of fuzzy compactness was introduced in $[0,1]$-topological spaces by R. Lowen [5]. Subsequently its characterization was given by G.J. Wang in terms of α-net in [12]. In 1988, it was extended to L-topological spaces [13], where L is a completely distributive DeMorgan algebra. In [9], a new definition of fuzzy compactness was presented by means of open L-sets and their inequality in L-topological spaces. When L is a completely distributive DeMorgan algebra, it is equivalent to the notion of fuzzy compactness in [4, 7, 13]. Recently the concept of L-fuzzy compactness was introduced by Shi and Li [10] in L-fuzzy topological spaces.

In this paper, our aim is to continue the research of L-fuzzy countable compactness and the L-fuzzy Lindelöf property of L-fuzzy sets.

2. Preliminaries

Throughout this paper $(L, \bigvee, \bigwedge, ')$ is a completely distributive DeMorgan algebra, X is a nonempty set. L^X is the set of all L-fuzzy sets on X. The smallest element and the largest element in L^X are denoted respectively by $\underline{\perp}$ and $\underline{\top}$. An L-fuzzy set is briefly written as an L-set. We often do not distinguish a crisp subset A from its characteristic function χ_A.

The set of nonunit prime elements in L is denoted by $P(L)$. The set of nonzero co-prime elements in L is denoted by $M(L)$. The set of nonzero co-prime elements in L^X is denoted by $M(L^X)$. The set all L-fuzzy points x_λ (i.e., an L-fuzzy set $A \in L^X$ such that $A(x) = \lambda \neq 0$ and $A(y) = 0$ for $y \neq x$) is denoted by $pt(L^X)$.

The binary relation $\prec$ in L is defined as follows: for $a, b \in L$, $a \prec b$ if and only if for every subset $D \subseteq L$, $b \leq \sup D$ always implies the existence of $d \in D$ with $a \leq d$ [1]. In a completely distributive DeMorgan algebra L, each member b is a sup of $\{a \in L \mid a \prec b\}$. In the sense of [4, 13], $\{a \in L \mid a \prec b\}$ is the greatest minimal family of b, denoted by $\beta(b)$, and $\beta^*(b) = \beta(b) \cap M(L)$. Moreover for $b \in L$, define $\alpha(b) = \{a \in L \mid a' \prec b'\}$ and $\alpha^*(b) = \alpha(b) \cap P(L)$.

For $a \in L$ and $A \in L^X$, we define $A_{[a]} = \{x \in X \mid A(x) \geq a\}$.

Definition 2.1 ([2, 3, 6, 11]). *An L-fuzzy topology on a set X is a map $\mathcal{T} : L^X \to L$ such that*

(1) $\mathcal{T}(\underline{\top}) = \mathcal{T}(\underline{\perp}) = \top$;

(2) $\forall U, V \in L^X$, $\mathcal{T}(U \wedge V) \geq \mathcal{T}(U) \wedge \mathcal{T}(V)$;

(3) $\forall U_j \in L^X$, $j \in J$, $\mathcal{T}(\bigvee_{j\in J} U_j) \geq \bigwedge_{j\in J} \mathcal{T}(U_j)$.

$\mathcal{T}(U)$can be interpreted as the degree to which U is an open set. $\mathcal{T}^*(U) = \mathcal{T}(U')$ will be called the degree of closedness of U. The pair $(X, \mathcal{T})$ is called an L-fuzzy topological space.

A mapping $f : (X, \mathcal{T}) \to (Y, \mathcal{U})$ is said to be L-fuzzy continuous if $\mathcal{T}(f_L^{\leftarrow}(B)) \geq \mathcal{U}(B)$ holds for all $B \in L^Y$, where $f_L^{\leftarrow}$ is defined by $f_L^{\leftarrow}(B)(x) = B(f(x))$ [6].

Theorem 2.2 ([14]). *Let $(X, \mathcal{T})$ and $(Y, \mathcal{U})$ be L-fuzzy topological spaces. Then $f : (X, \mathcal{T}) \to (Y, \mathcal{U})$ be L-fuzzy continuous if and only if $\forall a \in M(L)$, $f : (X, \mathcal{T}_{[a]}) \to (Y, \mathcal{U}_{[a]})$ be L-continuous.*

Definition 2.3 ([8, 9]). *Let $a \in L\backslash\{\top\}$ and $G \in L^X$. A subfamily U in L^X is said to be*

(1) *an a-shading of G if for any $x \in X$, it follows that $G'(x) \vee \bigvee\limits_{A\in\mathcal{U}} A(x) \not\leq a$.*

(2) *a strong a-shading of G if $\bigwedge\limits_{x\in X}\left(G'(x) \vee \bigvee\limits_{A\in\mathcal{U}} A(x)\right) \not\leq a$.*

Definition 2.4 ([8, 9]). *Let $a \in L\backslash\{\bot\}$ and $G \in L^X$. A subfamily $\mathcal{P}$ in L^X is said to be*

(1) *an a-remote family of G if for any $x \in X$, it follows that $G(x) \wedge \bigwedge\limits_{B\in\mathcal{P}} B(x) \not\geq a$.*

(2) *a strong a-remote family of G if $\bigvee\limits_{x\in X}\left(G(x) \wedge \bigwedge\limits_{B\in P} B(x)\right) \not\geq a$.*

Definition 2.5 ([8, 9]). *Let $a \in L\backslash\{\bot\}$ and $G \in L^X$. A subfamily $\mathcal{U}$ in L^X is called*

(1) *a β_a-cover of G if for any $x \in X$, it follows that $a \in \beta\left(G'(x) \vee \bigvee\limits_{A\in\mathcal{U}} A(x)\right)$.*

(2) *a strong β_a-cover of G if for any $x \in X$, it follows that $a \in \beta\left(\bigwedge\limits_{x\in X}\left(G'(x) \vee \bigvee\limits_{A\in\mathcal{U}} A(x)\right)\right)$.*

Definition 2.6 ([8, 9]). *Let* $a \in L\backslash\{\perp\}$ *and* $G \in L^X$. *A subfamily* $\mathcal{U}$ *in* L^X *is called a* Q_a*-cover of* G *if* $a \leq \bigwedge_{x\in X}\left(G'(x) \vee \bigvee_{A\in\mathcal{U}} A(x)\right)$.

For a subfamily $\Phi \subseteq L^X$, $2^{(\Phi)}$ denotes the set of all finite subfamilies of Φ. $2^{[\Phi]}$ denotes the set of countable subfamilies of Φ.

Definition 2.7 ([8]). *Let* $(X, \mathcal{T})$ *be an* L*-topological space.* $G \in L^X$ *is said to be countably compact if for every countable family* $\mathcal{U} \subseteq L^X$, *it follows that*

$$\bigwedge_{x\in X}\left(G'(x) \vee \bigvee_{F\in\mathcal{U}} F(x)\right) \leq \bigvee_{V\in 2^{(\mathcal{U})}} \bigwedge_{x\in X}\left(G'(x) \vee \bigvee_{F\in\mathcal{V}} F(x)\right).$$

Definition 2.8 ([8]). *Let* $(X, \mathcal{T})$ *be an* L*-topological space.* $G \in L^X$ *is said to have the Lindelöf property if for every family* $\mathcal{U} \subseteq L^X$, *it follows that*

$$\bigwedge_{x\in X}\left(G'(x) \vee \bigvee_{F\in\mathcal{U}} F(x)\right) \leq \bigvee_{V\in 2^{[\mathcal{U}]}} \bigwedge_{x\in X}\left(G'(x) \vee \bigvee_{F\in\mathcal{V}} F(x)\right).$$

Definition 2.9 ([10]). *Let* $(X, \mathcal{T})$ *be an* L*-fuzzy topological space.* $G \in L^X$ *is said to be* L*-fuzzy compact if for every family* $\mathcal{U} \subseteq L^X$, *it follows that*

$$\bigwedge_{F\in\mathcal{U}} \mathcal{T}(F) \wedge \left(\bigwedge_{x\in X}\left(G'(x) \vee \bigvee_{F\in\mathcal{U}} F(x)\right)\right) \leq \bigvee_{\mathcal{V}\in 2^{(\mathcal{U})}} \bigwedge_{x\in X}\left(G'(x) \vee \bigvee_{F\in\mathcal{V}} F(x)\right).$$

3. L-fuzzy countable compactness

Definition 3.1. *Let* $(X, \mathcal{T})$ *be an* L*-fuzzy topological space.* $G \in L^X$ *is said to be* L*-fuzzy countably compact if for every countable family* $\mathcal{U} \subseteq L^X$, *it follows that*

$$\bigwedge_{F\in\mathcal{U}} \mathcal{T}(F) \wedge \left(\bigwedge_{x\in X}\left(G'(x) \vee \bigvee_{F\in\mathcal{U}} F(x)\right)\right) \leq \bigvee_{\mathcal{V}\in 2^{(\mathcal{U})}} \bigwedge_{x\in X}\left(G'(x) \vee \bigvee_{F\in\mathcal{V}} F(x)\right).$$

Obviously L-fuzzy compactness implies L-fuzzy countable compactness.
Let $(X, \mathcal{T})$ be an L-topological space. Let $\chi_{\mathcal{T}} : L^X \to L$

$$\chi_{\mathcal{T}} = \begin{cases} 1, & A \in \mathcal{T}, \\ 0, & A \notin \mathcal{T}. \end{cases}$$

Obviously, $(X, \chi_{\mathcal{T}})$ is a special L-fuzzy topological spaces. So we can easily prove the following theorem.

Theorem 3.2. *Let $(X, \mathcal{T})$ be an L-topological space and $G \in L^X$. G is L-fuzzy countably compact in $(X, \chi_{\mathcal{T}})$ if and only if G is countably compact [8] in $(X, \mathcal{T})$.*

From Definition 2.1 we easily obtain the following theorem by simply using quasi-complement.

Theorem 3.3. *Let $(X, \mathcal{T})$ be an L-fuzzy topological space. $G \in L^X$ is L-fuzzy countably compact if and only if for every countably family $\mathcal{P} \subseteq L^X$ it follows that*

$$\bigvee_{F \in \mathcal{P}} \mathcal{T}'(F') \vee \left(\bigvee_{x \in X} (G(x) \wedge \bigwedge_{F \in P} F(x)) \right) \geq \bigwedge_{\mathcal{H} \in 2^{(P)}} \bigvee_{x \in X} \left(G(x) \wedge \bigwedge_{F \in H} F(x) \right).$$

By Definition 2.1 and Theorem 2.2 and analogous to [8] we immediately obtain the following result.

Theorem 3.4. *Let $(X, \mathcal{T})$ be an L-fuzzy topological space and $G \in L^X$. Then the following conditions are equivalent to each other.*

(1) *G is L-fuzzy countably compact.*

(2) *For any $a \in M(L)$, each countable strong a-remote family $\mathcal{P}$ of G with $\bigwedge_{F \in P} \mathcal{T}^*(F) \not\leq a'$ has a finite subfamily $\mathcal{H}$ which is a (strong) a-remote family of G.*

(3) *For any $a \in M(L)$, and any countable strong a-remote family $\mathcal{P}$ of G with $\bigwedge_{F \in P} \mathcal{T}^*(F) \not\leq a'$, there exists a finite subfamily $\mathcal{H}$ of $\mathcal{P}$ and $b \in \beta^*(a)$ such that $\mathcal{H}$ is a (strong) b-remote family of G.*

(4) *For any $a \in P(L)$, each countable strong a-shading $\mathcal{U}$ of G with $\bigwedge_{F \in \mathcal{U}} \mathcal{T}(F) \not\leq a$ has a finite subfamily $\mathcal{V}$ which is a (strong) a-shading of G.*

(5) *For any $a \in P(L)$ and any countable strong a-shading $\mathcal{U}$ of G with $\bigwedge_{F \in U} \mathcal{T}(F) \not\leq a$, there exists a finite subfamily $\mathcal{V}$ of $\mathcal{U}$ and $b \in \alpha^*(a)$ such that $\mathcal{V}$ is a (strong) b-shading of G.*

(6) *For any* $a \in M(L)$ *and any* $b \in \beta^*(a)$*, each countable* Q_a*-cover* $\mathcal{U}$ *of* G *with* $\mathcal{T}(F) \geq a$ $(\forall F \in \mathcal{U})$ *has a finite subfamily* $\mathcal{V}$ *which is a* Q_b*-cover of* G*.*

(7) *For any* $a \in M(L)$ *and any* $b \in \beta^*(a)$*, each countable* Q_a*-cover* $\mathcal{U}$ *of* G *with* $\mathcal{T}(F) \geq a$ $(\forall F \in \mathcal{U})$ *has a finite subfamily* $\mathcal{V}$ *which is a (strong)* β_b*-cover of* G*.*

Theorem 3.5. *Let* $(X, \mathcal{T})$ *be an* L*-fuzzy topological space and* $G \in L^X$*. If* $\beta(c \wedge d) = \beta(c) \cap \beta(d)$ $(\forall c, d \in L)$*, then the following conditions are equivalent to each other.*

(1) G *is* L*-fuzzy countably compact.*

(2) *For any* $a \in M(L)$*, each countable strong* β_a*-cover* $\mathcal{U}$ *of* G *with* $a \in \beta\left(\bigwedge_{F \in \mathcal{U}} \mathcal{T}(F)\right)$ *has a finite subfamily* $\mathcal{V}$ *which is a (strong)* β_a*-cover of* G*.*

(3) *For any* $a \in M(L)$ *and any countable strong* β_a*-cover* $\mathcal{U}$ *of* G *with* $a \in \beta\left(\bigwedge_{F \in \mathcal{U}} \mathcal{T}(F)\right)$*, there exists a finite subfamily* $\mathcal{V}$ *of* $\mathcal{U}$ *and* $b \in M(L)$ *with* $a \in \beta^*(b)$ *such that* $\mathcal{V}$ *is a (strong)* β_b*-cover of* G*.*

Now in order to research properties of L-fuzzy countably compactness, we introduce the following definition.

Definition 3.6. *Let* $(X, \mathcal{T})$ *be an* L*-topological space,* $a \in M(L)$ *and* $G \in L^X$*.* G *is said to be countably* a*-compact if and only if* $\forall b \in \beta(a)$*, each countable* Q_a*-open cover* $\mathcal{U}$ *of* G *has a finite subfamily* $\mathcal{V}$ *which is a* Q_b*-open cover of* G*.*

Theorem 3.7. *Let* $(X, \mathcal{T})$ *be an* L*-topological space.* $G \in L^X$ *is countably compact if and only if* $\forall a \in M(L)$*,* G *is countably* a*-compact.*

Theorem 3.8. *Let* $(X, \mathcal{T})$ *be an* L*-fuzzy topological space and* $G \in L^X$*.* G *is* L*-fuzzy countably compact in* $(X, \mathcal{T})$ *if and only if* $\forall a \in M(L)$*,* G *is countably* a*-compact in* $(X, \mathcal{T}_{[a]})$*.*

Proof. (Necessity) Since G is L-fuzzy countably compact in $(X, \mathcal{T})$, by Definition 2.1 we know that for every countable family $\mathcal{U} \subseteq L^X$, it follows that

$$\bigwedge_{F\in\mathcal{U}} \mathcal{T}(F) \wedge \left(\bigwedge_{x\in X} \left(G'(x) \vee \bigvee_{F\in\mathcal{U}} F(x) \right) \right) \leq \bigvee_{\mathcal{V}\in 2^{(\mathcal{U})}} \bigwedge_{x\in X} \left(G'(x) \vee \bigvee_{F\in\mathcal{V}} F(x) \right).$$

Hence $\forall a \in M(L)$ and for every countable family $\mathcal{U} \subseteq \mathcal{T}_{[a]}$, we have that

$$a \leq \bigwedge_{x\in X} \left(G'(x) \vee \bigvee_{F\in\mathcal{U}} F(x) \right) \Rightarrow a \leq \bigvee_{\mathcal{V}\in 2^{(\mathcal{U})}} \bigwedge_{x\in X} \left(G'(x) \vee \bigvee_{F\in\mathcal{V}} F(x) \right).$$

Thus $\forall b \in \beta(a)$, there exists $\mathcal{V} \in 2^{(\mathcal{U})}$ such that $b \leq \bigwedge_{x\in X} \left(G'(x) \vee \bigvee_{F\in\mathcal{V}} F(x) \right)$, i.e., $\forall a \in M(L)$, $\forall b \in \beta(a)$, each countable Q_a-cover $\mathcal{U}$ of G in $(X, \mathcal{T}_{[a]})$ has a finite subfamily $\mathcal{V}$ which is a Q_b-cover of G. Therefore $\forall a \in M(L)$, G is countably a-compact in $(X, \mathcal{T}_{[a]})$.

(Sufficiency) Suppose that $\forall a \in M(L)$, G is countably a-compact in $(X, \mathcal{T}_{[a]})$. Let $\mathcal{U} \subseteq L^X$ ($\mathcal{U}$ is countable family) and $a \leq \bigwedge_{F\in U} \mathcal{T}(F) \wedge \left(\bigwedge_{x\in X} \left(G'(x) \vee \bigvee_{F\in\mathcal{U}} F(x) \right) \right)$. Then $\mathcal{U} \subseteq \mathcal{T}_{[a]}$ and $a \leq \bigwedge_{x\in X} \left(G'(x) \vee \bigvee_{F\in\mathcal{U}} F(x) \right)$. Thus $\forall b \in \beta(a)$, there exists $\mathcal{V} \in 2^{(\mathcal{U})}$ such that $b \leq \bigwedge_{x\in X} \left(G'(x) \vee \bigvee_{F\in\mathcal{V}} F(x) \right)$. Hence $a \leq \bigvee_{\mathcal{V}\in 2^{(\mathcal{U})}} \bigwedge_{x\in X} \left(G'(x) \vee \bigvee_{F\in\mathcal{V}} F(x) \right)$. Therefore G is L-fuzzy countably compact in $(X, \mathcal{T})$. $\square$

Analogous to Shi's proof in [8], we can obtain the following Lemma 2.7.

Lemma 3.9. *Let $(X, \mathcal{T})$ be an L-topological space, $a \in M(L)$ and $G \in L^X$. If G is countably a-compact, then $G \wedge H$ is countably a-compact for each $H \in \mathcal{T}'$.*

Theorem 3.10. *Let $(X, \mathcal{T})$ be an L-fuzzy topological space and $G \in L^X$. If G is L-fuzzy countably compact, then for each $H \in L^X$ with $\mathcal{T}^*(H) = \top$, $G \wedge H$ is L-fuzzy countably compact.*

Proof. $\forall a \in M(L)$, since G is L-fuzzy countably compact in $(X, \mathcal{T})$, by Theorem 2.6, G is countably a-compact in $(X, \mathcal{T}_{[a]})$. By $\mathcal{T}^*(H) = \top$, we know that $H \in \mathcal{T}'_{[a]}$. Further by Lemma 2.7, $G \wedge H$ is countably a-compact in $(X, \mathcal{T}_{[a]})$. Then by Theorem 2.8, $G \wedge H$ is L-fuzzy countably compact in $(X, \mathcal{T})$. □

Analogous to Shi's proof in [8] , we can obtain the following Lemma 2.9.

Lemma 3.11. *Let $(X, \mathcal{T})$ be an L-topological space, $G, H \in L^X$ and $a \in M(L)$. If G and H are countably a-compact, then $G \vee H$ is countably a-compact as well.*

Theorem 3.12. *Let $(X, \mathcal{T})$ be an L-fuzzy topological space and $H, G \in L^X$. If G and H are L-fuzzy countably compact, then $G \vee H$ is L-fuzzy countably compact as well.*

Proof. *Since both G and H are L-fuzzy countably compact in $(X, \mathcal{T})$, by Theorem 2.6, $\forall a \in M(L)$, we know that both G and H are countably a-compact in $(X, \mathcal{T}_{[a]})$. By Lemma 2.9, $G \vee H$ is countably a-compact in $(X, \mathcal{T}_{[a]})$. So $G \vee H$ is L-fuzzy countably compact in $(X, \mathcal{T})$.* □

Analogous to Shi's proof in [8] , we can obtain the following Lemma 2.11.

Lemma 3.13. *Let $(X, \mathcal{T})$, $(Y, \mathcal{U})$ be two L-topological spaces and $a \in M(L)$. If G is countably a-compact in $(X, \mathcal{T})$ and $f : (X, \mathcal{T}) \to (Y, \mathcal{U})$ is an L-continuous mapping, then $f_L^{\rightarrow}(G)$ is countably a-compact in $(Y, \mathcal{U})$.*

Theorem 3.14. *Let $(X, \mathcal{T})$, $(Y, \mathcal{U})$ be two L-fuzzy topological spaces, and $f : (X, \mathcal{T}) \to (Y, \mathcal{U})$ be an L-fuzzy continuous mapping. If $G \in L^X$ is L-fuzzy countably compact in $(X, \mathcal{T})$, then so is $f_L^{\rightarrow}(G)$ in $(Y, \mathcal{U})$.*

Proof. *Since G is L-fuzzy countably compact in $(X, \mathcal{T})$, by Theorem 2.6, $\forall a \in M(L)$, G is countably a-compact in $(X, \mathcal{T}_{[a]})$. By Theorem 1.2, $f : (X, \mathcal{T}_{[a]}) \to (Y, \mathcal{U}_{[a]})$ is an L-continuous mapping. Hence $f_L^{\rightarrow}(G)$ is countably a-compact in $(Y, \mathcal{U}_{[a]})$. Therefore $f_L^{\rightarrow}(G)$ is L-fuzzy countably compact in $(Y, \mathcal{U})$.* □

4. The L-fuzzy Lindelöf property

Definition 4.1. *Let $(X, \mathcal{T})$ be an L-fuzzy topological space. $G \in L^X$ is said to have the L-fuzzy Lindelöf property if for every family $\mathcal{U} \subseteq L^X$, it* *follows that*

$$\bigwedge_{F\in\mathcal{U}}\mathcal{T}(F)\wedge\left(\bigwedge_{x\in X}\left(G'(x)\vee\bigvee_{F\in\mathcal{U}}F(x)\right)\right)\leq\bigvee_{\mathcal{V}\in 2^{[\mathcal{U}]}}\bigwedge_{x\in X}\left(G'(x)\vee\bigvee_{F\in\mathcal{V}}F(x)\right).$$

Obviously we have the following theorem.

Theorem 4.2. *Let $(X,\mathcal{T})$ be an L-fuzzy topological space and $G\in L^X$ has the L-fuzzy Lindelöf property. Then G is L-fuzzy compact if and only if it is L-fuzzy countably compact.*

Analogous to L-fuzzy countable compactness, we have the following results.

Theorem 4.3. *Let $(X,\mathcal{T})$ be an L-topological space and $G\in L^X$. G has the L-fuzzy Lindelöf property in $(X,\chi_{\mathcal{T}})$ if and only if G has the Lindelöf property in $(X,\mathcal{T})$.*

Theorem 4.4. *Let $(X,\mathcal{T})$ be an L-fuzzy topological space. $G\in L^X$ has the L-fuzzy Lindelöf property if and only if for every family $\mathcal{P}\subseteq L^X$, it follows that*

$$\bigvee_{F\in\mathcal{P}}\mathcal{T}'(F')\vee\left(\bigvee_{x\in X}(G(x)\wedge\bigwedge_{F\in P}F(x))\right)\geq\bigwedge_{\mathcal{H}\in 2^{[P]}}\bigvee_{x\in X}\left(G(x)\wedge\bigwedge_{F\in H}F(x)\right).$$

Theorem 4.5. *Let $(X,\mathcal{T})$ be an L-fuzzy topological space and $G\in L^X$. Then the following conditions are equivalent to each other.*

(1) *G has the L-fuzzy Lindelöf property.*

(2) *For any $a\in M(L)$, each strong a-remote family $\mathcal{P}$ of G with $\bigwedge_{F\in P}\mathcal{T}^*(F)\not\leq a'$ has a countable subfamily $\mathcal{H}$ which is a (strong) a-remote family of G.*

(3) *For any $a\in M(L)$, and any strong a-remote family $\mathcal{P}$ of G with $\bigwedge_{F\in P}\mathcal{T}^*(F)\not\leq a'$, there exists a countable subfamily $\mathcal{H}$ of $\mathcal{P}$ and $b\in\beta^*(a)$ such that $\mathcal{H}$ is a (strong) b-remote family of G.*

(4) *For any $a\in P(L)$, each strong a-shading U of G with $\bigwedge_{F\in\mathcal{U}}\mathcal{T}(F)\not\leq a$ has a countable subfamily $\mathcal{V}$ which is a (strong) a-shading of G.*

(5) *For any $a \in P(L)$ and any strong a-shading $\mathcal{U}$ of G with $\bigwedge_{F\in U} \mathcal{T}(F) \not\leq a$, there exists a countable subfamily $\mathcal{V}$ of $\mathcal{U}$ and $b \in \alpha^*(a)$ such that $\mathcal{V}$ is a (strong) b-shading of G.*

(6) *For any $a \in M(L)$ and any $b \in \beta^*(a)$, each Q_a-cover $\mathcal{U}$ of G with $\mathcal{T}(F) \geq a$ $(\forall F \in \mathcal{U})$ has a countable subfamily $\mathcal{V}$ which is a Q_b-cover of G.*

(7) *For any $a \in M(L)$ and any $b \in \beta^*(a)$, each Q_a-cover $\mathcal{U}$ of G with $\mathcal{T}(F) \geq a$ $(\forall F \in \mathcal{U})$ has a countable subfamily $\mathcal{V}$ which is a (strong) β_b-cover of G.*

Theorem 4.6. *Let $(X, \mathcal{T})$ be an L-fuzzy topological space and $G \in L^X$. If $\beta(c \wedge d) = \beta(c) \cap \beta(d)$ $(\forall c, d \in L)$, then the following conditions are equivalent to each other.*

(1) *G has the L-fuzzy Lindelöf property.*

(2) *For any $a \in M(L)$, each strong β_a-cover $\mathcal{U}$ of G with $a \in \beta\left(\bigwedge_{F\in\mathcal{U}} \mathcal{T}(F)\right)$ has a countable subfamily $\mathcal{V}$ which is a (strong) β_a-cover of G.*

(3) *For any $a \in M(L)$ and any strong β_a-cover $\mathcal{U}$ of G with $a \in \beta\left(\bigwedge_{F\in\mathcal{U}} \mathcal{T}(F)\right)$, there exists a countable subfamily $\mathcal{V}$ of $\mathcal{U}$ and $b \in M(L)$ with $a \in \beta^*(b)$ such that $\mathcal{V}$ is a (strong) β_b-cover of G.*

Definition 4.7. *Let $(X, \mathcal{T})$ be an L-topological space, $a \in M(L)$ and $G \in L^X$. G has the a-Lindelöf property if and only if $\forall b \in \beta(a)$, each Q_a-open cover $\mathcal{U}$ of G has a countable subfamily $\mathcal{V}$ which is a Q_b-open cover of G.*

Theorem 4.8. *Let $(X, \mathcal{T})$ be an L-topological space. Then $G \in L^X$ has the Lindelöf property if and only if $\forall a \in M(L)$, G has the a-Lindelöf property.*

Theorem 4.9. *Let $(X, \mathcal{T})$ be an L-fuzzy topological space and $G \in L^X$. Then G has the L-fuzzy Lindelöf property in $(X, \mathcal{T})$ if and only if $\forall a \in M(L)$, G has the a-Lindelöf property in $(X, \mathcal{T}_{[a]})$.*

Lemma 4.10. *Let $(X, \mathcal{T})$ be an L-topological space, $a \in M(L)$ and $G \in L^X$. If G has the a- Lindelöf property, then $G \wedge H$ has the a-Lindelöf property for each $H \in \mathcal{T}'$.*

Theorem 4.11. *Let $(X, \mathcal{T})$ be an L-fuzzy topological space and $G \in L^X$. If G has the L-fuzzy Lindelöf property, then for each $H \in L^X$ with $\mathcal{T}^*(H) = \top$, $G \wedge H$ has the L-fuzzy Lindelöf property.*

Lemma 4.12. *Let $(X, \mathcal{T})$ be an L-topological space, $G, H \in L^X$ and $a \in M(L)$. If G and H have the a-Lindelöf property, then $G \vee H$ has the a-Lindelöf property as well.*

Theorem 4.13. *Let $(X, \mathcal{T})$ be an L-fuzzy topological space and $H, G \in L^X$. If G and H have the L-fuzzy Lindelöf property, then $G \vee H$ has the L-fuzzy Lindelöf property as well.*

Lemma 4.14. *Let $(X, \mathcal{T})$, $(Y, \mathcal{U})$ be two L-topological spaces and $a \in M(L)$. If G has the a-Lindelöf property in $(X, \mathcal{T})$ and $f : (X, \mathcal{T}) \to (Y, \mathcal{U})$ is an L-continuous mapping, then $f_L^{\to}(G)$ has the a-Lindelöf property in $(Y, \mathcal{U})$.*

Theorem 4.15. *Let $(X, \mathcal{T})$, $(Y, \mathcal{U})$ be two L-fuzzy topological spaces, and $f : (X, \mathcal{T}) \to (Y, \mathcal{U})$ be an L-fuzzy continuous mapping. If $G \in L^X$ has the L-fuzzy Lindelöf property in $(X, \mathcal{T})$, then $f_L^{\to}(G)$ has the L-fuzzy Lindelöf property in $(Y, \mathcal{U})$.*

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