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Lie algebras with complex structures having nilpotent eigenspaces

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Abstract

Let $(\mathbf{g}, [\cdot,\cdot])$ *be a Lie algebra with an integrable complex structure* J*. The* $\pm i$ *eigenspaces of* J *are complex subalgebras of* $\mathbf{g}^{\mathbf{C}}$ *isomorphic to the algebra* $(\mathbf{g}, [*]_J)$ *with bracket* $[X * Y]_J = \frac{1}{2}([X,Y] - [JX, JY])$*. We consider here the case where these subalgebras are nilpotent and prove that the original* $(\mathbf{g}, [\cdot,\cdot])$ *Lie algebra must be solvable. We consider also the* 6*-dimensional case and determine explicitly the possible nilpotent Lie algebras* $(\mathbf{g}, [*]_J)$*. Finally we produce several examples illustrating different situations, in particular we show that for each given* s *there exists* $\mathbf{g}$ *with complex structure* J *such that* $(\mathbf{g}, [*]_J)$ *is* s*-step nilpotent. Similar examples of hypercomplex structures are also built.*

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1. Introduction

Given a real Lie algebra $(\mathbf{g}, [\cdot,\cdot])$ with a complex structure J, we can write $\mathbf{g}^{\mathbf{C}} = \mathbf{g}^{1,0} \oplus \mathbf{g}^{0,1}$, where $\mathbf{g}^{1,0}$ and $\mathbf{g}^{0,1}$ are eigen– spaces of J corresponding to the eigenvalues i and $-i$, respectively. These eigenspaces are subalgebras of $\mathbf{g}^{\mathbf{C}}$ if and only if J is an integrable complex structure.

In this paper we consider such complex structures for which the eigenspaces are nilpotent subalgebras of $\mathbf{g}^{\mathbf{C}}$. Our main result of general nature states that the Lie algebra $\mathbf{g}$ is solvable if $\mathbf{g}^{1,0}$ (or equivalently $\mathbf{g}^{0,1}$) is a nilpotent subalgebra (see Theorem 3.2). The proof of this fact relies on a result which goes back to Goto [10], namely, that a semi-simple Lie algebra cannot be written as the sum of two nilpotent subalgebras.

In another direction we apply the classification Salamon [13] to determine those 6-dimensional solvable Lie algebras admitting an integrable complex structure with nilpotent $\mathbf{g}^{1,0}$.

Complex structures on solvable and nilpotent Lie algebras have been extensively studied recently (see Dotti-Fino [6], [7], [8], [9], Barberis-Dotti [2] and Salamon [13] and references therein). In particular, abelian structures where considered in [8], where it is proved that abelian complex structures occur only on solvable Lie algebras, and in [2] where a characterization of solvable Lie algebras admiting abelian complex structures is given.

Before proceeding we remark that Cordero-Fernández-Gray-Ugarte [5], defined the concept of nilpotent complex structure by asking that the ascending series of ideals $\mathbf{a}_l(J)$ defined inductively by $\mathbf{a}_0(J) = \{0\}$ and

$$\mathbf{a}_l(J) = \{X \in \mathbf{g}; [X, \mathbf{g}] \subset \mathbf{a}_{l-1}(J) \ \ and \ \ [JX, \mathbf{g}] \subset \mathbf{a}_{l-1}(J)\}$$

ends in $\mathbf{g}$. We remark that if J is nilpotent then the Lie algebra $\mathbf{g}$ is also nilpotent. Hence this concept is stronger than the nilpotence of the eigenspaces $\mathbf{g}^{1,0}$ and $\mathbf{g}^{0,1}$.

We describe now the contents of the paper. In Section 2 we follow Bartolomeis [3] and define a new Lie bracket $[*]_J$ on $\mathbf{g}$ that is given $[X * Y]_J = \frac{1}{2}([X,Y] - [JX, JY])$. This bracket satisfies the Jacobi identity if and only if J is integrable. In this case we denote the Lie algebra obtained by $\mathbf{g}_*$. It turns out that $\mathbf{g}_*$ is a complex Lie algebra (with complex structure J) isomorphic to both $\mathbf{g}^{1,0}$ and $\mathbf{g}^{0,1}$. We remark that these algebras are also isomorphic to the Lie algebra in $\mathbf{g}$ given by the bracket $[X,Y]_J = \frac{1}{2}([JX,Y] + [X,JY])$.

In Section 3 we prove that $\mathbf{g}$ is solvable if $\mathbf{g}_*$ is nilpotent. As mentioned above the proof is based on a lemma of [10], whose proof we reproduce here with some modifications.

In Section 4 we consider the 6-dimensional solvable Lie algebras $\mathbf{g}$ with complex structure J such that $\mathbf{g}_*$ is s-step nilpotent. It turns out that in this case that either $s = 1$ (that is, $\mathbf{g}_*$ is abelian) or $s = 2$ and in this case there exists a basis $\{f_1, \ldots, f_6\}$ of $\mathbf{g}$ such that the non-zero brackets of $(\mathbf{g}, [*]_J)$ are given by

$$[f_1 * f_3] = [f_4 * f_2] = -f_5, \ \ [f_1 * f_4] = [f_2 * f_3] = -f_6.$$

Hence, there are exactly two possibilities for $\mathbf{g}_*$ in dimension 6. We note that 6 is the first low dimensional case where we can have nonabelian $\mathbf{g}_*$, since in dimension 4 is abelian if it is nilpotent.

Finally Section 5 is devoted to the construction of examples of Lie algebras $\mathbf{g}$ with complex structure J for which $\mathbf{g}_*$ is s-step for each given s. Similar examples of hypercomplex structures are also built.

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2. Complex structures on Lie algebras

Recall that a complex structure on a real Lie algebra $(\mathbf{g}, [\cdot,\cdot])$ is an endomorphism J of $\mathbf{g}$ such that $J^2 = -I$. We assume throughout that J is in integrable, that is, $N_J = 0$, where N_J is the Nijenhuis tensor of J:

$$N_J(X,Y) = J[X,Y] - [JX,Y] - [X,JY] - J[JX,JY].$$

Given a complex structure J on $\mathbf{g}$, its complexification to $\mathbf{g}^{\mathbf{C}} = \mathbf{g} \oplus i\mathbf{g}$ gives a splitting $\mathbf{g}^{\mathbf{C}} = \mathbf{g}^{1,0} \oplus \mathbf{g}^{0,1}$, where

$$\mathbf{g}^{1,0} = \{X \in \mathbf{g}^{\mathbf{C}}; JX = iX\} \text{ and } \mathbf{g}^{0,1} = \{X \in \mathbf{g}^{\mathbf{C}}; JX = -iX\}$$

are the $\pm i$-eigenspaces of J. It is well known (and easy to prove) that J is integrable if and only if the $\mathbf{g}^{1,0}$ and $\mathbf{g}^{0,1}$ are complex subalgebras of $\mathbf{g}^{\mathbf{C}}$.

To study J we define in $\mathbf{g}$ a new Lie bracket $[*]_J$, such that the realification of both $\mathbf{g}^{1,0}$ and $\mathbf{g}^{0,1}$ are Lie algebras isomorphic to $(\mathbf{g}, [*]_J)$.

Definition 2.1. *Let* $(\mathbf{g}, [,])$ *be a Lie algebra with a complex structure* J. *Define* $[*]_J : \mathbf{g} \times \mathbf{g} \longrightarrow \mathbf{g}$ *by*
$[X * Y]_J = \frac{1}{2}\left([X,Y] - [JX,JY]\right)$.

It is easy to see that $[*]_J$ is bilinear skew-symmetric and the annihilation of N_J implies the Jacobi identity of $[*]_J$ (for general J the cyclic permutation of $[X * [Y * Z]]$ equals the cyclic permutation of $\dfrac{1}{4}[JX, N_J(JY, Z)]$).

In the sequel we denote by $\mathbf{g}_*$ the Lie algebra obtained by endowing $\mathbf{g}$ with the bracket $[*]$.

In general the brackets $[\cdot,\cdot]$ and $[*]$ highly different. Clearly, $[\cdot,\cdot] = [*]$ if and only if $[X,Y] = -[JX,JY]$ for all $X,Y \in \mathbf{g}$ and this happens if and only if $\mathbf{g}$ is already a complex Lie algebra with complex structure given by J. In fact, recall that J is said to be *adapted* to $(\mathbf{g},[\cdot,\cdot])$ if $[X,JY] = J\,[X,Y]$ for all $X,Y \in \mathbf{g}$. In this case $\mathbf{g}$ becomes a complex Lie algebra where J is multiplication by i.

It is easy to see that $[X,Y] = -\,[JX,JY]$ for all $X,Y \in \mathbf{g}$ if J is adapted, so that in this case the Lie algebras $\mathbf{g}_*$ and $\mathbf{g}$ coincide. Conversely, we check next that $\mathbf{g}_*$ is complex with respect to J, so that if $[\cdot,\cdot] = [*]$ then $\mathbf{g}$ is complex.

Proposition 2.2. *Let $(\mathbf{g},[\cdot,\cdot])$ be a Lie algebra with a complex structure J. Then J is adapted to $(\mathbf{g},[*]_J)$. Hence $\mathbf{g}_*$ is a complex Lie algebra.*

Proof: Given $X,Y \in \mathbf{g}$ we have J$[X*Y]_J = \frac{1}{2}(J([X,Y]-[JX,JY])) = \frac{1}{2}([JX,Y]+[X,JY])$
$= \frac{1}{2}([JX,Y]-[JJX,JY]) = [JX*Y]_J\,.$

□

It is easy to see that J also satisfies the integrability condition with respect to $[*]_J$. Hence the above proposition implies that $[X**Y]_J = [X*Y]_J$ where

$$[X**Y]_J = \frac{1}{2}\left([X*Y]_J - [JX*JY]_J\right)$$

is the $*$-algebra of $\mathbf{g}_*$.

Remark: The above discussion shows that the condition $N_J = 0$ together with $[X,Y] = -[JX,JY]$ is equivalent to J be adapted. If G is a Lie group with Lie algebra $\mathbf{g}$ the integrability condition $N_J = 0$ means that J can be extended to a complex structure on G which is left invariant, that is, the left translations are holomorphic maps. The extra condition $[X,Y] = -[JX,JY]$ is the missing one for J to be bi-invariant in G.

Remark: One can build form J another Lie algebra structure on $\mathbf{g}$, namely

$$[X,Y]_J = \frac{1}{2}\left([JX,Y]+[X,JY]\right).$$

Again $[\cdot,\cdot]_J$ satisfies the Jacobi identity if and only if J is integrable (w.r.t. the original bracket). However the Lie algebra obtained this way is isomorphic to $\mathbf{g}_*$. In fact, a straightforward computation using $N_J = 0$ shows that $-J\,[X * Y]_J = [-JX, -JY]_J$, so that $-J$ is an isomorphism between $[*]_J$ and $[\cdot,\cdot]_J$.

A glance at the definition of $[*]$ shows that the subalgebras $\mathbf{g}^{1,0}$ and $\mathbf{g}^{0,1}$ are commuting ideals of the complexification $\mathbf{g}_*^{\mathbf{C}}$. Actually these ideals are isomorphic to $\mathbf{g}_*$:

Proposition 2.3. *Let J be a complex structure on $(\mathbf{g}, [\cdot,\cdot])$. Then the complex Lie algebras $(\mathbf{g}, [*]_J)$, $\mathbf{g}^{1,0}$ and $\mathbf{g}^{0,1}$ are isomorphic.*

Proof: Observe that $\mathbf{g}^{1,0} = \{X - iJX; X \in \mathbf{g}\}$. Then a straightforward computation (using that $N_J = 0$) shows that $\varphi : \mathbf{g} \to \mathbf{g}^{1,0}$ given by $\varphi(X) = \frac{1}{2}(X - iJX)$ is an isomorphism between $(\mathbf{g}, [*]_J)$ and $\mathbf{g}^{1,0}$. Analogously $\mathbf{g}^{0,1} = \{X + iJX; X \in \mathbf{g}\}$ and $\psi(X) = \frac{1}{2}(X + iJX)$ is an isomorphism between $(\mathbf{g}, [*]_J)$ and $(\mathbf{g}^{0,1})$. Note that for every $X \in \mathbf{g}$ we have $\varphi(JX) = i\varphi(X)$ and $\psi(JX) = i\psi(X)$, so that φ and ψ are complex linear maps. $\square$

This proposition implies at once that $\mathbf{g}_*$ is nilpotent if $\mathbf{g}$ is nilpotent. Also, if $\mathbf{g}$ is solvable then $\mathbf{g}_*$ is solvable as well. The converse to both these statements is not true. In fact, there are examples where $\mathbf{g}_*$ is nilpotent and $\mathbf{g}$ is solvable and not nilpotent and where $\mathbf{g}_*$ is solvable and $\mathbf{g}$ simple. In any case it will be proved below that $\mathbf{g}$ is solvable if $\mathbf{g}_*$ is nilpotent.

For later reference we write explicitly the lower central series of $\mathbf{g}_*$: We have, inductively $\mathbf{g}_*^0 = \mathbf{g}$ and

$$\mathbf{g}_*^k = span\left\{[X,Y] - [JX,JY]\,; X \in \mathbf{g}, Y \in \mathbf{g}_*^{k-1}\right\}.$$

In particular, $\mathbf{g}_*$ is abelian if and only if

$$[JX, JY] = [X, Y] \text{ for all } X, Y \in \mathbf{g}. \tag{2.1}$$

According to [2] a complex structure J in the Lie algebra $(\mathbf{g}, [\cdot,\cdot])$ is said to be abelian if condition (2.1) is satisfied. It is easy to see that this condition implies automatically the vanishing of the Nijenhuis tensor. Inspired by this we say that a complex structure J on $\mathbf{g}$ *s-step nilpotent* it is integrable and $\mathbf{g}_*$ *is* s-step nilpotent. We note that this different from the concept of nilpotent complex structure of [5].

3. g is solvable if $\mathbf{g}_*$ is nilpotent

In [8], Proposition 3.1, it was shown that abelian complex structures only occur in solvable Lie algebras. Our purpose in this section is to generalize that result by showing that if $\mathbf{g}_*$ is nilpotent then $\mathbf{g}$ is solvable.

Our proof is based in a lemma of Goto [10]. For the sake of completeness we present here a modified proof of it. Before stating the lemma define some notation. Let $\mathbf{g}$ be a complex semi-simple Lie algebra and $\mathbf{h} \subset \mathbf{g}$ is a Cartan subalgebra of $\mathbf{g}$. Denote by Π the corresponding root system and let Π^+ be the set of positive roots with respect to a choice of a lexicographic ordering in $\mathbf{h}$. Put $\mathbf{n}^+ = \sum_{\alpha\in\Pi^+} \mathbf{g}_\alpha$ where $\mathbf{g}_\alpha$ is the root space corresponding to α. Then

$$\mathbf{g} = \mathbf{h} \oplus \mathbf{n}^+ \oplus \mathbf{n}^-,$$

where $\mathbf{n}^- = \sum_{\alpha\in\Pi^+} \mathbf{g}_\alpha$ and $\Pi^- = -\Pi^+$. Also, $\mathbf{b} = \mathbf{h} \oplus \mathbf{n}^+$ is a Borel subalgebra of $\mathbf{g}$ and as is well known every solvable subalgebra of $\mathbf{g}$ is conjugate by an inner automorphism of $\mathbf{g}$ to a subalgebra of $\mathbf{b}$ (see [4], [15] or [11]).

Lemma 3.1. *Let $\mathbf{g}$ be a semi-simple Lie algebra over an algebraically closed field of zero characteristic. Put $d = \dim \mathbf{g}$ and let $l = \dim \mathbf{h}$ be the rank of $\mathbf{g}$. Suppose that $\mathbf{n} \subset \mathbf{g}$ is a nilpotent subalgebra. Then*

$$\dim \mathbf{n} \leq \frac{d-l}{2} = \dim \mathbf{b} - l$$

where $\mathbf{b}$ is a Borel subalgebra.

Proof: Since $\mathbf{b} = \mathbf{h} \oplus \mathbf{n}^+$ is a Borel subalgebra of $\mathbf{g}$ we can assume without loss of generality that $\mathbf{n} \subset \mathbf{b}$.

Denote by $\mathbf{a}$ the set all semi-simple elements in $\mathbf{n}$. If $A \in \mathbf{a}$ its adjoint $\mathrm{a}d(A)$ is a semi-simple linear transformation on $\mathbf{g}$. Since $\mathrm{a}d(A)$, restricted on $\mathbf{n}$ is nilpotent, we have $\mathrm{a}d\,(A) = 0$ on $\mathbf{n}$. Hence $\mathbf{a}$ is and ideal contained in the center $\mathbf{z}\,(\mathbf{n})$ of $\mathbf{n}$.

We claim that $\mathbf{a}$ is contained in a Cartan subalgebra of $\mathbf{b}$. To see this recall that the Cartan subalgebras of $\mathbf{g}$ are exactly the maximal abelian subalgebras consisting of semi-simple elements (see [15], Chapter III). This characterization works for the Cartan subalgebras of $\mathbf{b}$ as well. In fact, $\mathbf{h}$ is a also a Cartan subalgebra of $\mathbf{b}$ hence any Cartan subalgebra $\mathbf{h}_1$ of $\mathbf{b}$ is maximal abelian and consists of semi-simple elements, because $\mathbf{h}_1$ is conjugate to $\mathbf{h}$ by an inner automorphism of $\mathbf{b}$. Conversely, if $\mathbf{h}_2$ is maximal abelian and consists of semi-simple elements then $\mathbf{h}_2$ is a Cartan subalgebra of $\mathbf{g}$ and hence of $\mathbf{b}$. Now, $\mathbf{a}$ is abelian and its elements are semi-simple hence $\mathbf{a}$ is contained in a Cartan subalgebra of $\mathbf{b}$.

Therefore we can assume without loss of generality that

$$\mathbf{a} \subset \mathbf{h} \subset \mathbf{b}.$$

Furthermore, any element of $\mathbf{a}$ is semi-simple, we have

$$\mathbf{n} \cap \mathbf{h} = \mathbf{a}.$$

Let $\Theta = \{\alpha \in \Pi^+ : \alpha(H) = 0 \text{ for all } H \in \mathbf{a}\}$ and put $\mathbf{m}_1 = \mathbf{h} \oplus \sum_{\alpha \in \Theta} \mathbf{g}_\alpha$. Clearly, $\mathbf{m}_1$ is the centralizer of $\mathbf{a}$ in $\mathbf{b}$, so that $\mathbf{h} + \mathbf{n} \subset \mathbf{m}_1$. Hence

$$\dim \mathbf{m}_1 \geq \dim \mathbf{n} + \dim \mathbf{h} - \dim \mathbf{n} \cap \mathbf{h} = \dim \mathbf{n} + l - k \tag{3.1}$$

where $k = \dim \mathbf{a}$.

On the other hand, if we put $\mathbf{m}_2 = \sum_{\alpha \in \Pi^+ \setminus \Theta} \mathbf{g}_\alpha$ then $\mathbf{b} = \mathbf{m}_1 \oplus \mathbf{m}_2$. We claim that $\dim \mathbf{m}_2 \geq k$.

In fact,

let Σ be the simple system of roots contained in Π^+. It is a basis of the dual $\mathbf{h}^*$ of $\mathbf{h}$. Hence there exists at least k roots $\alpha_1, \ldots, \alpha_s \in \Sigma$, $s \geq k$, which do not belong to Θ. Hence the root spaces $\mathbf{g}_{\alpha_i}$, $i = 1, \ldots, s$, are contained in $\mathbf{m}_2$ showing that $\dim \mathbf{m}_2 \geq k$.

Combining this inequality with (3.1) we have

$$\dim \mathbf{b} = \dim \mathbf{m}_1 + \dim \mathbf{m}_2 \geq \dim \mathbf{n} + l$$

showing that $\dim \mathbf{n} \leq \dim \mathbf{b} - l$, as required. □

Now we can prove the main result of this section.

Theorem 3.2. *Let $\mathbf{g}$ be a Lie algebra with complex structure, such that $\mathbf{g}_*$ is nilpotent. Then $\mathbf{g}$ is solvable.*

Proof: We write the complexification $\mathbf{g}^{\mathbf{C}}$ of $\mathbf{g}$ as the direct sum of two nilpotent subalgebras:

$$\mathbf{g}^{\mathbf{C}} = \mathbf{g}^{1,0} \oplus \mathbf{g}^{0,1}.$$

Suppose that $\mathbf{g}^{\mathbf{C}}$ is not solvable and let $\mathbf{r}\left(\mathbf{g}^{\mathbf{C}}\right)$ be its solvable radical. The Lie algebra $\mathbf{s} = \mathbf{g}^{\mathbf{C}}/\mathbf{r}\left(\mathbf{g}^{\mathbf{C}}\right)$ is semi-simple (or $\{0\}$). Let $\mathbf{n}_1$ and $\mathbf{n}_2$ be the respective projections of $\mathbf{g}^{1,0}$ and $\mathbf{g}^{0,1}$ onto $\mathbf{s}$. Clearly, $\mathbf{s} = \mathbf{n}_1 + \mathbf{n}_2$. However both subalgebras $\mathbf{n}_1$ and $\mathbf{n}_2$ of $\mathbf{s}$ are nilpotent, hence by Lemma 3.1 their dimension is $< \frac{1}{2} \dim \mathbf{s}$. This is a contradiction unless $\mathbf{s} = \{0\}$, that is $\mathbf{g}$ is solvable. □

In view of the above result it is natural to get results relating the degree of nilpotence of $\mathbf{g}_*$ with the degree of solvability of $\mathbf{g}$. In this direction we mention that it was proved recently by Andrada-Barberis-Dotti-Ovando [1] (see also [12]) that $\mathbf{g}$ is 2-step solvable if it is sum of two abelian subalgebras.

4. 6-dimensional Lie algebras

The purpose of this section is to determine which nilpotent 6-dimensional Lie algebras are $\mathbf{g}_*$ for some integrable complex structure. We observe first that the 4-dimensional case is completely clear. In fact, if $\mathbf{g}_*$ is nilpotent then it is abelian since it is a 2-dimensional complex Lie algebra. In the 6-dimensional case $\mathbf{g}^{1,0}$, $\mathbf{g}^{0,1}$ and $\mathbf{g}_*$ are 3-dimensional complex subalgebras, and hence at most 2-step nilpotent. Despite this fact be known, we want to leave here a proof.

Lemma 4.1. *If* $\mathbf{g}$ *is a* 3*-dimensional nilpotent Lie algebra, then* $\mathbf{g}$ *is at most* 2*-step nilpotent. In particular, either* $\mathbf{g}$ *is abelian or* $\mathbf{g}$ *is isomorphic to Heisenberg Lie algebra.*

Proof: If $\dim \mathbf{g}^1 = 0$, then $\mathbf{g}$ is abelian. If $\dim \mathbf{g}^1 = 1$, then either $\mathbf{g}^2 = \{0\}$, that is, $\mathbf{g}$ is 2-step nilpotent, or $\mathbf{g}^2 = \mathbf{g}^1$, what contradicts the fact that $\mathbf{g}$ is nilpotent. If $\dim \mathbf{g}^1 = 2$, then either $\mathbf{g}^1$ is abelian or there exists a basis $\{X, Y\}$ of $\mathbf{g}^1$ such that $[X, Y] = Y$. In this last case $Y \in \mathbf{g}^k$, for all k, what contradicts the fact that $\mathbf{g}$ is nilpotent. Thus $\mathbf{g}^1$ is abelian. Now, take $X \in \mathbf{g} \setminus \mathbf{g}^1$. Then $ad(X)\mathbf{g}^1 \subset \mathbf{g}^1$, and since $ad(X)$ restricted to $\mathbf{g}^1$ is nilpotent there exists a basis $\{Y, Z\}$ of $\mathbf{g}^1$ such that $[X, Y] = 0$, $[X, Z] = Y$ and $[Y, Z] = 0$. But this contradicts the fact that $\dim \mathbf{g}^1 = 2$. Therefore $\mathbf{g}$ is 2-step nilpotent. $\square$

We consider a 6-dimensional real Lie algebra 6-dimensional $\mathbf{g}$ with complexa structure J, such that $\mathbf{g}_*$ is nilpotent, but not abelian. Then $\mathbf{g}^{1,0}$ is the Heisenberg Lie algebra because $\mathbf{g}^{1,0}$ is 3-dimensional and note abelian. Consequently there exists a basis

$$\{\omega_1 = e_1 - iJe_1, \omega_2 = e_2 - iJe_2, \omega_3 = e_3 - iJ_3\} \quad of \quad \mathbf{g}^{1,0},$$

where $\{e_1, e_2, e_3, Je_1, Je_2, J_3\}$ is a basis of $\mathbf{g}$, such that the non-zero bracket is

$$[\omega_1, \omega_3] = \omega_2.$$

Therefore,

$$[e_1 - iJe_1, e_3 - iJe_3] = e_2 - iJe_2.$$

Developing these equalities we obtain

$$[e_1, e_3] - [Je_1, Je_3] = e_2 \text{ and } [e_1, Je_3] + [Je_1, e_3] = Je_2.$$

Hence,

$$[e_1 * e_3] = \frac{1}{2}e_2 \text{ and } [e_1 * Je_3] = \frac{1}{2}Je_2.$$

Now, by Proposition 2.2 J is adapted to $\mathbf{g}_*$. Therefore,

$$[Je_1 * Je_3] = -\frac{1}{2}e_2 \text{ and } [Je_1 * e_3] = \frac{1}{2}Je_2.$$

The other brackets of $\mathbf{g}_*$ are zero because $[\omega_1, \omega_2] = [\omega_2, \omega_3] = 0$. We observe that $\mathbf{g}_*$ is isomorphic the Lie algebra $\mathbf{h}$, with non-zero brackets

$$[f_1 * f_3] = [f_4 * f_2] = -f_5, \;\; [f_1 * f_4] = [f_2 * f_3] = -f_6, \tag{4.1}$$

where $\{f_1, f_2, f_3, f_4, f_5, f_6\}$ is a basis of $\mathbf{h}$. Formally, we have following Theorem

Theorem 4.2. *Let $\mathbf{g}$ be a 6-dimensional Lie algebra with a non-abelian complex structure J. If J is s-step nilpotent, then $s = 2$ and there exists a basis $\{f_1, f_2, f_3, f_4, f_5, f_6\}$ of $\mathbf{g}$, such that the non-zero brackets of $(\mathbf{g}, [*]_J)$ are*

$$[f_1 * f_3] = [f_4 * f_2] = -f_5, \;\; [f_1 * f_4] = [f_2 * f_3] = -f_6. \tag{4.2}$$

Now, we exhibit a typical example.

Example 1. *Let $\mathbf{g}$ be the Lie algebra with non-zero brackets*

$$[e_1, e_2] = -e_3 \qquad [e_1, e_3] = -e_4 \qquad [e_2, e_3] = -e_5 \qquad [e_1, e_4] = [e_2, e_5] = -e_6,$$

where $\{e_1, e_2, e_3, e_4, e_5, e_6\}$ is a basis of $\mathbf{g}$. Let J be the complex structure given by

$$Je_1 = -e_2 \qquad Je_4 = -e_5 \qquad Je_3 = -e_6.$$

We have

$$\mathbf{h} = \mathbf{g}_J^{(1,0)} = \mathit{span}\,\{\omega_1 = e_1 - iJe_1, \omega_2 = e_4 - iJe_4, \omega_3 = e_3 - iJ_3\}.$$

Now,

$$[\omega_1, \omega_2] = 0 \qquad [\omega_1, \omega_3] = \omega_2 \qquad [\omega_2, \omega_3] = 0.$$

Therefore, $\mathbf{h}^1 = \mathit{span}\,\{w_2\}$, $\mathbf{h}^2 = 0$ and J is 2-step nilpotent.

In the example 1 we have

$$[e_1 * e_3]_J = [e_6 * e_2]_J = -\frac{1}{2}e_4 \qquad [e_2 * e_3]_J = [e_1 * e_6]_J = -\frac{1}{2}e_5.$$

We can easily see that this Lie algebra is isomorphic to Lie algebra $\mathbf{h}$, with non-zero brackets

$$[f_1 * f_3] = [f_4 * f_2] = -f_5, \;\; [f_1 * f_4] = [f_2 * f_3] = -f_6,$$

where $\{f_1, f_2, f_3, f_4, f_5, f_6\}$ is a basis $\mathbf{h}$.

We can also prove the Theorem 4.2 without the use of the Lemma 4.1, just using the classification given by Salamon. In fact, the subspaces $\mathbf{g}_*^k$ have even dimension since they are J-invariant. In our case we can rule out $\dim \mathbf{g}_*^1 = 6$ and $\dim \mathbf{g}_*^1 = 0$ because $\mathbf{g}_*$ is nilpotent and not abelian. We consider separately the cases $\dim \mathbf{g}_*^1 = 4$ and $\dim \mathbf{g}_*^1 = 2$ and apply the classification given in [13].

Suppose that $\dim \mathbf{g}_*^1 = 4$. By Theorem 3.1 in [13], there exists a basis $\{f_i, 1 \leq i \leq 6\}$ of $\mathbf{g}$, such that the non-zero brackets of $(\mathbf{g}, [*]_J)$ are

$$[f_1 * f_2] = -f_3, \;\; [f_1 * f_3] = -f_4, \;\; [f_2 * f_3] = -f_5, \;\; [f_1 * f_4] = [f_2 * f_5] = -f_6.$$

In this case, $\mathbf{g}_*^1 = span\,\{f_3, f_4, f_5, f_6\}$,
$\mathbf{g}_*^2 = span\,\{f_4, f_5, f_6\}$, what contradicts the fact that $\dim \mathbf{g}_*^2$ is even. Therefore, we can assume $\dim \mathbf{g}_*^1 \neq 4$.

In case $\dim \mathbf{g}_*^1 = 2$ we have by Theorem 3.3 in [13] five possibilities. In fact, there exists a basis $\{f_i, 1 \leq i \leq 6\}$ of $\mathbf{g}$, such that the non-zero brackets of $(\mathbf{g}, [*]_J)$ are one of the following

1. $[f_1 * f_2] = -f_5$, $[f_1 * f_4] = [f_2 * f_5] = -f_6$.
2. $[f_1 * f_2] = -f_5$, $[f_1 * f_3] = -f_6$.
3. $[f_1 * f_3] = [f_4 * f_2] = -f_5$, $[f_1 * f_4] = [f_2 * f_3] = -f_6$.
4. $[f_1 * f_2] = -f_5$, $[f_1 * f_4] = [f_2 * f_3] = -f_6$.
5. $[f_1 * f_2] = -f_5$, $[f_3 * f_4] = -f_6$.

By our Example 1 there exists a Lie algebra $\mathbf{g}$, with 2-step complex structure J, such that $(\mathbf{g}, [*]_J)$ is isomorphic to algebra given in the third line above. We prove now that the other cases cannot happen.

In the first case, $\mathbf{g}_*^1 = span\{f_5, f_6\}$,
$\mathbf{g}_*^k = span\{f_6\}$, which contradicts the fact that $\dim \mathbf{g}_*^2$ is even, ruling out this case.

In the other cases ((2), (4) and (5)) we have $\mathbf{g}_*^1 = span\{f_5, f_6\}$,
$\mathbf{g}_*^2 = \{0\}$. To get rid of case (2) we suppose that $[f_1 * f_2] = -f_5$, $[f_1 * f_3] = -f_6$. Since $\mathbf{g}_*^1$ is J-invariant we can write $\mathrm{Jf}_5 = af_5 + bf_6$,
$Jf_6 = cf_5 + df_6$,
$Jf_i = \sum_{j=1}^{6} a_{ij} f_j$, $1 \leq i \leq 4$. But $J^2 = -I$ implies that $a = -d$. Moreover, by Proposition 2.2 we have

$$-(af_5 + bf_6) = -Jf_5 = J[f_1 * f_2] = [Jf_1 * f_2] = -a_{11}f_5.$$

Hence $b = 0$ and $a = a_{11}$, and this contradicts the fact that $J^2 = -I$.

Now we suppose that $[f_1 * f_2] = -f_5$, $[f_1 * f_4] = [f_2 * f_3] = -f_6$ (case (4)). Taking J as above and applying Proposition 2.2 again we get

$$-(cf_5 - af_6) = -Jf_6 = J[f_2 * f_3] = [Jf_2 * f_3] = -a_{22}f_6.$$

Thus, $c = 0$ and $a = -a_{22}$, what contradicts the fact that $J^2 = -I$.

Finally suppose that $[f_1 * f_2] = -f_5$, $[f_3 * f_4] = -f_6$. Writing J as above we obtain $b = 0$ and $a = a_{11}$. Again this contradicts the fact that $J^2 = -I$. This finishes the cases completing the proof of the theorem.

5. Examples

In this section we produce some examples of complex and hypercomplex structures on solvable Lie algebras.

We start by applying Theorem 4.2 to produce examples relating the $\mathbf{g}$ and $\mathbf{g}_*$ structures on 6-dimensional solvable Lie algebras. This requires some preparations.

Let $\{f_1, f_2, f_3, f_4, f_5, f_6\}$ be a basis of $\mathbf{g}_*$ with bracket relations given by (**??**). Our objective is to get information about the complex structure J and the original bracket $[\cdot,\cdot]$. For this we write

$$Jf_i = \sum_{j=1}^{6} a_{ij} f_j, 1 \leq i \leq 4$$

and $\mathrm{Jf}_5 = af_5 + bf_6$
$Je_6 = cf_5 + df_6$. This is possible because $\mathbf{g}_*^1$ is J-invariant. From $J^2 = -I$ we have the restriction $a = -d$.

Now, by Proposition 2.2, we have $J\left[f_1 * f_2\right]_J = \left[Jf_1 * f_2\right]_J$. Therefore,
$0 = [a_{11}f_1 + a_{21}f_2 + a_{31}f_3 + a_{41}f_4 + a_{51}f_5 + a_{61}f_6 * f_2]_J$
$= a_{31}\left[f_3 * f_2\right]_J + a_{41}\left[f_4 * f_2\right]_J$
$= a_{31}f_6 - a_{41}f_5$, and consequently $a_{31} = a_{41} = 0$. Also, $J\left[f_1 * f_3\right]_J = \left[Jf_1 * f_3\right]_J$. Hence, $J(-f_5) = \left[Jf_1 * f_3\right]_J$ so that

$$J(f_5) = a_{11}f_5 + a_{21}f_6 = af_5 + bf_6,$$

which implies that $a_{11} = a$ and $a_{21} = b$. Applying $J\left[f_1 * f_3\right]_J = \left[Jf_1 * f_3\right]_J$ to the other brackets in (**??**) and using $J^2 = -I$, we obtain $a = d = 0$, $a_{11} = a_{31} = a_{41} = 0$, $a_{22} = a_{32} = a_{42} = 0$, $a_{13} = a_{23} = a_{33} = 0$, $a_{14} = a_{24} = a_{44} = 0$, $a_{43} = -a_{34} = b$, $b^2 = 1$, $a_{54} = a_{63}$, $a_{64} = -a_{53}$, $a_{52} = a_{61}$ and $a_{62} = -a_{51}$. In summary, $\mathrm{Jf}_1 = bf_2 + a_{51}f_5 + a_{61}f_6$
$Jf_2 = -bf_1 + a_{61}f_5 - a_{51}f_6$
$Jf_3 = bf_4 + a_{53}f_5 + a_{63}f_6$
$Jf_4 = -bf_3 + a_{63}f_5 - a_{53}f_6$
$Jf_5 = bf_6$
$Jf_6 = -bf_5$.

In particular, let us take $b = 1$ and $a_{51} = a_{61} = a_{53} = a_{63} = 0$ so that -$\mathrm{e}_5 = \left[e_1 * e_3\right]_J = \frac{1}{2}([e_1, e_3] - [Je_1, Je_3]$
$= \frac{1}{2}([e_1, e_3] - [e_2, e_4])$. In a similar way, we get the brackets $[e_1, e_3] = [e_2, e_4] - 2e_5$
$[e_1, e_4] = -\left[e_2, e_3\right] - 2e_6$
$[e_1, e_5] = [e_2, e_6]$
$[e_1, e_6] = -\left[e_2, e_5\right]$
$[e_3, e_5] = [e_4, e_6]$
$[e_3, e_6] = -\left[e_4, e_5\right]$.

Observing these equalities we can construct an example where $(\mathbf{g}, [*]_J)$ is nilpotent but $(\mathbf{g}, [\cdot,\cdot])$ is not nilpotent.

Example 2. *Let* $\mathbf{g}$ *be a 6-dimensional vector space and* $\{f_1, f_2, f_3, f_4, f_5, f_6\}$ *a fixed base of* $\mathbf{g}$. *Then* $(\mathbf{g}, [\cdot,\cdot])$ *is a Lie algebra if the non-zero brackets are given by*

$$[f_1, f_2] = f_1,\ [f_2, f_4] = 2f_5,\ [f_2, f_3] = -2f_6.$$

Moreover, the endomorphism $J : \mathbf{g} \longrightarrow \mathbf{g}$ *given by*

$$Jf_1 = f_2,\ Jf_3 = f_4,\ Jf_5 = f_6$$

is a complex structure on $\mathbf{g}$. *We have* $\mathbf{g}^1 = span\{f_1, f_5, f_6\}$ *and* $\mathbf{g}^k = span\{f_1\}$, *for all* $k \geq 2$. *This shows that* $(\mathbf{g}, [\cdot,\cdot])$ *is not nilpotent. However,* *the non-zero brackets of* $(\mathbf{g}, [*]_J)$ *are given by*

$$[f_1 * f_3]_J = [f_4 * f_2]_J = -f_5, \qquad [f_1 * f_4]_J = [f_2 * f_3]_J = -f_6.$$

Thus $(\mathbf{g}, [*]_J)$ *is 2-step nilpotent.*

Remark: In the example 1 we have a nilpotent Lie algebra with complex strucuture. Already in the example 2 we have a non nilpotent Lie algebra with another complex structure. Hence this are non isomorphics Lie algebras. However, the bracket * gives us isomorphics Lie algebras.

Next we provide examples proving that for any $s \geq 1$ there exists a nilpotent Lie algebra $\mathbf{g}$ with a complex structure J such that $\mathbf{g}_*$ is s-step nilpotent. As in [2] we work with certain affine Lie algebras $\mathbf{aff}\,(\mathcal{A})$ and natural complex structures on them.

Proposition 5.1. *For any* $s \geq 1$ *there exists a* s*-step nilpotent complex structure.*

Proof: We first work out in detail the case $s = 3$. Let $\mathcal{A}$ be the space of 4×4 upper triangular real matrices with zeros on the diagonal. Clearly, $\mathcal{A}$ is an associative and noncommutative algebra and in general $ABC \neq 0$ and $ABCD = 0$ for $A, B, C, D \in \mathcal{A}$.

We denote $\mathcal{A}^k = span\,\{A_1 A_2 \cdots A_k;\ A_j \in \mathcal{A}\}$. Let $\mathbf{aff}\,(\mathcal{A})$ be the Lie algebra $\mathcal{A} \oplus \mathcal{A}$ with bracket given by

$$[(A,B),(C,D)] = (AC - CA, AD - CB)\,, \forall\ A, B, C, D \in \mathcal{A}.$$

The algebra $\mathbf{aff}\,(\mathcal{A})$ is the semidirect product of $\mathcal{A}$ (considered as Lie subalgebra of $\mathbf{gl}(\mathcal{A})$, that is, for each $A \in \mathcal{A}$, $\mathcal{A} \ni B \mapsto AB \in \mathcal{A}$) and $\mathcal{A}$ (considered as an abelian Lie algebra), with canonical representation, that is, the inclusion.

If $\mathcal{B}$ and $\mathcal{C}$ are two subsets of $\mathcal{A}$, we denote $(\mathcal{B},\mathcal{C}) = \{(B,C)\,; B \in \mathcal{B}, C \in \mathcal{C}\}$. Let J be the endomorphism of $\mathbf{aff}\,(\mathcal{A})$ defined by

$$J(A,B) = (B, -A),\ \forall\ A, B \in \mathcal{A}.$$

Clearly, $J^2 = -I$, so it defines a complex structure on $\mathbf{aff}\,(\mathcal{A})$. Moreover, we have $2[(A,B) * (C,D)] = [(A,B),(C,D)] - [(B,-A),(D,-C)]$
$= (AC - CA, AD - CB) - (BD - DB, -BC + DA)$
$= (AC - CA + DB - BD, AD - DA + BC - CB)$. This shows that J is not an abelian complex structure.

If we put $(X, Y) = (AC - CA + DB - BD, AD - DA + BC - CB)$, we can see that

$$[(E, F), (X, Y)] - [J(E, F), J(X, Y)] \in \left(\mathcal{A}^3, \mathcal{A}^3\right)$$

and this bracket is in general not zero. By a similar computation, we obtain

$$\mathbf{g}_*^3 \subset \left(\mathcal{A}^4, \mathcal{A}^4\right) = \{(0,0)\}.$$

Therefore J is a 3-step nilpotent complex structure on $\mathbf{aff}(\mathcal{A})$.

In general we let $\mathcal{A}$ be the space of $(s+1) \times (s+1)$ upper triangular real matrices with zeros on the diagonal. Again $\mathcal{A}$ is an associative and noncommutative algebra. We take the bracket and J as above. By a similar computation we get $\mathbf{g}_*^{s-1} \subset (\mathcal{A}^s, \mathcal{A}^s)$ and $\mathbf{g}_*^{s-1}$ is not zero. Moreover, $\mathbf{g}_*^s \subset (\mathcal{A}^{s+1}, \mathcal{A}^{s+1}) = \{(0,0)\}$. Therefore J is a s-step nilpotent complex structure on $\mathbf{aff}(\mathcal{A})$. □

We give now examples of hypercomplex structures. By lemma 3.1 of [8] we have that if J_1, J_2 are anticommuting complex structures and J_1 is abelian then J_2 is also abelian. Also by Proposition 3.1 of [7] an 8-dimensional nilpotent Lie algebra which admits a hypercomplex structure is 2-step nilpotent. Since $\mathbf{g}_*^2 \subset \mathbf{g}^2 = \{0\}$, it follows that each J_i is s-step nilpotent and $s \leq 2$. Moreover, if one of the complex structures is 2-step nilpotent, then by Lemma 3.1 of [8] the other is 2-step nilpotent as well.

In general we say that hypercomplex structure given by a pair $\{J_1, J_2\}$ of anticommuting complex structure is s-step nilpotent if both J_1 and J_2 are s-step nilpotent for the same s.

We present a example of 2-step nilpotent hypercomplex structure.

Example 3. *Let* $\mathbf{n}$ *be the* 8*-dimensional Lie algebra with non-zero brackets given by*

$$[e_1, e_2] = [e_3, e_4] = -e_6 \quad [e_1, e_3] = -[e_2, e_4] = -e_7 \quad [e_1, e_4] = [e_2, e_3] = -e_8,$$

where $\{e_1, e_2, e_3, e_4, e_5, e_6, e_7, e_8\}$ *is a basis of* $\mathbf{n}$. *Let* $\{J_1, J_2\}$ *be the hypercomplex structure on* $\mathbf{n}$ *given by* $J_1 e_1 = e_2 \quad J_1 e_3 = e_4 \quad J_1 e_5 = e_6 \quad J_1 e_7 = e_8$
$J_2 e_1 = e_3 \quad J_2 e_2 = -e_4 \quad J_2 e_5 = e_7 \quad J_2 e_6 = -e_8$ *We have*

$$\mathbf{h} = \mathbf{n}_{J_2}^{(0,1)} = span\{\omega_j = e_j + iJ_2 e_j, 1 \leq j \leq 8\}$$

Now, $[\omega_1, \omega_2] = \omega_4 \qquad [\omega_1, \omega_3] = 0 \qquad [\omega_1, \omega_4] = 0$
$[\omega_2, \omega_3] = 0 \qquad [\omega_2, \omega_4] = 0 \qquad [\omega_3, \omega_4] = 0$ *Therefore,* $\mathbf{h}^1 = span\,\{\omega_4\}$ *and* $\mathbf{h}^2 = \{0\}$*. Thus* J_2 *is* 2*-step nilpotent. As remarked above we have that* $\{J_1, J_2\}$ *is a* 2*-step nilpotent hypercomplex structure.*

Theorem 3.1 in [9] says that the hypercomplex structure of a HKT structure on any 2-step nilpotent Lie algebra is abelian. Thus the hypercomplex structure $\{J_1, J_2\}$ in the above example is not a hypercomplex structure of a HKT structure.

In [2], Proposition 3.5, for any positive integer k is presented a k-step nilpotent Lie algebra carrying an abelian hypercomplex structure. Our purpose is to build a s-step nilpotent hypercomplex structure for each $s \geq 1$.

Similar to Proposition 5.1 let $\mathcal{A}$ be the space of $(s+1) \times (s+1)$ upper triangular complex matrices with zeros on the diagonal. Define an endomorphism K of $\mathbf{aff}\,(\mathcal{A})$ by

$$K(A, B) = (-iA, iB), \forall\ A, B \in \mathcal{A}.$$

Is is easy to see that K is a complex structure on $\mathbf{aff}\,(\mathcal{A})$ and that $KJ = -JK$, where J is defined as in the proof of Proposition 5.1. A straightforward computations shows that

$$2\,[(A, B) * (C, D)] = (2AC - 2CA, 0),$$

so that K also is s-step nilpotent on $\mathbf{aff}\,(\mathcal{A})$. Combining this remark with previous Proposition 5.1 we deduce:

Proposition 5.2. *For any* $s \geq 1$ *there exists a* s*-step nilpotent hypercomplex structure.*

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