Proyecciones Journal of Mathematics
Vol. 30, N^o 2, pp. 213-245, August 2011.
Universidad Católica del Norte
Antofagasta - Chile
DOI: 10.4067/S0716-09172011000200007

Triple generating functions of Jacobi polynomials of two variables

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Received : March 2011. Accepted : May 2011

Abstract

The present paper is a study of triple generating functions of Jacobi polynomials of two variables.

Keywords : *Triple generating functions.*

A. M. S. Subject Classification : *33C45.*

1. INTRODUCTION

In our paper [6] we discussed some generating and double generating functions of Jacobi polynomials of two variables. Now in this paper we obtain some triple generating functions of Jacobi polynomials of two variables.

In our investigation we require the following results:
Jacobi polynomials of two variables are defined by [10]

$$P_n^{(\alpha_1,\,\beta_1;\,\alpha_2,\,\beta_2)}(x,y) = \frac{(1+\alpha_1)_n(1+\alpha_2)_n}{(n!)^2}$$

(1.1)
$$\times F_2\left[-n, 1+\alpha_2+\beta_2+n, 1+\alpha_1+\beta_1+n; 1+\alpha_2, 1+\alpha_1; \frac{1-y}{2}, \frac{1-x}{2}\right].$$

The confluent hypergeometric functions of two variables are defined by [9]

$$\phi_1\left[\alpha,\beta;\gamma;x,y\right] = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(\alpha)_{m+n}(\beta)_m}{(\gamma)_{m+n}}\frac{x^m}{m!}\frac{y^n}{n!}, |x|<1, |y|<\infty \ , \tag{1.2}$$

$$\phi_2\left[\beta,\beta';\gamma;x,y\right] = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(\beta)_m(\beta')_n}{(\gamma)_{m+n}}\frac{x^m}{m!}\frac{y^n}{n!}, |x|<\infty, \quad |y|<\infty \ , \tag{1.3}$$

$$\phi_3\left[\beta;\gamma;x,y\right] = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(\beta)_m}{(\gamma)_{m+n}}\frac{x^m}{m!}\frac{y^n}{n!}, |x|<\infty, |y|<\infty \ , \tag{1.4}$$

$$\Psi_1\left[\alpha,\beta;\gamma,\gamma';x,y\right] = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(\alpha)_{m+n}(\beta)_m}{(\gamma)_m(\gamma')_n}\frac{x^m}{m!}\frac{y^n}{n!}, |x|<1, |y|<\infty \ , \tag{1.5}$$

$$\Psi_2\left[\alpha;\gamma,\gamma';x,y\right] = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(\alpha)_{m+n}}{(\gamma)_m(\gamma')_n}\frac{x^m}{m!}\frac{y^n}{n!}, |x|<\infty, \quad |y|<\infty \ , \tag{1.6}$$

(1.7)
$$\Xi_1\left[\alpha,\alpha',\beta;\gamma;x,y\right] = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(\alpha)_m(\alpha')_n(\beta)_m}{(\gamma)_{m+n}}\frac{x^m}{m!}\frac{y^n}{n!}, |x|<1, |y|<\infty.$$

2. TRIPLE GENERATING FUNCTIONS

The polynomials $P_n^{(\alpha_1,\,\beta_1;\,\alpha_2,\,\beta_2)}(x,y)$ admits the following triple generating functions:

$$\sum_{m=0}^{\infty}\sum_{k=0}^{\infty}\sum_{n=0}^{\infty}\frac{n!(1+\alpha_1+\beta_1)_m(\lambda)_k(1+\alpha_2+\beta_2)_k}{m!k!(1+\alpha_1)_n(1+\alpha_2)_{k+n}}$$

$$\times P_n^{(\alpha_1,\,\beta_1+m-n;\,\alpha_2+k,\,\beta_2-n)}(x,y)t^n u^m v^k$$

$$= e^t\ (1-u)^{-1-\alpha_1-\beta_1}\ {}_1F_1\left[1+\alpha_1+\beta_1; 1+\alpha_1; \frac{\frac{1}{2}(x-1)t}{(1-u)}\right]$$

$$\times\ \Phi_1\left[1+\alpha_2+\beta_2, \lambda; 1+\alpha_2; v, \frac{1}{2}(y-1)t\right], \tag{2.1}$$

$$\sum_{m=0}^{\infty}\sum_{k=0}^{\infty}\sum_{n=0}^{\infty}\frac{n!(\lambda)_m(1+\alpha_1+\beta_1)_m(1+\alpha_2+\beta_2)_k}{m!k!(1+\alpha_1)_{m+n}(1+\alpha_2)_n}$$

$$\times P_n^{(\alpha_1+m,\,\beta_1-n;\,\alpha_2,\,\beta_2+k-n)}(x,y)t^n u^m v^k$$

$$= e^t\ (1-v)^{-1-\alpha_2-\beta_2}\ {}_1F_1\left[1+\alpha_2+\beta_2; 1+\alpha_2; \frac{\frac{1}{2}(y-1)t}{(1-v)}\right]$$

$$\times\ \Phi_1\left[1+\alpha_1+\beta_1, \lambda; 1+\alpha_1; u, \frac{1}{2}(x-1)t\right], \tag{2.2}$$

$$\sum_{m=0}^{\infty}\sum_{k=0}^{\infty}\sum_{n=0}^{\infty}\frac{n!(1+\alpha_1+\beta_1)_m(\lambda)_k}{m!k!(1+\alpha_1)_n(1+\alpha_2)_{k+n}}$$

$$\times P_n^{(\alpha_1,\,\beta_1+m-n;\,\alpha_2+k,\,\beta_2-k-n)}(x,y)t^n u^m v^k$$

$$= e^t\ (1-u)^{-1-\alpha_1-\beta_1}\ {}_1F_1\left[1+\alpha_1+\beta_1; 1+\alpha_1; \frac{\frac{1}{2}(x-1)t}{(1-u)}\right]$$

$$\times\ \Phi_2\left[\lambda, 1+\alpha_2+\beta_2; 1+\alpha_2; v, \frac{1}{2}(y-1)t\right], \tag{2.3}$$

$$\sum_{m=0}^{\infty}\sum_{k=0}^{\infty}\sum_{n=0}^{\infty}\frac{n!(\lambda)_m(1+\alpha_2+\beta_2)_k}{m!k!(1+\alpha_1)_{m+n}(1+\alpha_2)_n}$$

$$\times P_n^{(\alpha_1+m,\ \beta_1-m-n;\ \alpha_2,\ \beta_2+k-n)}(x,y)t^n u^m v^k$$

$$= e^t\ (1-v)^{-1-\alpha_2-\beta_2}\ {}_1F_1\left[1+\alpha_2+\beta_2; 1+\alpha_2; \frac{\frac{1}{2}(y-1)t}{(1-v)}\right]$$

$$\times\ \Phi_2\left[\lambda, 1+\alpha_1+\beta_1; 1+\alpha_1; u, \frac{1}{2}(x-1)t\right], \tag{2.4}$$

$$\sum_{m=0}^{\infty}\sum_{k=0}^{\infty}\sum_{n=0}^{\infty}\frac{n!(1+\alpha_1+\beta_1)_m(\lambda)_k(1+\alpha_2+\beta_2)_k}{m!k!(\mu)_k(1+\alpha_1)_n(1+\alpha_2)_n}$$

$$\times P_n^{(\alpha_1,\ \beta_1+m-n;\ \alpha_2,\ \beta_2+k-n)}(x,y)t^n u^m v^k$$

$$= e^t\ (1-u)^{-1-\alpha_1-\beta_1}\ {}_1F_1\left[1+\alpha_1+\beta_1; 1+\alpha_1; \frac{\frac{1}{2}(x-1)t}{(1-u)}\right]$$

$$\times\ \Psi_1\left[1+\alpha_2+\beta_2, \lambda; \mu, 1+\alpha_2; v, \frac{1}{2}(y-1)t\right], \tag{2.5}$$

$$\sum_{m=0}^{\infty}\sum_{k=0}^{\infty}\sum_{n=0}^{\infty}\frac{n!(\lambda)_m(1+\alpha_1+\beta_1)_m(1+\alpha_2+\beta_2)_k}{m!k!(\mu)_m(1+\alpha_1)_n(1+\alpha_2)_n}$$

$$\times P_n^{(\alpha_1,\ \beta_1+m-n;\ \alpha_2,\ \beta_2+k-n)}(x,y)t^n u^m v^k$$

$$= e^t\ (1-v)^{-1-\alpha_2-\beta_2}\ {}_1F_1\left[1+\alpha_2+\beta_2; 1+\alpha_2; \frac{\frac{1}{2}(y-1)t}{(1-v)}\right]$$

$$\times\ \Psi_1\left[1+\alpha_1+\beta_1, \lambda; \mu, 1+\alpha_1; u, \frac{1}{2}(x-1)t\right], \tag{2.6}$$

$$\sum_{m=0}^{\infty}\sum_{k=0}^{\infty}\sum_{n=0}^{\infty}\frac{n!(1+\alpha_1+\beta_1)_m(1+\alpha_2+\beta_2)_k}{m!k!(\lambda)_k(1+\alpha_1)_n(1+\alpha_2)_n}$$

$$\times P_n^{(\alpha_1,\ \beta_1+m-n;\ \alpha_2,\ \beta_2+k-n)}(x,y)t^n u^m v^k$$

$$= e^t\ (1-u)^{-1-\alpha_1-\beta_1}\ {}_1F_1\left[1+\alpha_1+\beta_1; 1+\alpha_1; \frac{\frac{1}{2}(x-1)t}{(1-u)}\right]$$

$$\times\ \Psi_2\left[1+\alpha_2+\beta_2;\lambda,1+\alpha_2;v,\frac{1}{2}(y-1)t\right], \tag{2.7}$$

$$\sum_{m=0}^{\infty}\sum_{k=0}^{\infty}\sum_{n=0}^{\infty}\frac{n!(1+\alpha_1+\beta_1)_m(1+\alpha_2+\beta_2)_k}{m!k!(\lambda)_m(1+\alpha_1)_n(1+\alpha_2)_n}$$
$$\times P_n^{(\alpha_1,\,\beta_1+m-n;\,\alpha_2,\,\beta_2+k-n)}(x,y)t^n u^m v^k$$
$$=e^t\ (1-v)^{-1-\alpha_2-\beta_2}\ {}_1F_1\left[1+\alpha_2+\beta_2;1+\alpha_2;\frac{\frac{1}{2}(y-1)t}{(1-v)}\right]$$

$$\times\ \Psi_2\left[1+\alpha_1+\beta_1;\lambda,1+\alpha_1;u,\frac{1}{2}(x-1)t\right], \tag{2.8}$$

$$\sum_{m=0}^{\infty}\sum_{k=0}^{\infty}\sum_{n=0}^{\infty}\frac{n!(\lambda)_m(1+\alpha_1+\beta_1)_m(\mu)_k(1+\alpha_2+\beta_2)_k}{m!k!(1+\alpha_1)_{m+n}(1+\alpha_2)_{k+n}}$$
$$\times P_n^{(\alpha_1+m,\,\beta_1-n;\,\alpha_2+k,\,\beta_2-n)}(x,y)t^n u^m v^k$$
$$=e^t\ \Phi_1\left[1+\alpha_1+\beta_1,\lambda;1+\alpha_1;u,\frac{1}{2}(x-1)t\right]$$

$$\times\ \Phi_1\left[1+\alpha_2+\beta_2,\mu;1+\alpha_2;v,\frac{1}{2}(y-1)t\right], \tag{2.9}$$

$$\sum_{m=0}^{\infty}\sum_{k=0}^{\infty}\sum_{n=0}^{\infty}\frac{n!(\lambda)_m(\mu)_k}{m!k!(1+\alpha_1)_{m+n}(1+\alpha_2)_{k+n}}$$
$$\times P_n^{(\alpha_1+m,\,\beta_1-m-n;\,\alpha_2+k,\,\beta_2-k-n)}(x,y)t^n u^m v^k$$
$$=e^t\ \Phi_2\left[\lambda,1+\alpha_1+\beta_1;1+\alpha_1;u,\frac{1}{2}(x-1)t\right]$$

$$\times\ \Phi_2\left[\mu,1+\alpha_2+\beta_2;1+\alpha_2;v,\frac{1}{2}(y-1)t\right], \tag{2.10}$$

$$\sum_{m=0}^{\infty}\sum_{k=0}^{\infty}\sum_{n=0}^{\infty}\frac{n!(\lambda)_m(\mu)_k(1+\alpha_1+\beta_1)_m(1+\alpha_2+\beta_2)_k}{m!k!(\delta)_m(\nu)_k(1+\alpha_1)_n(1+\alpha_2)_n}$$

$$\times P_n^{(\alpha_1,\,\beta_1+m-n;\,\alpha_2,\,\beta_2+k-n)}(x,y)t^n u^m v^k$$
$$= e^t\ \Psi_1\left[1+\alpha_1+\beta_1,\lambda;\delta,1+\alpha_1;u,\frac{1}{2}(x-1)t\right]$$
$$\times\ \Psi_1\left[1+\alpha_2+\beta_2,\mu;\nu,1+\alpha_2;v,\frac{1}{2}(y-1)t\right], \tag{2.11}$$

$$\sum_{m=0}^{\infty}\sum_{k=0}^{\infty}\sum_{n=0}^{\infty}\frac{n!(1+\alpha_1+\beta_1)_m(1+\alpha_2+\beta_2)_k}{m!k!(\lambda)_m(\mu)_k(1+\alpha_1)_n(1+\alpha_2)_n}$$
$$\times P_n^{(\alpha_1,\,\beta_1+m-n;\,\alpha_2,\,\beta_2+k-n)}(x,y)t^n u^m v^k$$
$$= e^t\ \Psi_2\left[1+\alpha_1+\beta_1;\lambda,1+\alpha_1;u,\frac{1}{2}(x-1)t\right]$$
$$\times\ \Psi_2\left[1+\alpha_2+\beta_2;\mu,1+\alpha_2;v,\frac{1}{2}(y-1)t\right], \tag{2.12}$$

$$\sum_{m=0}^{\infty}\sum_{k=0}^{\infty}\sum_{n=0}^{\infty}\frac{n!(\lambda)_m(1+\alpha_1+\beta_1)_m(\mu)_k}{m!k!(1+\alpha_1)_{m+n}(1+\alpha_2)_{k+n}}$$
$$\times P_n^{(\alpha_1+m,\,\beta_1-n;\,\alpha_2+k,\,\beta_2-k-n)}(x,y)t^n u^m v^k$$
$$= e^t\ \Phi_1\left[1+\alpha_1+\beta_1,\lambda;1+\alpha_1;u,\frac{1}{2}(x-1)t\right]$$
$$\times\ \Phi_2\left[\mu,1+\alpha_2+\beta_2;1+\alpha_2;v,\frac{1}{2}(y-1)t\right], \tag{2.13}$$

$$\sum_{m=0}^{\infty}\sum_{k=0}^{\infty}\sum_{n=0}^{\infty}\frac{n!(\lambda)_m(\mu)_k(1+\alpha_2+\beta_2)_k}{m!k!(1+\alpha_1)_{m+n}(1+\alpha_2)_{k+n}}$$
$$\times P_n^{(\alpha_1+m,\,\beta_1-m-n;\,\alpha_2+k,\,\beta_2-n)}(x,y)t^n u^m v^k$$
$$= e^t\ \Phi_2\left[\lambda,1+\alpha_1+\beta_1;1+\alpha_1;u,\frac{1}{2}(x-1)t\right]$$
$$\times\ \Phi_1\left[1+\alpha_2+\beta_2,\mu;1+\alpha_2;v,\frac{1}{2}(y-1)t\right], \tag{2.14}$$

$$\sum_{m=0}^{\infty}\sum_{k=0}^{\infty}\sum_{n=0}^{\infty}\frac{n!(\lambda)_m(1+\alpha_1+\beta_1)_m}{m!k!(1+\alpha_1)_{m+n}(1+\alpha_2)_{k+n}}$$
$$\times P_n^{(\alpha_1+m,\ \beta_1-n;\ \alpha_2+k,\ \beta_2-k-n)}(x,y)t^n u^m v^k$$
$$= e^t\ \Phi_1\left[1+\alpha_1+\beta_1,\lambda;1+\alpha_1;u,\frac{1}{2}(x-1)t\right]$$
$$\times\ \Phi_3\left[1+\alpha_2+\beta_2;1+\alpha_2;\frac{1}{2}(y-1)t,v\right], \tag{2.15}$$

$$\sum_{m=0}^{\infty}\sum_{k=0}^{\infty}\sum_{n=0}^{\infty}\frac{n!(\lambda)_k(1+\alpha_2+\beta_2)_k}{m!k!(1+\alpha_1)_{m+n}(1+\alpha_2)_{k+n}}$$
$$\times P_n^{(\alpha_1+m,\ \beta_1-m-n;\ \alpha_2+k,\ \beta_2-n)}(x,y)t^n u^m v^k$$
$$= e^t\ \Phi_3\left[1+\alpha_1+\beta_1;1+\alpha_1;\frac{1}{2}(x-1)t,u\right]$$
$$\times\ \Phi_1\left[1+\alpha_2+\beta_2,\lambda;1+\alpha_2;v,\frac{1}{2}(y-1)t\right], \tag{2.16}$$

$$\sum_{m=0}^{\infty}\sum_{k=0}^{\infty}\sum_{n=0}^{\infty}\frac{n!(\lambda)_m}{m!k!(1+\alpha_1)_{m+n}(1+\alpha_2)_{k+n}}$$
$$\times P_n^{(\alpha_1+m,\ \beta_1-m-n;\ \alpha_2+k,\ \beta_2-k-n)}(x,y)t^n u^m v^k$$
$$= e^t\ \Phi_2\left[\lambda,1+\alpha_1+\beta_1;1+\alpha_1;u,\frac{1}{2}(x-1)t\right]$$
$$\times\ \Phi_3\left[1+\alpha_2+\beta_2;1+\alpha_2;\frac{1}{2}(y-1)t,v\right], \tag{2.17}$$

$$\sum_{m=0}^{\infty}\sum_{k=0}^{\infty}\sum_{n=0}^{\infty}\frac{n!(\lambda)_k}{m!k!(1+\alpha_1)_{m+n}(1+\alpha_2)_{k+n}}$$
$$\times P_n^{(\alpha_1+m,\ \beta_1-m-n;\ \alpha_2+k,\ \beta_2-k-n)}(x,y)t^n u^m v^k$$

$$= e^t \ \Phi_3 \left[1+\alpha_1+\beta_1; 1+\alpha_1; \frac{1}{2}(x-1)t, u\right]$$

$$\times \ \Phi_2 \left[\lambda, 1+\alpha_2+\beta_2; 1+\alpha_2; v, \frac{1}{2}(y-1)t\right], \tag{2.18}$$

$$\sum_{m=0}^{\infty}\sum_{k=0}^{\infty}\sum_{n=0}^{\infty} \frac{n!(\lambda)_m(1+\alpha_1+\beta_1)_m(\mu)_k(1+\alpha_2+\beta_2)_k}{m!k!(\delta)_k(1+\alpha_1)_{m+n}(1+\alpha_2)_n}$$

$$\times P_n^{(\alpha_1+m,\ \beta_1-n;\ \alpha_2,\ \beta_2+k-n)}(x,y)t^n u^m v^k$$

$$= e^t \ \Phi_1 \left[1+\alpha_1+\beta_1, \lambda; 1+\alpha_1; u, \frac{1}{2}(x-1)t\right]$$

$$\times \ \Psi_1 \left[1+\alpha_2+\beta_2, \mu; \delta, 1+\alpha_2; v, \frac{1}{2}(y-1)t\right], \tag{2.19}$$

$$\sum_{m=0}^{\infty}\sum_{k=0}^{\infty}\sum_{n=0}^{\infty} \frac{n!(\lambda)_m(1+\alpha_1+\beta_1)_m(\mu)_k(1+\alpha_2+\beta_2)_k}{m!k!(\delta)_m(1+\alpha_1)_n(1+\alpha_2)_{k+n}}$$

$$\times P_n^{(\alpha_1,\ \beta_1+m-n;\ \alpha_2+k,\ \beta_2-n)}(x,y)t^n u^m v^k$$

$$= e^t \ \Psi_1 \left[1+\alpha_1+\beta_1, \lambda; \delta, 1+\alpha_1; u, \frac{1}{2}(x-1)t\right]$$

$$\times \ \Phi_1 \left[1+\alpha_2+\beta_2, \mu; 1+\alpha_2; v, \frac{1}{2}(y-1)t\right], \tag{2.20}$$

$$\sum_{m=0}^{\infty}\sum_{k=0}^{\infty}\sum_{n=0}^{\infty} \frac{n!(\lambda)_m(1+\alpha_1+\beta_1)_m(1+\alpha_2+\beta_2)_k}{m!k!(1+\alpha_1)_{m+n}(\mu)_k(1+\alpha_2)_n}$$

$$\times P_n^{(\alpha_1+m,\ \beta_1-n;\ \alpha_2,\ \beta_2+k-n)}(x,y)t^n u^m v^k$$

$$= e^t \ \Phi_1 \left[1+\alpha_1+\beta_1, \lambda; 1+\alpha_1; u, \frac{1}{2}(x-1)t\right]$$

$$\times \ \Psi_2 \left[1+\alpha_2+\beta_2, \mu; 1+\alpha_2; v, \frac{1}{2}(y-1)t\right], \tag{2.21}$$

$$\sum_{m=0}^{\infty}\sum_{k=0}^{\infty}\sum_{n=0}^{\infty}\frac{n!(1+\alpha_1+\beta_1)_m(\lambda)_k(1+\alpha_2+\beta_2)_k}{m!k!(\mu)_m(1+\alpha_1)_n(1+\alpha_2)_{k+n}}$$
$$\times P_n^{(\alpha_1,\,\beta_1+m-n;\,\alpha_2+k,\,\beta_2-n)}(x,y)t^n u^m v^k$$
$$= e^t\ \Psi_2\left[1+\alpha_1+\beta_1,\mu;1+\alpha_1;u,\frac{1}{2}(x-1)t\right]$$
$$\times\ \Phi_1\left[1+\alpha_2+\beta_2,\lambda;1+\alpha_2;v,\frac{1}{2}(y-1)t\right], \tag{2.22}$$

$$\sum_{m=0}^{\infty}\sum_{k=0}^{\infty}\sum_{n=0}^{\infty}\frac{n!(\lambda)_m(1+\alpha_1+\beta_1)_m(\mu)_k(\nu)_k}{m!k!(1+\alpha_1)_{m+n}(1+\alpha_2)_{k+n}}$$
$$\times P_n^{(\alpha_1+m,\,\beta_1-n;\,\alpha_2+k,\,\beta_2-k-n)}(x,y)t^n u^m v^k$$
$$= e^t\ \Phi_1\left[1+\alpha_1+\beta_1,\lambda;1+\alpha_1;u,\frac{1}{2}(x-1)t\right]$$
$$\times\ \Xi_1\left[\mu,1+\alpha_2+\beta_2,\nu;1+\alpha_2;v,\frac{1}{2}(y-1)t\right], \tag{2.23}$$

$$\sum_{m=0}^{\infty}\sum_{k=0}^{\infty}\sum_{n=0}^{\infty}\frac{n!(\lambda)_m(\mu)_m(\nu)_k(1+\alpha_2+\beta_2)_k}{m!k!(1+\alpha_1)_{m+n}(1+\alpha_2)_{k+n}}$$
$$\times P_n^{(\alpha_1+m,\,\beta_1-m-n;\,\alpha_2+k,\,\beta_2-n)}(x,y)t^n u^m v^k$$
$$= e^t\ \Xi_1\left[\lambda,1+\alpha_1+\beta_1,\mu;1+\alpha_1;u,\frac{1}{2}(x-1)t\right]$$
$$\times\ \Phi_1\left[1+\alpha_2+\beta_2,\nu;1+\alpha_2;v,\frac{1}{2}(y-1)t\right], \tag{2.24}$$

$$\sum_{m=0}^{\infty}\sum_{k=0}^{\infty}\sum_{n=0}^{\infty}\frac{n!(\lambda)_m(\mu)_k(1+\alpha_2+\beta_2)_k}{m!k!(1+\alpha_1)_{m+n}(\delta)_k(1+\alpha_2)_n}$$
$$\times P_n^{(\alpha_1+m,\,\beta_1-m-n;\,\alpha_2,\,\beta_2+k-n)}(x,y)t^n u^m v^k$$

$$= e^t \ \Phi_2 \left[\lambda, 1+\alpha_1+\beta_1; 1+\alpha_1; u, \frac{1}{2}(x-1)t\right]$$

$$\times \ \Psi_1 \left[1+\alpha_2+\beta_2, \mu; \delta, 1+\alpha_2; v, \frac{1}{2}(y-1)t\right], \tag{2.25}$$

$$\sum_{m=0}^{\infty}\sum_{k=0}^{\infty}\sum_{n=0}^{\infty} \frac{n!(\lambda)_m(1+\alpha_1+\beta_1)_m(\mu)_k}{m!k!(\delta)_m(1+\alpha_1)_n(1+\alpha_2)_{k+n}}$$

$$\times P_n^{(\alpha_1,\ \beta_1+m-n;\ \alpha_2+k,\ \beta_2-k-n)}(x,y)t^n u^m v^k$$

$$= e^t \ \Psi_1 \left[1+\alpha_1+\beta_1, \lambda; \delta, 1+\alpha_1; u, \frac{1}{2}(x-1)t\right]$$

$$\times \ \Phi_2 \left[\mu, 1+\alpha_2+\beta_2; 1+\alpha_2; v, \frac{1}{2}(y-1)t\right], \tag{2.26}$$

$$\sum_{m=0}^{\infty}\sum_{k=0}^{\infty}\sum_{n=0}^{\infty} \frac{n!(\lambda)_m(1+\alpha_2+\beta_2)_k}{m!k!(\mu)_k(1+\alpha_1)_{m+n}(1+\alpha_2)_n}$$

$$\times P_n^{(\alpha_1+m,\ \beta_1-m-n;\ \alpha_2,\ \beta_2+k-n)}(x,y)t^n u^m v^k$$

$$= e^t \ \Phi_2 \left[\lambda, 1+\alpha_1+\beta_1; 1+\alpha_1; u, \frac{1}{2}(x-1)t\right]$$

$$\times \ \Psi_2 \left[1+\alpha_2+\beta_2; \mu, 1+\alpha_2; v, \frac{1}{2}(y-1)t\right], \tag{2.27}$$

$$\sum_{m=0}^{\infty}\sum_{k=0}^{\infty}\sum_{n=0}^{\infty} \frac{n!(1+\alpha_1+\beta_1)_m(\lambda)_k}{m!k!(\mu)_m(1+\alpha_1)_n(1+\alpha_2)_{k+n}}$$

$$\times P_n^{(\alpha_1,\ \beta_1+m-n;\ \alpha_2+k,\ \beta_2-k-n)}(x,y)t^n u^m v^k$$

$$= e^t \ \Psi_2 \left[1+\alpha_1+\beta_1; \mu, 1+\alpha_1; u, \frac{1}{2}(x-1)t\right]$$

$$\times \ \Phi_2 \left[\lambda, 1+\alpha_2+\beta_2; 1+\alpha_2; v, \frac{1}{2}(y-1)t\right], \tag{2.28}$$

$$\sum_{m=0}^{\infty}\sum_{k=0}^{\infty}\sum_{n=0}^{\infty}\frac{n!(\lambda)_m(\mu)_k(\nu)_k}{m!k!(1+\alpha_1)_{m+n}(1+\alpha_2)_{k+n}}$$
$$\times P_n^{(\alpha_1+m,\,\beta_1-m-n;\,\alpha_2+k,\,\beta_2-k-n)}(x,y)t^n u^m v^k$$
$$=e^t\ \Phi_2\left[\lambda,1+\alpha_1+\beta_1;1+\alpha_1;u,\frac{1}{2}(x-1)t\right]$$
$$\times\ \Xi_1\left[\mu,1+\alpha_2+\beta_2,\nu;1+\alpha_2;v,\frac{1}{2}(y-1)t\right], \tag{2.29}$$

$$\sum_{m=0}^{\infty}\sum_{k=0}^{\infty}\sum_{n=0}^{\infty}\frac{n!(\lambda)_m(\mu)_m(\nu)_k}{m!k!(1+\alpha_1)_{m+n}(1+\alpha_2)_{k+n}}$$
$$\times P_n^{(\alpha_1+m,\,\beta_1-m-n;\,\alpha_2+k,\,\beta_2-k-n)}(x,y)t^n u^m v^k$$
$$=e^t\ \Xi_1\left[\lambda,1+\alpha_1+\beta_1,\mu;1+\alpha_1;u,\frac{1}{2}(x-1)t\right]$$
$$\times\ \Phi_2\left[\nu,1+\alpha_2+\beta_2;1+\alpha_2;v,\frac{1}{2}(y-1)t\right], \tag{2.30}$$

$$\sum_{m=0}^{\infty}\sum_{k=0}^{\infty}\sum_{n=0}^{\infty}\frac{n!(\lambda)_k(1+\alpha_2+\beta_2)_k}{m!k!(\mu)_k(1+\alpha_1)_{m+n}(1+\alpha_2)_n}$$
$$\times P_n^{(\alpha_1+m,\,\beta_1-m-n;\,\alpha_2,\,\beta_2+k-n)}(x,y)t^n u^m v^k$$
$$=e^t\ \Phi_3\left[1+\alpha_1+\beta_1;1+\alpha_1;\frac{1}{2}(x-1)t,u\right]$$
$$\times\ \Psi_1\left[1+\alpha_2+\beta_2,\lambda;\mu,1+\alpha_2;v,\frac{1}{2}(y-1)t\right], \tag{2.31}$$

$$\sum_{m=0}^{\infty}\sum_{k=0}^{\infty}\sum_{n=0}^{\infty}\frac{n!(\lambda)_m(1+\alpha_1+\beta_1)_m}{m!k!(\mu)_m(1+\alpha_1)_n(1+\alpha_2)_{k+n}}$$
$$\times P_n^{(\alpha_1,\,\beta_1+m-n;\,\alpha_2+k,\,\beta_2-k-n)}(x,y)t^n u^m v^k$$

$$= e^t\ \Psi_1\left[1+\alpha_1+\beta_1,\lambda;\mu,1+\alpha_1;u,\frac{1}{2}(x-1)t\right]$$

$$\times\ \Phi_3\left[1+\alpha_2+\beta_2;1+\alpha_2;\frac{1}{2}(y-1)t,v\right], \tag{2.32}$$

$$\sum_{m=0}^{\infty}\sum_{k=0}^{\infty}\sum_{n=0}^{\infty}\frac{n!(1+\alpha_2+\beta_2)_k}{m!k!(1+\alpha_1)_{m+n}(\lambda)_k(1+\alpha_2)_n}$$

$$\times P_n^{(\alpha_1+m,\ \beta_1-m-n;\ \alpha_2,\ \beta_2+k-n)}(x,y)t^n u^m v^k$$

$$= e^t\ \Phi_3\left[1+\alpha_1+\beta_1;1+\alpha_1;\frac{1}{2}(x-1)t,u\right]$$

$$\times\ \Psi_2\left[1+\alpha_2+\beta_2;\lambda,1+\alpha_2;v,\frac{1}{2}(y-1)t\right], \tag{2.33}$$

$$\sum_{m=0}^{\infty}\sum_{k=0}^{\infty}\sum_{n=0}^{\infty}\frac{n!(1+\alpha_1+\beta_1)_m}{m!k!(\lambda)_m(1+\alpha_1)_n(1+\alpha_2)_{k+n}}$$

$$\times P_n^{(\alpha_1,\ \beta_1+m-n;\ \alpha_2+k,\ \beta_2-k-n)}(x,y)t^n u^m v^k$$

$$= e^t\ \Psi_2\left[1+\alpha_1+\beta_1;\lambda,1+\alpha_1;u,\frac{1}{2}(x-1)t\right]$$

$$\times\ \Phi_3\left[1+\alpha_2+\beta_2;1+\alpha_2;\frac{1}{2}(y-1)t,v\right], \tag{2.34}$$

$$\sum_{m=0}^{\infty}\sum_{k=0}^{\infty}\sum_{n=0}^{\infty}\frac{n!(\lambda)_m(1+\alpha_1+\beta_1)_m(1+\alpha_2+\beta_2)_k}{m!k!(\mu)_m(\delta)_k(1+\alpha_1)_n(1+\alpha_2)_n}$$

$$\times P_n^{(\alpha_1,\ \beta_1+m-n;\ \alpha_2,\ \beta_2+k-n)}(x,y)t^n u^m v^k$$

$$= e^t\ \Psi_1\left[1+\alpha_1+\beta_1,\lambda;\mu,1+\alpha_1;u,\frac{1}{2}(x-1)t\right]$$

$$\times\ \Psi_2\left[1+\alpha_2+\beta_2;\delta,1+\alpha_2;v,\frac{1}{2}(y-1)t\right], \tag{2.35}$$

$$\sum_{m=0}^{\infty}\sum_{k=0}^{\infty}\sum_{n=0}^{\infty}\frac{n!(\lambda)_k(1+\alpha_1+\beta_1)_m(1+\alpha_2+\beta_2)_k}{m!k!(\mu)_m(\delta)_k(1+\alpha_1)_n(1+\alpha_2)_n}$$
$$\times P_n^{(\alpha_1,\, \beta_1+m-n;\, \alpha_2,\, \beta_2+k-n)}(x,y)t^n u^m v^k$$
$$= e^t\ \Psi_2\left[1+\alpha_1+\beta_1;\mu,1+\alpha_1;u,\frac{1}{2}(x-1)t\right]$$
$$\times\ \Psi_1\left[1+\alpha_2+\beta_2,\lambda;\delta,1+\alpha_2;v,\frac{1}{2}(y-1)t\right], \tag{2.36}$$

$$\sum_{m=0}^{\infty}\sum_{k=0}^{\infty}\sum_{n=0}^{\infty}\frac{n!(\lambda)_m(\mu)_k(\delta)_k(1+\alpha_1+\beta_1)_m}{m!k!(\nu)_m(1+\alpha_1)_n(1+\alpha_2)_{k+n}}$$
$$\times P_n^{(\alpha_1,\, \beta_1+m-n;\, \alpha_2+k,\, \beta_2-k-n)}(x,y)t^n u^m v^k$$
$$= e^t\ \Psi_1\left[1+\alpha_1+\beta_1,\lambda;\nu,1+\alpha_1;u,\frac{1}{2}(x-1)t\right]$$
$$\times\ \Xi_1\left[\mu,1+\alpha_2+\beta_2,\delta;1+\alpha_2;v,\frac{1}{2}(y-1)t\right], \tag{2.37}$$

$$\sum_{m=0}^{\infty}\sum_{k=0}^{\infty}\sum_{n=0}^{\infty}\frac{n!(\lambda)_m(\mu)_m(\delta)_k(1+\alpha_2+\beta_2)_k}{m!k!(\nu)_k(1+\alpha_1)_{m+n}(1+\alpha_2)_n}$$
$$\times P_n^{(\alpha_1+m,\, \beta_1-m-n;\, \alpha_2,\, \beta_2+k-n)}(x,y)t^n u^m v^k$$
$$= e^t\ \Xi_1\left[\lambda,1+\alpha_1+\beta_1,\mu;1+\alpha_1;u,\frac{1}{2}(x-1)t\right]$$
$$\times\ \Psi_1\left[1+\alpha_2+\beta_2,\delta;\nu,1+\alpha_2;v,\frac{1}{2}(y-1)t\right], \tag{2.38}$$

$$\sum_{m=0}^{\infty}\sum_{k=0}^{\infty}\sum_{n=0}^{\infty}\frac{n!(1+\alpha_1+\beta_1)_m(\lambda)_k(\mu)_k}{m!k!(\nu)_m(1+\alpha_1)_n(1+\alpha_2)_{k+n}}$$
$$\times P_n^{(\alpha_1,\, \beta_1+m-n;\, \alpha_2+k,\, \beta_2-k-n)}(x,y)t^n u^m v^k$$

$$= e^t \ \Psi_2 \left[1+\alpha_1+\beta_1;\nu,1+\alpha_1;u,\frac{1}{2}(x-1)t\right]$$

$$\times \ \Xi_1 \left[\lambda,1+\alpha_2+\beta_2,\mu;1+\alpha_2;v,\frac{1}{2}(y-1)t\right], \tag{2.39}$$

$$\sum_{m=0}^{\infty}\sum_{k=0}^{\infty}\sum_{n=0}^{\infty}\frac{n!(\lambda)_m(\mu)_m(1+\alpha_2+\beta_2)_k}{m!k!(\nu)_k(1+\alpha_1)_{m+n}(1+\alpha_2)_n}$$

$$\times P_n^{(\alpha_1+m,\ \beta_1-m-n;\ \alpha_2,\ \beta_2+k-n)}(x,y)t^n u^m v^k$$

$$= e^t \ \Xi_1 \left[\lambda,1+\alpha_1+\beta_1,\mu;1+\alpha_1;u,\frac{1}{2}(x-1)t\right]$$

$$\times \ \Psi_2 \left[1+\alpha_2+\beta_2;\nu,1+\alpha_2;v,\frac{1}{2}(y-1)t\right]. \tag{2.40}$$

PROOF OF (2.1)

$$\sum_{m=0}^{\infty}\sum_{k=0}^{\infty}\sum_{n=0}^{\infty}\frac{n!(1+\alpha_1+\beta_1)_m(\lambda)_k(1+\alpha_2+\beta_2)_k}{m!k!(1+\alpha_1)_n(1+\alpha_2)_{k+n}}$$

$$\times P_n^{(\alpha_1,\ \beta_1+m-n;\ \alpha_2+k,\ \beta_2-n)}(x,y)t^n u^m v^k$$

$$= \sum_{m=0}^{\infty}\sum_{k=0}^{\infty}\sum_{n=0}^{\infty}\frac{n!(1+\alpha_1+\beta_1)_m(\lambda)_k(1+\alpha_2+\beta_2)_k}{m!k!(1+\alpha_1)_n(1+\alpha_2+k)_n(1+\alpha_2)_k}\times\frac{(1+\alpha_1)_n(1+\alpha_2+k)_n}{(n!)^2}$$

$$\times\times\sum_{r=0}^{n}\sum_{s=0}^{n-r}\frac{(-n)_{r+s}(1+\alpha_1+\beta_1+m)_r(1+\alpha_2+\beta_2+k)_s}{r!s!(1+\alpha_1)_r(1+\alpha_2+k)_s}$$

$$\left(\frac{1-x}{2}\right)^r\left(\frac{1-y}{2}\right)^s t^n u^m v^k$$

$$= \sum_{m=0}^{\infty}\sum_{k=0}^{\infty}\sum_{n=0}^{\infty}\sum_{r=0}^{n}\sum_{s=0}^{n-r}\frac{(-1)^{r+s}n!(1+\alpha_1+\beta_1)_{m+r}(1+\alpha_2+\beta_2)_{k+s}(\lambda)_k}{m!k!n!r!s!(n-r-s)!(1+\alpha_1)_r(1+\alpha_2)_{k+s}}$$

$$\times\left(\frac{1-x}{2}\right)^r\left(\frac{1-y}{2}\right)^s t^n u^m v^k$$

$$= \sum_{m=0}^{\infty}\sum_{k=0}^{\infty}\sum_{n=0}^{\infty}\sum_{r=0}^{\infty}\sum_{s=0}^{\infty} \frac{(1+\alpha_1+\beta_1)_{m+r}(1+\alpha_2+\beta_2)_{k+s}(\lambda)_k}{m!k!n!r!s!(1+\alpha_1)_r(1+\alpha_2)_{k+s}}$$

$$\times \left(\tfrac{x-1}{2}\right)^r \left(\tfrac{y-1}{2}\right)^s t^{n+r+s} u^m v^k$$

$$= e^t \sum_{r=0}^{\infty} \frac{(1+\alpha_1+\beta_1)_r}{r!(1+\alpha_1)_r} \left(\frac{1}{2}(x-1)t\right)^r \sum_{m=0}^{\infty} \frac{(1+\alpha_1+\beta_1+r)_m}{m!} u^m$$

$$\sum_{k=0}^{\infty}\sum_{s=0}^{\infty} \tfrac{(1+\alpha_2+\beta_2)_{k+s}(\lambda)_k}{k!s!(1+\alpha_2)_{k+s}} \times v^k \left(\tfrac{1}{2}(y-1)t\right)^s$$

$$= e^t \; (1-u)^{-1-\alpha_1-\beta_1} \sum_{r=0}^{\infty} \frac{(1+\alpha_1+\beta_1)_r}{r!(1+\alpha_1)_r} \left(\frac{\frac{1}{2}(x-1)t}{(1-u)}\right)^r$$

$$\sum_{k=0}^{\infty}\sum_{s=0}^{\infty} \tfrac{(1+\alpha_2+\beta_2)_{k+s}(\lambda)_k}{k!s!(1+\alpha_2)_{k+s}} v^k \left(\tfrac{1}{2}(y-1)t\right)^s$$

$$= e^t \; (1-u)^{-1-\alpha_1-\beta_1} \; {}_1F_1\left[1+\alpha_1+\beta_1; +\alpha_1; \frac{\frac{1}{2}(x-1)t}{(1-u)}\right]$$

$$\times \Phi_1\left[1+\alpha_2+\beta_2; \lambda; 1+\alpha_2; v, \frac{1}{2}(y-1)t\right]$$

which proves (2.1).

The proof of results (2.2) to (2.8) are similar to that of (2.1).

PROOF OF (2.9)

$$\sum_{m=0}^{\infty}\sum_{k=0}^{\infty}\sum_{n=0}^{\infty} \frac{n!(\lambda)_m(1+\alpha_1+\beta_1)_m(\mu)_k(1+\alpha_2+\beta_2)_k}{m!k!(1+\alpha_1)_{m+n}(1+\alpha_2)_{k+n}}$$

$$\times P_n^{(\alpha_1+m,\, \beta_1-n;\, \alpha_2+k,\, \beta_2-n)}(x,y) t^n u^m v^k$$

$$= \sum_{m=0}^{\infty}\sum_{k=0}^{\infty}\sum_{n=0}^{\infty} \frac{n!(\lambda)_m(1+\alpha_1+\beta_1)_m(\mu)_k(1+\alpha_2+\beta_2)_k}{m!k!(1+\alpha_1+m)_n(1+\alpha_1)_m(1+\alpha_2+k)_n(1+\alpha_2)_k}$$

$$\times \frac{(1+\alpha_1+m)_n(1+\alpha_2+k)_n}{(n!)^2}$$

$$\times \sum_{r=0}^{n} \sum_{s=0}^{n-r} \frac{(-n)_{r+s}(1+\alpha_1+\beta_1+m)_r(1+\alpha_2+\beta_2+k)_s}{r!s!(1+\alpha_1+m)_r(1+\alpha_2+k)_s}$$

$$\times \left(\tfrac{1-x}{2}\right)^r \left(\tfrac{1-y}{2}\right)^s t^n u^m v^k$$

$$= \sum_{m=0}^{\infty} \sum_{k=0}^{\infty} \sum_{n=0}^{\infty} \sum_{r=0}^{n} \sum_{s=0}^{n-r} \frac{(-1)^{r+s} n!(\lambda)_m(1+\alpha_1+\beta_1)_{m+r}(\mu)_k(1+\alpha_2+\beta_2)_{k+s}}{m!k!n!r!s!(n-r-s)!(1+\alpha_1)_{m+r}(1+\alpha_2)_{k+s}}$$

$$\times \left(\frac{1-x}{2}\right)^r \left(\frac{1-y}{2}\right)^s \times t^n u^m v^k$$

$$= \sum_{m=0}^{\infty} \sum_{k=0}^{\infty} \sum_{n=0}^{\infty} \sum_{r=0}^{\infty} \sum_{s=0}^{\infty} \frac{(\lambda)_m(1+\alpha_1+\beta_1)_{m+r}(\mu)_k(1+\alpha_2+\beta_2)_{k+s}}{m!k!n!r!s!(1+\alpha_1)_{m+r}(1+\alpha_2)_{k+s}}$$

$$\times \left(\frac{x-1}{2}\right)^r \left(\frac{y-1}{2}\right)^s t^{n+r+s} u^m v^k$$

$$= e^t \sum_{m=0}^{\infty} \sum_{r=0}^{\infty} \frac{(1+\alpha_1+\beta_1)_{m+r}(\lambda)_m}{m!r!(1+\alpha_1)_{m+r}} u^m \left(\frac{1}{2}(x-1)t\right)^r$$

$$\sum_{k=0}^{\infty} \sum_{s=0}^{\infty} \frac{(1+\alpha_2+\beta_2)_{k+s}(\mu)_k}{k!s!(1+\alpha_2)_{k+s}} v^k \left(\frac{1}{2}(y-1)t\right)^s$$

$$= e^t \ \Phi_1 \left[1+\alpha_1+\beta_1, \lambda; 1+\alpha_1; u, \frac{1}{2}(x-1)t\right]$$

$$\times \Phi_1 \left[1+\alpha_2+\beta_2, \mu; 1+\alpha_2; v, \frac{1}{2}(y-1)t\right]$$

which proves (2.9).

The proof of results (2.10) to (2.40) are similar to that of (2.9).

Further we have the following triple generating functions of Jacobi polynomials of two variables:

$$\sum_{m=0}^{\infty} \sum_{k=0}^{\infty} \sum_{n=0}^{\infty} \frac{n!(1+\alpha_1+\beta_1)_m(1+\alpha_2+\beta_2)_k}{m!k!(1+\alpha_1)_n(1+\alpha_2)_n} P_n^{(\alpha_1,\, \beta_1+m-n;\, \alpha_2,\, \beta_2+k-n)}(x,y) t^n u^m v^k$$

$$= e^t\,(1-u)^{-1-\alpha_1-\beta_1}\,(1-v)^{-1-\alpha_2-\beta_2}\,{}_1F_1\left[1+\alpha_1+\beta_1; 1+\alpha_1; \frac{\frac{1}{2}(x-1)t}{(1-u)}\right]$$

$$\times {}_1F_1\left[1+\alpha_2+\beta_2; 1+\alpha_2; \frac{\frac{1}{2}(y-1)t}{(1-v)}\right], \tag{2.41}$$

$$\sum_{m=0}^{\infty}\sum_{k=0}^{\infty}\sum_{n=0}^{\infty}\frac{n!(1+\alpha_1+\beta_1)_m(1+\alpha_2+\beta_2)_k}{m!k!(1+\alpha_1)_n(1+\alpha_2+k)_n}P_n^{(\alpha_1,\,\beta_1+m-n;\,\alpha_2+k,\,\beta_2-n)}(x,y)t^n u^m v^k$$

$$= \mathrm{e}^{\mathrm{t}}\,(1-\mathrm{u})^{-1-\alpha_1-\beta_1}\,{}_1\mathrm{F}_1\left[1+\alpha_1+\beta_1; 1+\alpha_1; \frac{\frac{1}{2}(\mathrm{x}-1)\mathrm{t}}{(1-\mathrm{u})}\right]$$

$$\times\ \Phi_1\left[1+\alpha_2+\beta_2, 1+\alpha_2; 1+\alpha_2; v, \frac{1}{2}(y-1)t\right], \tag{2.42}$$

$$\sum_{m=0}^{\infty}\sum_{k=0}^{\infty}\sum_{n=0}^{\infty}\frac{n!(1+\alpha_1+\beta_1)_m(1+\alpha_2+\beta_2)_k}{m!k!(1+\alpha_1+m)_n(1+\alpha_2)_n}P_n^{(\alpha_1+m,\,\beta_1-n;\,\alpha_2,\,\beta_2+k-n)}(x,y)t^n u^m v^k$$

$$= e^t\,(1-v)^{-1-\alpha_2-\beta_2}\,{}_1F_1\left[1+\alpha_2+\beta_2; 1+\alpha_2; \frac{\frac{1}{2}(y-1)t}{(1-v)}\right]$$

$$\times \Phi_1\left[1+\alpha_1+\beta_1, 1+\alpha_1; 1+\alpha_1; u, \frac{1}{2}(x-1)t\right], \tag{2.43}$$

$$\sum_{m=0}^{\infty}\sum_{k=0}^{\infty}\sum_{n=0}^{\infty}\frac{n!(1+\alpha_1+\beta_1)_m}{m!k!(1+\alpha_1)_n(1+\alpha_2+k)_n}P_n^{(\alpha_1,\,\beta_1+m-n;\,\alpha_2+k,\,\beta_2-k-n)}(x,y)t^n u^m v^k$$

$$= \mathrm{e}^{\mathrm{t}}\,(1-\mathrm{u})^{-1-\alpha_1-\beta_1}\,{}_1\mathrm{F}_1\left[1+\alpha_1+\beta_1; 1+\alpha_1; \frac{\frac{1}{2}(\mathrm{x}-1)\mathrm{t}}{(1-\mathrm{u})}\right]$$

$$\Phi_2\left[1+\alpha_2, 1+\alpha_2+\beta_2; 1+\alpha_2; v, \frac{1}{2}(y-1)t\right], \tag{2.44}$$

$$\sum_{m=0}^{\infty}\sum_{k=0}^{\infty}\sum_{n=0}^{\infty}\frac{n!(1+\alpha_2+\beta_2)_k}{m!k!(1+\alpha_1+m)_n(1+\alpha_2)_n}P_n^{(\alpha_1+m,\,\beta_1-m-n;\,\alpha_2,\,\beta_2+k-n)}(x,y)t^n u^m v^k$$

$$= \mathrm{e}^{\mathrm{t}}\,(1-\mathrm{v})^{-1-\alpha_2-\beta_2}\,{}_1\mathrm{F}_1\left[1+\alpha_2+\beta_2; 1+\alpha_2; \frac{\frac{1}{2}(\mathrm{y}-1)\mathrm{t}}{(1-\mathrm{v})}\right]$$

$$\times\ \Phi_2\left[1+\alpha_1, 1+\alpha_1+\beta_1; 1+\alpha_1; u, \frac{1}{2}(x-1)t\right], \tag{2.45}$$

$$\sum_{m=0}^{\infty}\sum_{k=0}^{\infty}\sum_{n=0}^{\infty}\frac{n!(1+\alpha_1+\beta_1)_m}{m!k!(1+\alpha_1)_n(1+\alpha_2)_{k+n}}P_n^{(\alpha_1,\,\beta_1+m-n;\,\alpha_2+k,\,\beta_2-k-n)}(x,y)t^n u^m v^k$$

$$= e^t\ (1-u)^{-1-\alpha_1-\beta_1}\ {}_1F_1\left[1+\alpha_1+\beta_1; 1+\alpha_1; \frac{\frac{1}{2}(x-1)t}{(1-u)}\right]$$

$$\times\ \Phi_3\left[1+\alpha_2+\beta_2; 1+\alpha_2; \frac{1}{2}(y-1)t, v\right], \tag{2.46}$$

$$\sum_{m=0}^{\infty}\sum_{k=0}^{\infty}\sum_{n=0}^{\infty}\frac{n!(1+\alpha_2+\beta_2)_k}{m!k!(1+\alpha_1)_{m+n}(1+\alpha_2)_n}P_n^{(\alpha_1+m,\,\beta_1-m-n;\,\alpha_2,\,\beta_2+k-n)}(x,y)t^n u^m v^k$$

$$= e^t\ (1-v)^{-1-\alpha_2-\beta_2}\ {}_1F_1\left[1+\alpha_2+\beta_2; 1+\alpha_2; \frac{\frac{1}{2}(y-1)t}{(1-v)}\right]$$

$$\times\ \Phi_3\left[1+\alpha_1+\beta_1; 1+\alpha_1; \frac{1}{2}(x-1)t, u\right], \tag{2.47}$$

$$\sum_{m=0}^{\infty}\sum_{k=0}^{\infty}\sum_{n=0}^{\infty}\frac{n!(1+\alpha_1+\beta_1)_m(\lambda)_k}{m!k!(1+\alpha_1)_n(1+\alpha_2)_n}P_n^{(\alpha_1,\,\beta_1+m-n;\,\alpha_2,\,\beta_2+k-n)}(x,y)t^n u^m v^k$$

$$= e^t\ (1-u)^{-1-\alpha_1-\beta_1}\ {}_1F_1\left[1+\alpha_1+\beta_1; 1+\alpha_1; \frac{\frac{1}{2}(x-1)t}{(1-u)}\right]$$

$$\times\Psi_1\left[1+\alpha_2+\beta_2, \lambda; 1+\alpha_2+\beta_2, 1+\alpha_2; v, \frac{1}{2}(y-1)t\right], \tag{2.48}$$

$$\sum_{m=0}^{\infty}\sum_{k=0}^{\infty}\sum_{n=0}^{\infty}\frac{n!(\lambda)_m(1+\alpha_2+\beta_2)_k}{m!k!(1+\alpha_1)_n(1+\alpha_2)_n}P_n^{(\alpha_1,\,\beta_1+m-n;\,\alpha_2,\,\beta_2+k-n)}(x,y)t^n u^m v^k$$

$$= e^t\ (1-v)^{-1-\alpha_2-\beta_2}\ {}_1F_1\left[1+\alpha_2+\beta_2; 1+\alpha_2; \frac{\frac{1}{2}(y-1)t}{(1-v)}\right]$$

$$\times\Psi_1\left[1+\alpha_1+\beta_1, \lambda; 1+\alpha_1+\beta_1, 1+\alpha_1; u, \frac{1}{2}(x-1)t\right], \tag{2.49}$$

$$\sum_{m=0}^{\infty}\sum_{k=0}^{\infty}\sum_{n=0}^{\infty}\frac{n!(1+\alpha_1+\beta_1)_m}{m!k!(1+\alpha_1)_n(1+\alpha_2)_n}P_n^{(\alpha_1,\,\beta_1+m-n;\,\alpha_2,\,\beta_2+k-n)}(x,y)t^nu^mv^k$$

$$= e^t\,(1-u)^{-1-\alpha_1-\beta_1}\,{}_1F_1\left[1+\alpha_1+\beta_1;1+\alpha_1;\frac{\frac{1}{2}(x-1)t}{(1-u)}\right]$$

$$\times\Psi_2\left[1+\alpha_2+\beta_2;1+\alpha_2+\beta_2,1+\alpha_2;v,\frac{1}{2}(y-1)t\right], \tag{2.50}$$

$$\sum_{m=0}^{\infty}\sum_{k=0}^{\infty}\sum_{n=0}^{\infty}\frac{n!(1+\alpha_2+\beta_2)_k}{m!k!(1+\alpha_1)_n(1+\alpha_2)_n}P_n^{(\alpha_1,\,\beta_1+m-n;\,\alpha_2,\,\beta_2+k-n)}(x,y)t^nu^mv^k$$

$$= e^t\,(1-v)^{-1-\alpha_2-\beta_2}\,{}_1F_1\left[1+\alpha_2+\beta_2;1+\alpha_2;\frac{\frac{1}{2}(y-1)t}{(1-v)}\right]$$

$$\times\Psi_2\left[1+\alpha_1+\beta_1;1+\alpha_1+\beta_1,1+\alpha_1;u,\frac{1}{2}(x-1)t\right], \tag{2.51}$$

$$\sum_{m=0}^{\infty}\sum_{k=0}^{\infty}\sum_{n=0}^{\infty}\frac{n!(1+\alpha_1+\beta_1)_m(\lambda)_k(\mu)_k}{m!k!(1+\alpha_1)_n(1+\alpha_2)_{k+n}}P_n^{(\alpha_1,\,\beta_1+m-n;\,\alpha_2+k,\,\beta_2-k-n)}(x,y)t^nu^mv^k$$

$$=\mathrm{e}^t\,(1-u)^{-1-\alpha_1-\beta_1}\,{}_1F_1\left[1+\alpha_1+\beta_1;1+\alpha_1;\tfrac{\frac{1}{2}(x-1)t}{(1-u)}\right]$$

$$\times\,\Xi_1\left[\lambda,1+\alpha_2+\beta_2,\mu;1+\alpha_2;v,\frac{1}{2}(y-1)t\right], \tag{2.52}$$

$$\sum_{m=0}^{\infty}\sum_{k=0}^{\infty}\sum_{n=0}^{\infty}\frac{n!(\lambda)_m(\mu)_m(1+\alpha_2+\beta_2)_k}{m!k!(1+\alpha_1)_{m+n}(1+\alpha_2)_n}P_n^{(\alpha_1+m,\,\beta_1-m-n;\,\alpha_2,\,\beta_2+k-n)}(x,y)t^nu^mv^k$$

$$=\mathrm{e}^t\,(1-v)^{-1-\alpha_2-\beta_2}\,{}_1F_1\left[1+\alpha_2+\beta_2;1+\alpha_2;\tfrac{\frac{1}{2}(y-1)t}{(1-v)}\right]$$

$$\Xi_1\left[\lambda,1+\alpha_1+\beta_1,\mu;1+\alpha_1;u,\frac{1}{2}(x-1)t\right], \tag{2.53}$$

$$\sum_{m=0}^{\infty}\sum_{k=0}^{\infty}\sum_{n=0}^{\infty}\frac{n!(1+\alpha_1+\beta_1)_m(1+\alpha_2+\beta_2)_k}{m!k!(1+\alpha_1+m)_n(1+\alpha_2+k)_n}$$

$$P_n^{(\alpha_1+m,\,\beta_1-n;\,\alpha_2+k,\,\beta_2-n)}(x,y)t^nu^mv^k$$

$$=\mathrm{e}^t\Phi_1\left[1+\alpha_1+\beta_1,1+\alpha_1;1+\alpha_1;u,\tfrac{1}{2}(x-1)t\right]$$

$$\times\ \Phi_1\left[1+\alpha_2+\beta_2,1+\alpha_2;1+\alpha_2;v,\frac{1}{2}(y-1)t\right], \tag{2.54}$$

$$\sum_{m=0}^{\infty}\sum_{k=0}^{\infty}\sum_{n=0}^{\infty}\frac{n!}{m!k!(1+\alpha_1+m)_n(1+\alpha_2+k)_n}$$

$$\times P_n^{(\alpha_1+m,\,\beta_1-m-n;\,\alpha_2+k,\,\beta_2-k-n)}(x,y)t^nu^mv^k$$

$$=\mathrm{e}^t\ \Phi_2\left[1+\alpha_1,1+\alpha_1+\beta_1;1+\alpha_1;u,\tfrac{1}{2}(x-1)t\right]$$

$$\times\ \Phi_2\left[1+\alpha_2,1+\alpha_2+\beta_2;1+\alpha_2;v,\frac{1}{2}(y-1)t\right], \tag{2.55}$$

$$\sum_{m=0}^{\infty}\sum_{k=0}^{\infty}\sum_{n=0}^{\infty}\frac{n!}{m!k!(1+\alpha_1)_{m+n}(1+\alpha_2)_{k+n}}$$

$$\mathrm{P}_n^{(\alpha_1+m,\,\beta_1-m-n;\,\alpha_2+k,\,\beta_2-k-n)}(x,y)t^nu^mv^k$$

$$=e^t\ \Phi_3\left[1+\alpha_1+\beta_1;1+\alpha_1;\frac{1}{2}(x-1)t,u\right]$$

$$\Phi_3\left[1+\alpha_2+\beta_2;1+\alpha_2;\frac{1}{2}(y-1)t,v\right], \tag{2.56}$$

$$\sum_{m=0}^{\infty}\sum_{k=0}^{\infty}\sum_{n=0}^{\infty}\frac{n!(\lambda)_m(\mu)_k}{m!k!((1+\alpha_1)_n(1+\alpha_2)_n}P_n^{(\alpha_1,\,\beta_1+m-n;\,\alpha_2,\,\beta_2+k-n)}(x,y)t^nu^mv^k$$

$$=e^t\ \Psi_1\left[1+\alpha_1+\beta_1,\lambda;1+\alpha_1+\beta_1,1+\alpha_1;u,\frac{1}{2}(x-1)t\right]$$

$$\times\Psi_1\left[1+\alpha_2+\beta_2,\mu;1+\alpha_2+\beta_2,1+\alpha_2;v,\tfrac{1}{2}(y-1)t\right],\quad(2.57)$$

$$\sum_{m=0}^{\infty}\sum_{k=0}^{\infty}\sum_{n=0}^{\infty}\frac{n!}{m!k!(1+\alpha_1)_n(1+\alpha_2)_n}P_n^{(\alpha_1,\,\beta_1+m-n;\,\alpha_2,\,\beta_2+k-n)}(x,y)t^nu^mv^k$$

$$=e^t\;\Psi_2\left[1+\alpha_1+\beta_1;1+\alpha_1+\beta_1,1+\alpha_1;u,\frac{1}{2}(x-1)t\right]$$

$$\times\Psi_2\left[1+\alpha_2+\beta_2;1+\alpha_2+\beta_2,1+\alpha_2;v,\frac{1}{2}(y-1)t\right],\tag{2.58}$$

$$\sum_{m=0}^{\infty}\sum_{k=0}^{\infty}\sum_{n=0}^{\infty}\frac{n!(\lambda)_m(\mu)_m(\delta)_k(\nu)_k}{m!k!(1+\alpha_1)_{m+n}(1+\alpha_2)_{k+n}}P_n^{(\alpha_1+m,\,\beta_1-m-n;\,\alpha_2+k,\,\beta_2-k-n)}(x,y)t^nu^mv^k$$

$$=e^t\;\Xi_1\left[\lambda,1+\alpha_1+\beta_1,\mu;1+\alpha_1;u,\frac{1}{2}(x-1)t\right]$$

$$\times\;\Xi_1\left[\delta,1+\alpha_2+\beta_2,\nu;1+\alpha_2;v,\frac{1}{2}(y-1)t\right],\tag{2.59}$$

$$\sum_{m=0}^{\infty}\sum_{k=0}^{\infty}\sum_{n=0}^{\infty}\frac{n!(1+\alpha_1+\beta_1)_m}{m!k!(1+\alpha_1+m)_n(1+\alpha_2+k)_n}P_n^{(\alpha_1+m,\,\beta_1-n;\,\alpha_2+k,\,\beta_2-k-n)}(x,y)t^nu^mv^k$$

$$=e^t\;\Phi_1\left[1+\alpha_1+\beta_1,1+\alpha_1;1+\alpha_1;u,\frac{1}{2}(x-1)t\right]$$

$$\times\;\Phi_2\left[1+\alpha_2,1+\alpha_2+\beta_2;1+\alpha_2;v,\frac{1}{2}(y-1)t\right],\tag{2.60}$$

$$\sum_{m=0}^{\infty}\sum_{k=0}^{\infty}\sum_{n=0}^{\infty}\frac{n!(1+\alpha_2+\beta_2)_k}{m!k!(1+\alpha_1+m)_n(1+\alpha_2+k)_n}P_n^{(\alpha_1+m,\,\beta_1-m-n;\,\alpha_2+k,\,\beta_2-n)}(x,y)t^nu^mv^k$$

$$=e^t\;\Phi_2\left[1+\alpha_1,1+\alpha_1+\beta_1;1+\alpha_1;u,\frac{1}{2}(x-1)t\right]$$

$$\times\;\Phi_1\left[1+\alpha_2+\beta_2,1+\alpha_2;1+\alpha_2;v,\frac{1}{2}(y-1)t\right],\tag{2.61}$$

$$\sum_{m=0}^{\infty}\sum_{k=0}^{\infty}\sum_{n=0}^{\infty}\frac{n!(1+\alpha_1+\beta_1)_m}{m!k!(1+\alpha_1+m)_n(1+\alpha_2)_{k+n}}P_n^{(\alpha_1+m,\,\beta_1-n;\,\alpha_2+k,\,\beta_2-k-n)}(x,y)t^n u^m v^k$$

$$= e^t\ \Phi_1\left[1+\alpha_1+\beta_1, 1+\alpha_1; 1+\alpha_1; u, \frac{1}{2}(x-1)t\right]$$

$$\times\ \Phi_3\left[1+\alpha_2+\beta_2; 1+\alpha_2; \frac{1}{2}(y-1)t, v\right], \tag{2.62}$$

$$\sum_{m=0}^{\infty}\sum_{k=0}^{\infty}\sum_{n=0}^{\infty}\frac{n!(1+\alpha_2+\beta_2)_k}{m!k!(1+\alpha_1)_{m+n}(1+\alpha_2+k)_n}$$

$$\mathrm{P}_n^{(\alpha_1+m,\,\beta_1-m-n;\,\alpha_2+k,\,\beta_2-n)}(x,y)t^n u^m v^k$$

$$= e^t\Phi_3\left[1+\alpha_1+\beta_1; 1+\alpha_1; \frac{1}{2}(x-1)t, u\right]$$

$$\times\ \Phi_1\left[1+\alpha_2+\beta_2, 1+\alpha_2; 1+\alpha_2; v, \frac{1}{2}(y-1)t\right], \tag{2.63}$$

$$\sum_{m=0}^{\infty}\sum_{k=0}^{\infty}\sum_{n=0}^{\infty}\frac{n!}{m!k!(1+\alpha_1+m)_n(1+\alpha_2)_{k+n}}$$

$$\mathrm{P}_n^{(\alpha_1+m,\,\beta_1-m-n;\,\alpha_2+k,\,\beta_2-k-n)}(x,y)t^n u^m v^k$$

$$= e^t\Phi_2\left[1+\alpha_1, 1+\alpha_1+\beta_1; 1+\alpha_1; u, \frac{1}{2}(x-1)t\right]$$

$$\times\Phi_3\left[1+\alpha_2+\beta_2; 1+\alpha_2; \frac{1}{2}(y-1)t, v\right], \tag{2.64}$$

$$\sum_{m=0}^{\infty}\sum_{k=0}^{\infty}\sum_{n=0}^{\infty}\frac{n!}{m!k!(1+\alpha_1)_{m+n}(1+\alpha_2+k)_n}$$

$$\mathrm{P}_n^{(\alpha_1+m,\,\beta_1-m-n;\,\alpha_2+k,\,\beta_2-k-n)}(x,y)t^n u^m v^k$$

$$= e^t \ \Phi_3 \left[1+\alpha_1+\beta_1; 1+\alpha_1; \frac{1}{2}(x-1)t, u\right]$$

$$\times \ \Phi_2 \left[1+\alpha_2, 1+\alpha_2+\beta_2; 1+\alpha_2; v, \frac{1}{2}(y-1)t\right], \tag{2.65}$$

$$\sum_{m=0}^{\infty}\sum_{k=0}^{\infty}\sum_{n=0}^{\infty} \frac{n!(1+\alpha_1+\beta_1)_m(\lambda)_k}{m!k!(1+\alpha_1+m)_n(1+\alpha_2)_n} P_n^{(\alpha_1+m,\, \beta_1-n;\, \alpha_2,\, \beta_2+k-n)}(x,y)t^n u^m v^k$$

$$= e^t \ \Phi_1 \left[1+\alpha_1+\beta_1, 1+\alpha_1; 1+\alpha_1; u, \frac{1}{2}(x-1)t\right]$$

$$\times \Psi_1 \left[1+\alpha_2+\beta_2, \lambda; 1+\alpha_2+\beta_2, 1+\alpha_2; v, \frac{1}{2}(y-1)t\right], \tag{2.66}$$

$$\sum_{m=0}^{\infty}\sum_{k=0}^{\infty}\sum_{n=0}^{\infty} \frac{n!(\lambda)_m(1+\alpha_2+\beta_2)_k}{m!k!(1+\alpha_1)_n(1+\alpha_2+k)_n} P_n^{(\alpha_1,\, \beta_1+m-n;\, \alpha_2+k,\, \beta_2-n)}(x,y)t^n u^m v^k$$

$$= e^t \ \Psi_1 \left[1+\alpha_1+\beta_1, \lambda; 1+\alpha_1+\beta_1, 1+\alpha_1; u, \frac{1}{2}(x-1)t\right]$$

$$\times \Phi_1 \left[1+\alpha_2+\beta_2, 1+\alpha_2; 1+\alpha_2; v, \frac{1}{2}(y-1)t\right], \tag{2.67}$$

$$\sum_{m=0}^{\infty}\sum_{k=0}^{\infty}\sum_{n=0}^{\infty} \frac{n!(1+\alpha_1+\beta_1)_m}{m!k!(1+\alpha_1+m)_n(1+\alpha_2)_n} P_n^{(\alpha_1+m,\, \beta_1-n;\, \alpha_2,\, \beta_2+k-n)}(x,y)t^n u^m v^k$$

$$= e^t \ \Phi_1 \left[1+\alpha_1+\beta_1, 1+\alpha_1; 1+\alpha_1; u, \frac{1}{2}(x-1)t\right]$$

$$\times \Psi_2 \left[1+\alpha_2+\beta_2; 1+\alpha_2+\beta_2, 1+\alpha_2; v, \frac{1}{2}(y-1)t\right], \tag{2.68}$$

$$\sum_{m=0}^{\infty}\sum_{k=0}^{\infty}\sum_{n=0}^{\infty} \frac{n!(1+\alpha_2+\beta_2)_k}{m!k!(1+\alpha_1)_n(1+\alpha_2+k)_n} P_n^{(\alpha_1,\, \beta_1+m-n;\, \alpha_2+k,\, \beta_2-n)}(x,y)t^n u^m v^k$$

$$= e^t\ \Psi_2\left[1+\alpha_1+\beta_1; 1+\alpha_1+\beta_1, 1+\alpha_1; u, \frac{1}{2}(x-1)t\right]$$

$$\times\Phi_1\left[1+\alpha_2+\beta_2, 1+\alpha_2; 1+\alpha_2; v, \frac{1}{2}(y-1)t\right], \tag{2.69}$$

$$\sum_{m=0}^{\infty}\sum_{k=0}^{\infty}\sum_{n=0}^{\infty}\frac{n!(1+\alpha_1+\beta_1)_m(\lambda)_k(\mu)_k}{m!k!(1+\alpha_1+m)_n(1+\alpha_2)_{k+n}}$$

$$P_n^{(\alpha_1+m,\ \beta_1-n;\ \alpha_2+k,\ \beta_2-k-n)}(x,y)t^n u^m v^k$$

$$=\mathrm{e}^t\ \Phi_1\left[1+\alpha_1+\beta_1, 1+\alpha_1; 1+\alpha_1; u, \tfrac{1}{2}(x-1)t\right]$$

$$\times\ \Xi_1\left[\lambda, 1+\alpha_2+\beta_2, \mu; 1+\alpha_2; v, \frac{1}{2}(y-1)t\right], \tag{2.70}$$

$$\sum_{m=0}^{\infty}\sum_{k=0}^{\infty}\sum_{n=0}^{\infty}\frac{n!(\lambda)_m(\mu)_m(1+\alpha_2+\beta_2)_k}{m!k!(1+\alpha_1)_{m+n}(1+\alpha_2+k)_n}$$

$$P_n^{(\alpha_1+m,\ \beta_1-m-n;\ \alpha_2+k,\ \beta_2-n)}(x,y)t^n u^m v^k$$

$$= e^t\ \Xi_1\left[\lambda, 1+\alpha_1+\beta_1, \mu; 1+\alpha_1; u, \frac{1}{2}(x-1)t\right]$$

$$\times\Phi_1\left[1+\alpha_2+\beta_2, 1+\alpha_2; 1+\alpha_2; v, \frac{1}{2}(y-1)t\right], \tag{2.71}$$

$$\sum_{m=0}^{\infty}\sum_{k=0}^{\infty}\sum_{n=0}^{\infty}\frac{n!(\lambda)_k}{m!k!(1+\alpha_1+m)_n(1+\alpha_2)_n}P_n^{(\alpha_1+m,\ \beta_1-m-n;\ \alpha_2,\ \beta_2+k-n)}(x,y)t^n u^m v^k$$

$$= e^t\ \Phi_2\left[1+\alpha_1, 1+\alpha_1+\beta_1; 1+\alpha_1; u, \frac{1}{2}(x-1)t\right]$$

$$\times\Psi_1\left[1+\alpha_2+\beta_2, \lambda; 1+\alpha_2+\beta_2, 1+\alpha_2; v, \frac{1}{2}(y-1)t\right], \tag{2.72}$$

$$\sum_{m=0}^{\infty}\sum_{k=0}^{\infty}\sum_{n=0}^{\infty}\frac{n!(\lambda)_m}{m!k!(1+\alpha_1)_n(1+\alpha_2+k)_n}P_n^{(\alpha_1,\,\beta_1+m-n;\,\alpha_2+k,\,\beta_2-k-n)}(x,y)t^n u^m v^k$$
$$= e^t\ \Psi_1\left[1+\alpha_1+\beta_1,\lambda;1+\alpha_1+\beta_1,1+\alpha_1;u,\frac{1}{2}(x-1)t\right]$$
$$\times\Phi_2\left[1+\alpha_2,1+\alpha_2+\beta_2;1+\alpha_2;v,\frac{1}{2}(y-1)t\right], \tag{2.73}$$

$$\sum_{m=0}^{\infty}\sum_{k=0}^{\infty}\sum_{n=0}^{\infty}\frac{n!}{m!k!(1+\alpha_1+m)_n(1+\alpha_2)_n}P_n^{(\alpha_1+m,\,\beta_1-m-n;\,\alpha_2,\,\beta_2+k-n)}(x,y)t^n u^m v^k$$
$$= e^t\ \Phi_2\left[1+\alpha_1,1+\alpha_1+\beta_1;1+\alpha_1;u,\frac{1}{2}(x-1)t\right]$$
$$\times\Psi_2\left[1+\alpha_2+\beta_2;1+\alpha_2+\beta_2,1+\alpha_2;v,\frac{1}{2}(y-1)t\right], \tag{2.74}$$

$$\sum_{m=0}^{\infty}\sum_{k=0}^{\infty}\sum_{n=0}^{\infty}\frac{n!}{m!k!(1+\alpha_1)_n(1+\alpha_2+k)_n}P_n^{(\alpha_1,\,\beta_1+m-n;\,\alpha_2+k,\,\beta_2-k-n)}(x,y)t^n u^m v^k$$
$$= e^t\ \Psi_2\left[1+\alpha_1+\beta_1;1+\alpha_1+\beta_1,1+\alpha_1;u,\frac{1}{2}(x-1)t\right]$$
$$\times\Phi_2\left[1+\alpha_2,1+\alpha_2+\beta_2;1+\alpha_2;v,\frac{1}{2}(y-1)t\right], \tag{2.75}$$

$$\sum_{m=0}^{\infty}\sum_{k=0}^{\infty}\sum_{n=0}^{\infty}\frac{n!(\lambda)_k(\mu)_k}{m!k!(1+\alpha_1+m)_n(1+\alpha_2)_{k+n}}P_n^{(\alpha_1+m,\,\beta_1-m-n;\,\alpha_2+k,\,\beta_2-k-n)}(x,y)t^n u^m v^k$$
$$= e^t\ \Phi_2\left[1+\alpha_1,1+\alpha_1+\beta_1;1+\alpha_1;u,\frac{1}{2}(x-1)t\right]$$
$$\Xi_1\left[\lambda,1+\alpha_2+\beta_2,\mu;1+\alpha_2;v,\frac{1}{2}(y-1)t\right], \tag{2.76}$$

$$\sum_{m=0}^{\infty}\sum_{k=0}^{\infty}\sum_{n=0}^{\infty}\frac{n!(\lambda)_m(\mu)_m}{m!k!(1+\alpha_1)_{m+n}(1+\alpha_2+k)_n}P_n^{(\alpha_1+m,\,\beta_1-m-n;\,\alpha_2+k,\,\beta_2-k-n)}(x,y)t^nu^mv^k$$

$$= e^t\ \Xi_1\left[\lambda,1+\alpha_1+\beta_1,\mu;1+\alpha_1;u,\frac{1}{2}(x-1)t\right]$$

$$\Phi_2\left[1+\alpha_2,1+\alpha_2+\beta_2;1+\alpha_2;v,\frac{1}{2}(y-1)t\right], \tag{2.77}$$

$$\sum_{m=0}^{\infty}\sum_{k=0}^{\infty}\sum_{n=0}^{\infty}\frac{n!(\lambda)_k}{m!k!(1+\alpha_1)_{m+n}(1+\alpha_2)_n}P_n^{(\alpha_1+m,\,\beta_1-m-n;\,\alpha_2,\,\beta_2+k-n)}(x,y)t^nu^mv^k$$

$$= e^t\ \Phi_3\left[1+\alpha_1+\beta_1;1+\alpha_1;\frac{1}{2}(x-1)t,u\right]$$

$$\Psi_1\left[1+\alpha_2+\beta_2,\lambda;1+\alpha_2+\beta_2,1+\alpha_2;v,\frac{1}{2}(y-1)t\right], \tag{2.78}$$

$$\sum_{m=0}^{\infty}\sum_{k=0}^{\infty}\sum_{n=0}^{\infty}\frac{n!(\lambda)_m}{m!k!(1+\alpha_1)_n(1+\alpha_2)_{k+n}}P_n^{(\alpha_1,\,\beta_1+m-n;\,\alpha_2+k,\,\beta_2-k-n)}(x,y)t^nu^mv^k$$

$$= e^t\ \Psi_1\left[1+\alpha_1+\beta_1,\lambda;1+\alpha_1+\beta_1,1+\alpha_1;u,\frac{1}{2}(x-1)t\right]$$

$$\Phi_3\left[1+\alpha_2+\beta_2;1+\alpha_2;\frac{1}{2}(y-1)t,v\right], \tag{2.79}$$

$$\sum_{m=0}^{\infty}\sum_{k=0}^{\infty}\sum_{n=0}^{\infty}\frac{n!}{m!k!(1+\alpha_1)_{m+n}(1+\alpha_2)_n}P_n^{(\alpha_1+m,\,\beta_1-m-n;\,\alpha_2,\,\beta_2+k-n)}(x,y)t^nu^mv^k$$

$$= e^t\ \Phi_3\left[1+\alpha_1+\beta_1;1+\alpha_1;\frac{1}{2}(x-1)t,u\right]$$

$$\times\ \Psi_2\left[1+\alpha_2+\beta_2;1+\alpha_2+\beta_2,1+\alpha_2;v,\frac{1}{2}(y-1)t\right], \tag{2.80}$$

$$\sum_{m=0}^{\infty}\sum_{k=0}^{\infty}\sum_{n=0}^{\infty}\frac{n!}{m!k!(1+\alpha_1)_n(1+\alpha_2)_{k+n}}P_n^{(\alpha_1,\,\beta_1+m-n;\,\alpha_2+k,\,\beta_2-k-n)}(x,y)t^nu^mv^k$$

$$= e^t\ \Psi_2\left[1+\alpha_1+\beta_1;1+\alpha_1+\beta_1,1+\alpha_1;u,\frac{1}{2}(x-1)t\right]$$

$$\times\ \Phi_3\left[1+\alpha_2+\beta_2;1+\alpha_2;\frac{1}{2}(y-1)t,v\right], \tag{2.81}$$

$$\sum_{m=0}^{\infty}\sum_{k=0}^{\infty}\sum_{n=0}^{\infty}\frac{n!(\lambda)_k(\mu)_k}{m!k!(1+\alpha_1)_{m+n}(1+\alpha_2)_{k+n}}P_n^{(\alpha_1+m,\,\beta_1-m-n;\,\alpha_2+k,\,\beta_2-k-n)}(x,y)t^nu^mv^k$$

$$= e^t\ \Phi_3\left[1+\alpha_1+\beta_1;1+\alpha_1;\frac{1}{2}(x-1)t,u\right]$$

$$\Xi_1\left[\lambda,1+\alpha_2+\beta_2,\mu;1+\alpha_2;v,\frac{1}{2}(y-1)t\right], \tag{2.82}$$

$$\sum_{m=0}^{\infty}\sum_{k=0}^{\infty}\sum_{n=0}^{\infty}\frac{n!(\lambda)_m(\mu)_m}{m!k!(1+\alpha_1)_{m+n}(1+\alpha_2)_{k+n}}P_n^{(\alpha_1+m,\,\beta_1-m-n;\,\alpha_2+k,\,\beta_2-k-n)}(x,y)t^nu^mv^k$$

$$= e^t\ \Xi_1\left[\lambda,1+\alpha_1+\beta_1,\mu;1+\alpha_1;u,\frac{1}{2}(x-1)t\right]$$

$$\Phi_3\left[1+\alpha_2+\beta_2;1+\alpha_2;\frac{1}{2}(y-1)t,v\right], \tag{2.83}$$

$$\sum_{m=0}^{\infty}\sum_{k=0}^{\infty}\sum_{n=0}^{\infty}\frac{n!(\lambda)_m}{m!k!((1+\alpha_1)_n(1+\alpha_2)_n}P_n^{(\alpha_1,\,\beta_1+m-n;\,\alpha_2,\,\beta_2+k-n)}(x,y)t^nu^mv^k$$

$$= e^t\ \Psi_1\left[1+\alpha_1+\beta_1,\lambda;1+\alpha_1+\beta_1,1+\alpha_1;u,\frac{1}{2}(x-1)t\right]$$

$$\times\Psi_2\left[1+\alpha_2+\beta_2;1+\alpha_2+\beta_2,1+\alpha_2;v,\frac{1}{2}(y-1)t\right], \tag{2.84}$$

$$\sum_{m=0}^{\infty}\sum_{k=0}^{\infty}\sum_{n=0}^{\infty}\frac{n!(\lambda)_k}{m!k!(1+\alpha_1)_n(1+\alpha_2)_n}P_n^{(\alpha_1,\,\beta_1+m-n;\,\alpha_2,\,\beta_2+k-n)}(x,y)t^nu^mv^k$$

$$= e^t\,\Psi_2\left[1+\alpha_1+\beta_1;1+\alpha_1+\beta_1,1+\alpha_1;u,\frac{1}{2}(x-1)t\right]$$

$$\times\Psi_1\left[1+\alpha_2+\beta_2,\lambda;1+\alpha_2+\beta_2,1+\alpha_2;v,\frac{1}{2}(y-1)t\right], \tag{2.85}$$

$$\sum_{m=0}^{\infty}\sum_{k=0}^{\infty}\sum_{n=0}^{\infty}\frac{n!(\lambda)_m(\mu)_k(\delta)_k}{m!k!(1+\alpha_1)_n(1+\alpha_2)_{k+n}}P_n^{(\alpha_1,\,\beta_1+m-n;\,\alpha_2+k,\,\beta_2-k-n)}(x,y)t^nu^mv^k$$

$$= e^t\;\Psi_1\left[1+\alpha_1+\beta_1,\lambda;1+\alpha_1+\beta_1,1+\alpha_1;u,\frac{1}{2}(x-1)t\right]$$

$$\times\Xi_1\left[\mu,1+\alpha_2+\beta_2,\delta;1+\alpha_2;v,\frac{1}{2}(y-1)t\right], \tag{2.86}$$

$$\sum_{m=0}^{\infty}\sum_{k=0}^{\infty}\sum_{n=0}^{\infty}\frac{n!(\lambda)_m(\mu)_m(\delta)_k}{m!k!(1+\alpha_1)_{m+n}(1+\alpha_2)_n}P_n^{(\alpha_1+m,\,\beta_1-m-n;\,\alpha_2,\,\beta_2+k-n)}(x,y)t^nu^mv^k$$

$$= e^t\;\Xi_1\left[\lambda,1+\alpha_1+\beta_1,\mu;1+\alpha_1;u,\frac{1}{2}(x-1)t\right]$$

$$\times\Psi_1\left[1+\alpha_2+\beta_2,\delta;1+\alpha_2+\beta_2,1+\alpha_2;v,\frac{1}{2}(y-1)t\right], \tag{2.87}$$

$$\sum_{m=0}^{\infty}\sum_{k=0}^{\infty}\sum_{n=0}^{\infty}\frac{n!(\lambda)_k(\mu)_k}{m!k!(1+\alpha_1)_n(1+\alpha_2)_{k+n}}P_n^{(\alpha_1,\,\beta_1+m-n;\,\alpha_2+k,\,\beta_2-k-n)}(x,y)t^nu^mv^k$$

$$= e^t\;\Psi_2\left[1+\alpha_1+\beta_1;1+\alpha_1+\beta_1,1+\alpha_1;u,\frac{1}{2}(x-1)t\right]$$

$$\times\;\Xi_1\left[\lambda,1+\alpha_2+\beta_2,\mu;1+\alpha_2;v,\frac{1}{2}(y-1)t\right], \tag{2.88}$$

$$\sum_{m=0}^{\infty}\sum_{k=0}^{\infty}\sum_{n=0}^{\infty}\frac{n!(\lambda)_m(\mu)_m}{m!k!(1+\alpha_1)_{m+n}(1+\alpha_2)_n}P_n^{(\alpha_1+m,\,\beta_1-m-n;\,\alpha_2,\,\beta_2+k-n)}(x,y)t^n u^m v^k$$

$$= e^t\ \Xi_1\left[\lambda, 1+\alpha_1+\beta_1, \mu; 1+\alpha_1; u, \frac{1}{2}(x-1)t\right]$$

$$\times\Psi_2\left[1+\alpha_2+\beta_2; 1+\alpha_2+\beta_2, 1+\alpha_2; v, \frac{1}{2}(y-1)t\right]. \tag{2.89}$$

PROOF OF (2.41)

$$\sum_{m=0}^{\infty}\sum_{k=0}^{\infty}\sum_{n=0}^{\infty}\frac{n!(1+\alpha_1+\beta_1)_m(1+\alpha_2+\beta_2)_k}{m!k!(1+\alpha_1)_n(1+\alpha_2)_n}P_n^{(\alpha_1,\beta_1+m-n;\alpha_2,\beta_2+k-n)}(x,y)t^n u^m v^k$$

$$= \sum_{m=0}^{\infty}\sum_{k=0}^{\infty}\sum_{n=0}^{\infty}\frac{n!(1+\alpha_1+\beta_1)_m(1+\alpha_2+\beta_2)_k}{m!k!(1+\alpha_1)_n(1+\alpha_2)_n}\times\frac{(1+\alpha_1)_n(1+\alpha_2)_n}{(n!)^2}$$

$$\times\sum_{r=0}^{n}\sum_{s=0}^{n-r}\frac{(-n)_{r+s}(1+\alpha_1+\beta_1+m)_r(1+\alpha_2+\beta_2+k)_s}{r!s!(1+\alpha_1)_r(1+\alpha_2)_s}$$

$$\times\left(\frac{1-x}{2}\right)^r\left(\frac{1-y}{2}\right)^s t^n u^m v^k$$

$$= \sum_{m=0}^{\infty}\sum_{k=0}^{\infty}\sum_{n=0}^{\infty}\sum_{r=0}^{n}\sum_{s=0}^{n-r}\frac{(-1)^{r+s}n!(1+\alpha_1+\beta_1)_{m+r}(1+\alpha_2+\beta_2)_{k+s}}{m!k!n!r!s!(n-r-s)!(1+\alpha_1)_r(1+\alpha_2)_s}$$

$$\times\left(\frac{1-x}{2}\right)^r\left(\frac{1-y}{2}\right)^s t^n u^m v^k$$

$$= \sum_{m=0}^{\infty}\sum_{k=0}^{\infty}\sum_{n=0}^{\infty}\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\frac{(1+\alpha_1+\beta_1)_{m+r}(1+\alpha_2+\beta_2)_{k+s}}{m!k!n!r!s!(1+\alpha_1)_r(1+\alpha_2)_s}$$

$$\times\left(\frac{x-1}{2}\right)^r\left(\frac{y-1}{2}\right)^s t^{n+r+s}u^m v^k$$

$$= e^t\ \sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\frac{(1+\alpha_1+\beta_1)_r(1+\alpha_2+\beta_2)_s}{r!s!(1+\alpha_1)_r(1+\alpha_2)_s}\left(\frac{1}{2}(x-1)t\right)^r\left(\frac{1}{2}(y-1)t\right)^s$$

$$\times \sum_{m=0}^{\infty} \frac{(1+\alpha_1+\beta_1+r)_m}{m!} u^m \sum_{k=0}^{\infty} \frac{(1+\alpha_2+\beta_2+s)_k}{k!} v^k$$

$$= e^t \ (1-u)^{-1-\alpha_1-\beta_1} \ (1-v)^{-1-\alpha_2-\beta_2} \sum_{r=0}^{\infty} \frac{(1+\alpha_1+\beta_1)_r}{r!(1+\alpha_1)_r} \left(\frac{\frac{1}{2}(x-1)t}{(1-u)} \right)^r$$

$$\times \sum_{s=0}^{\infty} \frac{(1+\alpha_2+\beta_2)_s}{s!(1+\alpha_2)_s} \left(\frac{\frac{1}{2}(y-1)t}{(1-v)} \right)^s$$

$$= e^t \ (1-u)^{-1-\alpha_1-\beta_1} \ (1-v)^{-1-\alpha_2-\beta_2} \ {}_1F_1 \left[1+\alpha_1+\beta_1; 1+\alpha_1; \frac{\frac{1}{2}(x-1)t}{(1-u)} \right]$$

$$\times \ {}_1F_1 \left[1+\alpha_2+\beta_2; 1+\alpha_2; \frac{\frac{1}{2}(y-1)t}{(1-v)} \right]$$

which proves (2.41).

The proof of results (2.42) to (2.53) are similar to that of (2.41).

PROOF OF (2.54) :

$$\sum_{m=0}^{\infty} \sum_{k=0}^{\infty} \sum_{n=0}^{\infty} \frac{n!(1+\alpha_1+\beta_1)_m (1+\alpha_2+\beta_2)_k}{m!k!(1+\alpha_1+m)_n (1+\alpha_2+k)_n} P_n^{(\alpha_1+m,\beta_1-n;\alpha_2+k,\beta_2-n)}(x,y) t^n u^m v^k$$

$$= \sum_{m=0}^{\infty} \sum_{k=0}^{\infty} \sum_{n=0}^{\infty} \frac{n!(1+\alpha_1+\beta_1)_m (1+\alpha_2+\beta_2)_k}{m!k!(1+\alpha_1+m)_n (1+\alpha_2+k)_n} \times \frac{(1+\alpha_1+m)_n (1+\alpha_2+k)_n}{(n!)^2}$$

$$\times \sum_{r=0}^{n} \sum_{s=0}^{n-r} \frac{(-n)_{r+s} (1+\alpha_1+\beta_1+m)_r (1+\alpha_2+\beta_2+k)_s}{r!s!(1+\alpha_1+m)_r (1+\alpha_2+k)_s}$$

$$\times \left(\tfrac{1-x}{2}\right)^r \left(\tfrac{1-y}{2}\right)^s t^n u^m v^k$$

$$= \sum_{m=0}^{\infty} \sum_{k=0}^{\infty} \sum_{n=0}^{\infty} \sum_{r=0}^{n} \sum_{s=0}^{n-r} \frac{(-1)^{r+s} n! (1+\alpha_1+\beta_1)_{m+r} (1+\alpha_2+\beta_2)_{k+s} (1+\alpha_1)_m (1+\alpha_2)_k}{m!k!n!r!s!(n-r-s)!(1+\alpha_1)_{m+r} (1+\alpha_2)_{k+s}}$$

$$\left(\frac{1-x}{2} \right)^r$$

$$\times \left(\frac{1-y}{2}\right)^s t^n u^m v^k$$

$$= \sum_{m=0}^{\infty}\sum_{k=0}^{\infty}\sum_{n=0}^{\infty}\sum_{r=0}^{\infty}\sum_{s=0}^{\infty} \frac{(1+\alpha_1+\beta_1)_{m+r}(1+\alpha_2+\beta_2)_{k+s}(1+\alpha_1)_m(1+\alpha_2)_k}{m!k!n!r!s!(1+\alpha_1)_{m+r}(1+\alpha_2)_{k+s}}$$

$$\left(\tfrac{x-1}{2}\right)^r \left(\tfrac{y-1}{2}\right)^s \times\ t^{n+r+s}u^m v^k$$

$$= \mathrm{e}^t \sum_{m=0}^{\infty}\sum_{r=0}^{\infty} \tfrac{(1+\alpha_1+\beta_1)_{m+r}(1+\alpha_1)_m}{m!r!(1+\alpha_1)_{m+r}} u^m \left(\tfrac{1}{2}(x-1)t\right)^r$$

$$\sum_{k=0}^{\infty}\sum_{s=0}^{\infty} \tfrac{(1+\alpha_2+\beta_2)_{k+s}(1+\alpha_2)_k}{k!s!(1+\alpha_2)_{k+s}} \times\ v^k \left(\tfrac{1}{2}(y-1)t\right)^s$$

$$= \mathrm{e}^t \Phi_1\left[1+\alpha_1+\beta_1, 1+\alpha_1; 1+\alpha_1; u, \tfrac{1}{2}(x-1)t\right]$$

$$\Phi_1\left[1+\alpha_2+\beta_2, 1+\alpha_2; 1+\alpha_2; v, \tfrac{1}{2}(y-1)t\right]$$

which proves (2.54).

The proof of results (2.55) to (2.89) are similar to that of (2.54).

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