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Partial Actions and Quotient Rings *

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Abstract

In this paper we study the Martindale ring of $\alpha-$quotients $\mathcal{Q}$ associated with the partial action (R, α). Among other results we extend the partial action to $\mathcal{Q}$ and prove that it can be identified with an ideal of Q, the Martindale ring of β-quotients of T, where (T, β) denotes the enveloping action of (R, α). We prove that, in general, (Q, β) is not the enveloping action of $(\mathcal{Q}, \alpha)$ and study the relationship between the rings R, $\mathcal{Q}$, T and Q. Finally, we establish some properties related to the center of $\mathcal{Q}$ and the extended $\alpha-$centroid of R.

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Introduction

Partial actions of groups have been considered in many contexts. This theory were introduced in the theory of operator algebras (see [6], [7] and the literature quoted therein). The partial actions on algebras in a purely algebraic framework were introduced by Dokuchaev and Exel in [6]. They are a powerful tool in the generalization of known results of global actions in several areas as partial Galois theory, skew polynomial rings, skew group rings, fixed rings, Hopf algebras and entwining structures.

According to [9], Martindale's theory was originally introduced for prime rings and was primarily intended to deal with applications to rings satisfying a polynomial identity. The generalization to semiprime rings was due to Amitsur. These rings of quotients associated with semiprime rings have since proved to be useful not only for the theory of rings with polynomial identities, but also for the Galois theory of noncommutative rings and for the study of prime ideals under ring extensions in general.

The Martindale ring of quotients has been used successfully applied by several authors to the study of partial actions. For example, Ferrero ([7]) proved that any proper partial action α on a semiprime ring R possesses a weak enveloping action. Also, Cortes et al. ([5]) proved that R is right Goldie if and only if $R[x;\alpha]$ is right Goldie if and only if $R\langle x;\alpha\rangle$ is right Goldie, where R is a semiprime ring, α is a partial action on R, $R[x;\alpha]$ is the partial skew polynomial ring and $R\langle x;\alpha\rangle$ is the partial skew Laurent polynomial ring. More recently, the Martindale ring of $\alpha-$quotients $\mathcal{Q}$ for a partial action (R,α) has been introduced in [2] to study the correspondence between all closed (R-disjoint prime) ideals of $R \star_\alpha G$ and all closed ($\mathcal{Q}$-disjoint prime) ideals of $\mathcal{Q} \star_\alpha G$. So, the ring $\mathcal{Q}$ is shown to be a suitable environment to generalize many well-known results of global actions.

$\mathcal{Q}$ has been only employed as a tool for studying prime ideals in some ring extensions. The main goal of this paper is to present several properties of $\mathcal{Q}$ and study the relationship between all the rings of the diagram

$$\begin{array}{ccccc} R & \rightarrow & T & & \\ \downarrow & & \downarrow & \searrow & \\ \mathcal{Q} & \rightarrow & E & \rightarrow & Q \end{array}$$

where R is an α-prime ring, (T,β) is the enveloping action of (R,α), $\mathcal{Q}$ the ring of α-quotients of R, (E,β) the enveloping action of $(\mathcal{Q},\alpha)$ and Q the ring of β-quotients of T.

In Section 1 we present partial actions and some notions related to enveloping actions and $\alpha-$invariant ideals, which will be used in other sections. In Section 2 we sketch the construction of the ring $\mathcal{Q}$ and extend the partial action (R, α) to a partial action on $\mathcal{Q}$. In addition, we prove that the ring $\mathcal{Q}$ can be identified with an ideal of Q, where Q is the ring of β-quotients of the enveloping action (T, β) of (R, α). Moreover, we determine under which assumptions (Q, β) is the enveloping action of $(\mathcal{Q}, \alpha)$. Among other results in Section 3 we prove some properties related to the center of $\mathcal{Q}$ and the extended $\alpha-$centroid of R.

1. Preliminaries

In this section we present the basic theory of partial actions related to enveloping actions and $\alpha-$invariant ideals. More details can be found in [6].

Definition 1.1. *Let G be a group and R a unital k-algebra, where k is a commutative ring. A partial action α of G on R is a collection of ideals S_g, $g \in G$ of R and isomorphisms of (non-necessarily unital) k-algebras $\alpha_g : S_{g^{-1}} \to S_g$ such that for all $g, h \in G$ the following statements hold:*

1. *$S_1 = R$ and α_1 is the identity mapping of R.*
2. *$S_{(gh)^{-1}} \supseteq \alpha_h^{-1}(S_h \cap S_{g^{-1}})$.*
3. *$\alpha_g \circ \alpha_h(x) = \alpha_{gh}(x)$, for any $x \in \alpha_h^{-1}(S_h \cap S_{g^{-1}})$.*

Natural examples of partial actions can be obtained by restricting a global action to an ideal. More precisely, suppose that the group G acts on an algebra T by automorphisms $\beta_g : T \to T$ and let R be an ideal of T. Set $S_g = R \cap \beta_g(R)$ and let α_g be the restriction of β_g to $S_{g^{-1}}$, for every $g \in G$. Then it is easy to see that $\alpha = \{\alpha_g : S_{g^{-1}} \to S_g \mid g \in G\}$ is a partial action of G on R. In this case, we say that α is the restriction of β to R. In additon, if T is generated by $\bigcup_{g \in G} \beta_g(R)$ then α is called an *admissible* restriction of β.

In what follows, we present the definition of equivalent partial actions and enveloping action ([6], Definitions 4.1 and 4.2 respectively).

Definition 1.2. *Given the partial actions $\alpha = \{\alpha_g : S_{g^{-1}} \to S_g \mid g \in G\}$ and $\alpha' = \{\alpha'_g : S'_{g^{-1}} \to S'_g \mid g \in G\}$ of a group G on rings R and R',* *respectively, we say that α and α' are equivalent if there exists an algebra isomorphism $\varphi : R \to R'$ such that for every $g \in G$ the following conditions hold:*

1. $\varphi(S_g) = S'_g$.
2. $\alpha'_g \circ \varphi(x) = \varphi \circ \alpha_g(x)$, *for all* $x \in S_{g^{-1}}$.

Definition 1.3. *An action β of G on an algebra T is said to be an enveloping action for the partial action α of G on R if α is equivalent to an admissible restriction of β to an ideal of T.*

In other words, β is an enveloping action for α if there exists an algebra isomorphism φ of R onto an ideal of T such that for all $g \in G$ the following properties hold:

1. $\varphi(S_g) = \varphi(R) \cap \beta_g(\varphi(R))$.
2. $\varphi \circ \alpha_g(x) = \beta_g \circ \varphi(x)$, for all $x \in S_{g^{-1}}$.
3. T is generated by $\bigcup_{g \in G} \beta_g(\varphi(R))$.

Thus it is natural to ask when a given partial action can be obtained as the restriction of a global action. The solution to this problem was given in [6] (Theorem 4.5) as follows.

Theorem 1.4. *Le R be an unital algebra. Then a partial action α of a group G on R admits an enveloping action β if and only if each ideal S_g, $g \in G$ is a unital algebra. Moreover, if such an enveloping action exists, it is unique up equivalence.*

In this paper we assume that each ideal S_g, $g \in G$ is generated by a central idempotent of R, denoted by 1_g. This condition guarantees the existence of the enveloping action (T, β) for (R, α) (Theorem 1.4). This means that there exists an algebra T together with a global action $\beta = \{\beta_g \mid g \in G\}$ of G on T, where each β_g is an automorphism of T, such that the partial action is given by restriction of the global action (Definition 1.3). Then we may consider that R is an ideal of T, $T = \sum_{g \in G} \beta_g(R)$, $S_g = R \cap \beta_g(R)$, for every $g \in G$ and $\alpha_g(x) = \beta_g(x)$, for all $g \in G$ and $x \in S_{g^{-1}}$.

Let (R, α) be a partial action of the group G on R. An ideal I of R is said to be α-invariant if $\alpha_g(I \cap S_{g-1}) = I \cap S_g$, for all $g \in G$. We will denote this by $I_\alpha R$. The ring R is said to be α-prime if the condition $AB = 0$, where A and B are α-invariant ideals of R, implies either $A = 0$ or $B = 0$ and it is said to be α-semiprime if the condition $A^2 = 0$, where A is an α-invariant ideal of R, implies $A = 0$. In the global case, similar notions are defined.

2. The Martindale ring of $\alpha-$quotients $\mathcal{Q}$ of R

In this section we assume that α is a partial action of a group G on R and R is an α-semiprime ring. If H is an $\alpha-$invariant ideal of R then the left (right) annihilator of H, $An_l(H)$ $(An_r(H))$, is an $\alpha-$invariant ideal of R and $An_l(H) = An_r(H)$. We consider the filter F(R)=$\{\mathrm{H}_\alpha R \mid An(H) = 0\}$, where $An(H)$ denotes the annihilator of H in R.

As in the global case, given $U \in F(R)$ and the homomorphism of left R-modules $f : \ _RU \to \ _RR$, either uf or $(u)f$ denotes f applied to u, for $u \in U$. Moreover, note that each element $x \in R$ induces an $R-$homomorphism of R via right multiplication, which will be denoted by x_r.

Let $T = \{(U, f) \mid U \in F(R)$ and $f : \ _RU \to \ _RR\}$. On the set T we define an equivalence relation by $(U, f) \sim (V, g)$ if and only if $f \mid_{U \cap V} = g \mid_{U \cap V}$. Given $(U, f) \in T$, we denote by $[U, f]$ the coset of (U, f). Then the set $T \sim = \{[A, f] \mid (A, f) \in T\}$ with the operations [A,f]+[B,g]=[A$\cap B, f + g$] [A,f].[B,g]=[BA,f$\circ g$], is a ring with identity $[R, (1_R)_r]$. This ring is called the (Martindale) ring of α-quotients of R, which will be denoted by $\mathcal{Q}$.

It is easy to see that R is a subring of $\mathcal{Q}$ via right multiplication (Proposition 3.1 (1)). Then, each element $x \in R$ can be identified with the coset $[R, x_r]$ and so we can assume that the identity element of $\mathcal{Q}$ is 1_R.

Recall that in the global case, for T a $\beta-$prime ring, the action of G was naturally extended to the ring of $\beta-$quotients Q of T ([10], Section 1). In the following theorem we extend the partial action of G on R to a partial action on $\mathcal{Q}$. For each ideal S_g, $g \in G$ of R, we define

$\mathrm{S}_g^\star = \{q \in \mathcal{Q} \mid$ there exists $H \in F(R)$ such that $Hq \subseteq S_g\}$.

Theorem 2.1. *Let R be an α-semiprime ring. The following statements hold:*

1. *The set $S_g^\star$, $g \in G$ is an ideal of $\mathcal{Q}$, which is generated by a central idempotent.*

2. *The function $\alpha_g^\star : S_{g^{-1}}^\star \longrightarrow S_g^\star$, defined by $\alpha_g^\star(q) = [H, f_{g,q}]$, where $H \in F(R)$ is such that $Hq \subseteq S_{g^{-1}}$ and for each $h \in H$, $(h)f_{g,q} = \alpha_g(\alpha_{g^{-1}}(h1_g)q)$, is an isomorphism of rings.*

3. *The set $\alpha^\star = \Big\{(S_g^\star, \alpha_g^\star) \mid g \in G\Big\}$ defines a partial action of G on $\mathcal{Q}$.*

Proof. 1. If $q \in S_g^\star$ and $p \in \mathcal{Q}$, then there exist $H, J \in F(R)$ such that $Hq \subseteq S_g$ and $Jp \subseteq R$. Thus, $(HJ)pq \subseteq H(Jp)q \subseteq Hq \subseteq S_g$. So, $pq \in S_g^\star$. Symmetrically we obtain that $qp \in S_g$.

Since each 1_g is a central idempotent of R, then it is clear that each 1_g is a central idempotent of $\mathcal{Q}$. Given $q \in \mathcal{Q}$, there exists $H \in F(R)$ such that $Hq \subseteq R$ and so $Hq1_g \subseteq S_g$ and $q1_g = 1_g q \in S_g^\star$. If $p \in S_g^\star$, then $Jp \subseteq S_g$ for some $J \in F(R)$. Thus, for all $j \in J$ we have that $jp1_g = jp$ which implies that $J(p1_g - p) = 0$. Hence, $p1_g = p$.
2. It is easy to see that $\alpha_g^\star$, $g \in G$ is a well defined function. Moreover, note that $f_{g,p}$ is a homomorphism of left R-modules for all $g \in G$ and all $p = [U, h] \in S_{g^{-1}}^\star$. In fact, for each $r \in R$ and each $u \in U$ we have that $(ru)f_{g,p} = \alpha_g(\alpha_{g^{-1}}(ru1_g)p) = \alpha_g(\alpha_{g^{-1}}(r1_g)\alpha_{g^{-1}}(u1_g)p) = r\alpha_g(\alpha_{g^{-1}}(u1_g)p) = r(u)f_{g,p}$.

Note that for all $u' \in U$, we have that $[R, u'_r]\,[U, f_{g,p}] = [U, u'_r \circ f_{g,p}]$. Thus for all $u \in U$ we have that $(u)(u'_r \circ f_{g,p}) = (uu')f_{g,p} = u\alpha_g(\alpha_{g^{-1}}(u'1_g)p)$ and it follows that $(u)(u'_r \circ f_{g,p}) = (u)(\alpha_g(\alpha_{g^{-1}}(u'1_g)p))_r$. Hence, $U\,[U, f_{g,p}] = [U, (\alpha_g(\alpha_{g^{-1}}(u'1_g)p))_r] = [R, (\alpha_g(\alpha_{g^{-1}}(u'1_g)p))_r]$. Since $\alpha_g(\alpha_{g^{-1}}(u'1_g)p) \in S_g$, we have that $U\alpha_g^\star(p) \subseteq S_g$. So, $\alpha_g^\star(p) \in S_g^\star$.

Finally, let $p = [J, f]$ and $q = [H, g] \in S_{g^{-1}}^\star$ with $Hq \subseteq S_{g^{-1}}$ and $Jp \subseteq S_{g^{-1}}$. Then, $\alpha_g^\star(pq) = [HJ, f_{g,pq}]$ and for $hj \in HJ$ we have that $(hj)f_{g,pq} = \alpha_g(\alpha_{g^{-1}}(hj1_g)pq)$. Since $\alpha_g^\star(p)\alpha_g^\star(q) = [J, f_{g,p}][H, f_{g,q}] = [HJ, f_{g,p} \circ f_{g,q}]$, then for $hj \in HJ$ we have that $(hj)f_{g,p} \circ f_{g,q} = (\alpha_g(\alpha_{g^{-1}}(hj1_g)p))f_{g,q} = \alpha_g(\alpha_{g^{-1}}(\alpha_g(\alpha_{g^{-1}}(hj1_g)p)1_g)q) = \alpha_g(\alpha_{g^{-1}}(hj1_g)pq) = (hj)f_{g,pq}$. Hence, $\alpha_g^\star(pq) = \alpha_g^\star(p)\alpha_g^\star(q)$. Analogously it is proved that $\alpha_g^\star(p+q) = \alpha_g^\star(p) + \alpha_g^\star(q)$ and so $\alpha_g^\star$ is a homomorphism of rings for each $g \in G$.

Let $p = [U, \eta] \in S_{g^{-1}}^\star$ and $H \in F(R)$ such that $Hp \subseteq S_{g^{-1}}$. Then, $p = [H \cap U, \eta]$ and for $q = \alpha_g^\star(p) = [H \cap U, f_{g,p}]$ we have that $\alpha_{g^{-1}}^\star\left(\alpha_g^\star(p)\right) = \alpha_{g^{-1}}^\star(q) = \Big[H \cap U, f_{g^{-1},q}\Big]$. For $h \in H \cap U$ we have that $(h)f_{g^{-1},q} = \alpha_{g^{-1}}(\alpha_g(h1_{g^{-1}})q) = \alpha_{g^{-1}}(\alpha_g(\alpha_{g^{-1}}(\alpha_g(h1_{g^{-1}}))p)) = hp = (h)\eta$. So,

$\alpha^\star_{g^{-1}}\left(\alpha^\star_g(p)\right) = \left[H\cap U, f_{g^{-1},q}\right] = [H\cap U,\eta] = p$. Thus, by symmetry we have that $\alpha^\star_g$, $g\in G$ is an isomorphism of rings.
3. It is analogous to that of Theorem 3.1 of [7]. □

Each isomorphism $\alpha^\star_g$, $g\in G$ of the above theorem will be denoted by α_g. In the following theorem we show that the ring $\mathcal{Q}$ can be identified with an ideal of the ring of $\beta-$quotients Q of T, where (T,β) is the enveloping action of (R,α). In addition, we find an enveloping action for $(\mathcal{Q},\alpha)$, which exists by virtue of Theorems 1.4 and 2.1.

First, note that if H is a nonzero α-invariant ideal of R, then $H^\star=\{t\in T \mid \beta_g(t)1_R\in H \text{ for all } g\in G\}$ is a nonzero $\beta-$invariant ideal of T and if $An_R(H)=0$, then $An_T(H^\star)=0$.

Theorem 2.2. *Let R be an α-semiprime ring. The following statements hold:*

1. *The function $\varphi:\mathcal{Q}\to Q$, defined by $\varphi([H,f]) = \left[H^\star,\overline{f}\right]$ where $(h^\star)\overline{f}=(h^\star 1_R)f$ for $h^\star\in H^\star$, is a monomorphism of rings.*
2. *$\mathcal{Q}\simeq Im\ \varphi$ is an ideal of Q.*
3. *$E=\sum_{g\in G}\beta_g(\varphi(\mathcal{Q}))$ is a G-invariant ideal of Q.*
4. *The enveloping action of $(\mathcal{Q},\alpha)$ is (E,β), where each β_g, $g\in G$ acts on E by restriction.*

Proof. 1. First note that $H^\star\in F(T)$. If $t\in T$ and $h^\star\in H^\star$, then $(th^\star)\overline{f}=(th^\star 1_R)f=(t1_Rh^\star 1_R)f=t(h^\star 1_R)f=t(h^\star)\overline{f}$. So, $\overline{f}$ is a homomorphism of left $T-$modules and $[H^\star,\overline{f}]\in Q$. Moreover, it is easy to see that φ is a well defined function.

If $[H,f],[H_1,f_1]\in\mathcal{Q}$, then $\varphi([H,f][H_1,f_1]) = \left[(H_1H)^\star,\overline{f\circ f_1}\right]$ and $\varphi([H,f])\varphi([H_1,f_1])=[H_1^\star H^\star,\overline{f}\circ\overline{f_1}]$. Since $H_1^\star H^\star\subseteq(H_1H)^\star$, for $tv\in H_1^\star H^\star$ we have that $(tv)\overline{f\circ f_1}=(tv)\overline{f}\circ\overline{f_1}$. Hence, by linearity we have that $\overline{f\circ f_1}\mid_{H_1^\star H^\star}=\overline{f}\circ\overline{f_1}\mid_{H_1^\star H^\star}$ which implies that $[(H_1H)^\star,\overline{f\circ f_1}]=[H_1^\star H^\star,\overline{f}\circ\overline{f_1}]$. So, $\varphi([H,f][H_1,f_1])=\varphi([H,f])\varphi([H_1,f_1])$. Analogously it is proved that $\varphi([H,f]+[H_1,f_1])=\varphi([H,f])+\varphi([H_1,f_1])$ and so φ is a homomorphism of rings.

Finally, $\varphi([H,f])=0$ implies $(H^\star)\overline{f}=0$ and thus $[H,f]=0$. Hence, φ is a monomorphism of rings.

2. Let $\left[H^\star, \overline{f}\right] \in Im\ \varphi$ and $0 \neq [J, j] \in Q$, where $J \in F(T)$ and $j : J \to T$ is a homomorphism of left $T-$modules. Then,
$\left[H^\star, \overline{f}\right][J, j] = \left[JH^\star, \overline{f} \circ j\right]$. Note that $J \cap R \in F(R)$ and $j_1 = j\mid_{J\cap R}$ is a homomorphism of left $R-$modules. Then, $[J \cap R, j_1] \in \mathcal{Q}$ and so $\varphi([(J \cap R)H, f \circ j_1]) = [(J \cap R)^\star H^\star, \overline{f} \circ \overline{j_1}]$. Since $J \subseteq (J \cap R)^\star$, for $tv \in JH^\star$ we have that $(tv)\overline{f} \circ \overline{j_1} = (tv)\overline{f} \circ j$. Thus by linearity we have that $[(J\cap R)^\star H^\star, \overline{f}\circ\overline{j_1}] = [JH^\star, \overline{f}\circ j]$, that is, $\varphi([(J\cap R)H, f\circ j_1]) = \left[H^\star, \overline{f}\right][J, j]$. Then, $Im\ \varphi$ is a right ideal of Q.

Similarly we have that $\varphi\left([H(J \cap R), j_1 \circ f]\right) = \left[H^\star\left(J \cap R\right)^\star, \overline{j_1} \circ \overline{f}\right] = \left[H^\star J, j \circ \overline{f}\right] = [J, j]\left[H^\star, \overline{f}\right]$, that is, $Im\ \varphi$ is a left ideal of Q. Therefore, $Im\ \varphi$ is an ideal of Q.

3. It is evident.

4. We must prove 1, 2 and 3 of Definition 1.3. 3 is consequence of item 3. To prove 2, let $q = [A, \eta] \in S^\star_{g^{-1}}$. Then there exists $H \in F(R)$ such that $Hq \subseteq S_{g^{-1}}$. Moreover, $\varphi \circ \alpha_g(q) = \left[H^\star, \overline{f_{g,q}}\right]$ and $\beta_g \circ \varphi(q) = \left[A^\star, \beta_{g^{-1}} \circ \overline{\eta} \circ \beta_g\right]$. For $h^\star \in H^\star \cap A^\star$ we have that $(h^\star)\overline{f_{g,q}} = \alpha_g(\alpha_{g^{-1}}(h^\star 1_g)q) = \beta_g(\beta_{g^{-1}}(h^\star 1_g)q) = \beta_g(\beta_{g^{-1}}(h^\star)1_{g^{-1}}q) = (h^\star)\beta_{g^{-1}} \circ \overline{\eta} \circ \beta_g$. Hence, $\varphi \circ \alpha_g(q) = \beta_g \circ \varphi(q)$.

To prove 1, let $q \in S^\star_g$. Then, $\varphi(q) \in \varphi(\mathcal{Q})$ and $\beta_{g^{-1}}(\varphi(q)) = \varphi(\alpha_{g^{-1}}(q))$ with $\varphi(\alpha_{g^{-1}}(q)) \in \varphi(\mathcal{Q})$. Hence, $\varphi(q) = \beta_g(\varphi(\alpha_{g^{-1}}(q))) \in \beta_g(\varphi(\mathcal{Q}))$. Therefore, $\varphi(S^\star_g) \subseteq \varphi(\mathcal{Q}) \cap \beta_g(\varphi(\mathcal{Q}))$ for all $g \in G$.

For the other inclusion, we can identify $\mathcal{Q}$ with $\varphi(\mathcal{Q})$. Let $q, q_1 \in \mathcal{Q}$ such that $q = \beta_g(q_1) \in Q \cap \beta_g(Q)$. Then, $q = 1_R\beta_g(1_R)\beta_g(q_1) = 1_g\beta_g(q_1) \in S^\star_g$. $\square$

Since $R \subseteq \mathcal{Q}$, we have $\beta_g(R) \subseteq \beta_g(\mathcal{Q})$ for all $g \in G$. Then, $\sum_g \beta_g(R) \subseteq \sum_g \beta_g(\mathcal{Q})$. Hence, $T \subseteq E$ because (T, β) is the enveloping action of (R, α) and (E, β) the enveloping action of $(\mathcal{Q}, \alpha)$. In conclusion, we obtain the following diagram of ring extensions

$$\begin{array}{ccccc} R & \rightarrow & T & & \\ \downarrow & & \downarrow & \searrow & \\ \mathcal{Q} & \rightarrow & E & \rightarrow & Q \end{array}$$

Thus it is natural to ask when (Q, β) is the enveloping action of $(\mathcal{Q}, \alpha)$, or equivalently, in which case $E = Q$. In the following proposition we give a necessary and sufficient condition to solve this problem. First, we recall that the partial action (R, α) with enveloping action (T, β) is said to be *of finite type* if T is a ring with identity 1_T (see [8], Proposition 1.2 for equivalences).

Proposition 2.3. *Let R be an α-semiprime ring. (Q,β) is the enveloping action of $(\mathcal{Q},\alpha)$ iff α is a partial action of finite type.*

Proof. If (Q,β) is the enveloping action of $(\mathcal{Q},\alpha)$, then $Q=\sum_{g\in G}\beta_g(Q)$. In particular, $1_Q\in\sum_{i=1}^n\beta_{g_i}(\mathcal{Q})$ for some $g_i\in G$ $i\in\{1,...,n\}$. Since $\beta_{g_i}(1_R)$ is the identity of $\beta_{g_i}(Q)$ and each $\beta_{g_i}(\mathcal{Q})$ is an ideal of Q for $i\in\{1,...,n\}$, we have that

$$1_Q=\sum_{1\leq l\leq n}\sum_{i_1<i_2<...<i_l}(-1)^{l+1}\beta_{i_1}(1_R)\beta_{i_2}(1_R)...\beta_{i_{l-1}}(1_R)\beta_{i_l}(1_R)$$

([8], Proposition 1.10). Hence, T is a ring with identity $1_T=1_Q$ and thus α is of finite type.
Conversely, if α is of finite type then T is a ring with identity 1_T. Thus, $1_T=1_Q$ and since $T\subseteq E$ we have that $1_Q\in E$. Finally, since E is an ideal of Q we conclude that $E=Q$ and so the result follows. □

3. Other properties

In this section we present some properties of the ring $\mathcal{Q}$ related to the center of $\mathcal{Q}$ and the extended $\alpha-$centroid of R, which extend to the partial case well-known results in the theory of quotient rings. For the sake of completeness, we start with the following proposition ([2], Proposition 1.1).

Proposition 3.1. *Let R be an α-semiprime ring. The following statements hold:*

1. *R is a subring of $\mathcal{Q}$, via right multiplication.*
2. *If $(U,f)\in T$, then for every $u\in U$ we have that $u_r\circ f=(uf)_r$, where $xu_r=xu$, for all $x\in R$.*
3. *If $q=[U,f]\in\mathcal{Q}$ then $(u)f=uq$ for all $u\in U$. In particular, $Uq\subseteq R$.*
4. *For any $q_1,...,q_n\in\mathcal{Q}$, there exists $U\in F(R)$ such that $Uq_i\subseteq R$, for all $i\in\{1,...,n\}$.*

5. *If $U \in F(R)$ and $f : U \longrightarrow R$ is a homomorphism of left $R-$modules, then there exists $q \in \mathcal{Q}$ such that $uf = uq$, for all $u \in U$.*

6. *Let $q \in \mathcal{Q}$ and $U \in F(R)$. If $Uq = 0$ or $qU = 0$, then $q = 0$.*

The set of elements $q \in \mathcal{Q}$ such that $qr = rq$ for all $r \in R$ is called the centralizer of R in $\mathcal{Q}$ and it is denoted by $C_{\mathcal{Q}}(R)$. The center of $\mathcal{Q}$ is the set $Z(\mathcal{Q}) = \{q \in \mathcal{Q} \mid qp = pq \text{ for all } p \in \mathcal{Q}\}$ and the extended α-centroid of R, $C_\alpha(R) = \{q \in Z(Q) \mid \alpha_g(q1_{g^{-1}}) = q1_g \text{ for all } g \in G\}$.

Proposition 3.2. *Let R be an α-semiprime ring and $q = [U, f] \in \mathcal{Q}$. The following statements are equivalent:*

1. *$q \in Z(\mathcal{Q})$.*

2. *f is a homomorphism of $R-$bimodules.*

3. *$q \in C_{\mathcal{Q}}(R)$.*

Proof. $1 \Rightarrow 3$. It is evident.

$3 \Rightarrow 2$. By Proposition 3.1(3), we have that $sq = (s)f$ for all $s \in U$. If $r \in R$, then $(sr)f = (sr)q = s(rq) = s(qr) = (sq)r = (s)fr$.

$2 \Rightarrow 1$. Let $p = [V, g]$. If $x \in UV \cap VU$, then we can assume that $x = uv$. Thus, $(uv)(g \circ f) = (u(v)g)f = (u)f(v)g = ((u)fv)g = ((uv)f)g = (uv)(f \circ g)$. Hence, $g \circ f\ |_{UV \cap VU} = f \circ g\ |_{UV \cap VU}$ and we have that $[UV, g \circ f] = [VU, f \circ g]$, that is, $pq = qp$. $\square$

Proposition 3.3. *Let R be an α-prime ring. If $0 \neq q \in \mathcal{Q}$ is such that $qR = Rq$ and $\alpha_g(q1_{g^{-1}}) = q1_g$ for all $g \in G$, then q is invertible in $\mathcal{Q}$. In particular, $C_\alpha(R)$ is a field.*

Proof. Consider $H \in F(R)$, such that $Hq \subseteq R$. Then $f : Hq \longrightarrow R$ defined by $(hq)f = h$ for each $hq \in Hq$, is a well defined function and it is a homomorphism of left $R-$modules. In addition, for $p = [Hq, f]$ and $h \in H$ we have that $h = (hq)f = hqp$ which implies that $qp = 1_R$.
Let $y \in \mathcal{Q}$ and $J \in F(R)$ such that $qy = 0$ and $Jy \subseteq R$ and consider the nonzero α-invariant ideal $I = Rq \cap R$ of R. Then for all $rq \in I$ and for all $jy \in Jy$, $(rq)(jy) = r(qj)y = (rj')(qy) = 0$ where $j' \in R$ and $qj = j'q$.

Hence, $IJy = 0$ which implies $y = 0$. In conclusion, q is a right invertible element and it is not a left zero divisor of $\mathcal{Q}$. Hence, q is an invertible element in $\mathcal{Q}$. $\square$

Proposition 3.4. *Let $R, T, \mathcal{Q}, Q, E$ be the rings with the above assumptions and consider the following statements:*

1. *R is $\alpha-$prime ($\alpha-$semiprime).*
2. *T is $\beta-$prime ($\beta-$semiprime).*
3. *$\mathcal{Q}$ is $\alpha-$prime ($\alpha-$semiprime).*
4. *E is $\beta-$prime ($\beta-$semiprime).*
5. *Q is $\beta-$prime ($\beta-$semiprime).*

Then, the following implications hold: $1 \Leftrightarrow 2$, $3 \Leftrightarrow 4$, $1 \Rightarrow 3$, $2 \Rightarrow 4$, $2 \Rightarrow 5$ *and* $4 \Rightarrow 5$.

References

[1] J. Ávila, *Ideais Fechados e Primos em Skew Anéis de Grupos Parciais*, Tese de Doutorado, Universidade Federal do Rio Grande do Sul, Porto Alegre (Brasil), (2008).

[2] J. Ávila and M. Ferrero, *Closed and Prime Ideals in Partial Skew Group Rings of Abelian Groups*, to appear in J. Algebra Appl., DOI N 10.1142/S0219498811005063,

[3] W. Cortes, *Partial Skew Polynomial Rings and Jacobson Rings*, Comm. Algebra 38 (4), pp. 1526-1548, (2010).

[4] W. Cortes, M. Ferrero, Y. Hirano and H. Marubayashi, *Partial Skew Polynomial Rings over Semisimple Artinian Rings*, Comm. Algebra 38 (5), pp. 1663-1676, (2010).

[5] W. Cortes, M. Ferrero and H. Marubayashi, *Partial Skew Polynomial Rings and Goldie Rings*, Comm. Algebra 36 (11), pp. 4284-4295, (2008).

[6] M. Dokuchaev and R. Exel, *Associativity of Crossed Products by Partial Actions, Enveloping Actions and Partial Representations*, Trans. Amer. Math. Society 357 (5), pp. 1931-1952, (2005).

[7] M. Ferrero, *Partial Actions of Groups on Semiprime Rings*, A Series of Lectures Notes in Pure and Applied Mathematics, Volume 248, Chapman and Hall, (2006).

[8] M. Ferrero and J. Lazzarin ; Partial actions and partial skew group rings, J. Algebra 319, pp. 5247-5264, (2008).

[9] T. Y. Lam, *Lectures on Modules and Rings*, Graduate Text in Mathematics, Springer-Verlag, New York, (1998).

[10] M. Malásquez, *Prime Ideals in Crossed Products of Abelian Groups*, Comm. Algebra 22 (5), pp. 1861-1876, (1994).

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