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Difference sequence spaces defined by a sequence of modulus functions

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Abstract

In the present paper we study difference sequence spaces defined by a sequence of modulus functions and examine some topological properties of these spaces.

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1. Introduction and Preliminaries

A modulus function is a function $f : [0, \infty) \to [0, \infty)$ such that

1. $f(x) = 0$ if and only if $x = 0$,
2. $f(x + y) \leq f(x) + f(y)$ for all $x \geq 0$, $y \geq 0$,
3. f is increasing,
4. f is continuous from right at 0.

It follows that f must be continuous everywhere on $[0, \infty)$. The modulus function may be bounded or unbounded. For example, if we take $f(x) = \frac{x}{x+1}$, then $f(x)$ is bounded. If $f(x) = x^p$, $0 < p < 1$, then the modulus $f(x)$ is unbounded. Subsequentially, modulus function has been discussed in ([1], [7], [8]) and many others.

Let X be a linear metric space. A function $p : X \to \mathbf{R}$ is called paranorm, if

1. $p(x) \geq 0$, for all $x \in X$,
2. $p(-x) = p(x)$, for all $x \in X$,
3. $p(x + y) \leq p(x) + p(y)$, for all $x, y \in X$,
4. if (λ_n) is a sequence of scalars with $\lambda_n \to \lambda$ *as* $n \to \infty$ and (x_n) is a sequence of vectors with $p(x_n - x) \to 0$ *as* $n \to \infty$, then $p(\lambda_n x_n - \lambda x) \to 0$ *as* $n \to \infty$.

A paranorm p for which $p(x) = 0$ implies $x = 0$ is called total paranorm and the pair (X, p) is called a total paranormed space. It is well known that the metric of any linear metric space is given by some total paranorm (see [9], Theorem 10.4.2, P-183).

Let w be the set of all sequences, real or complex numbers and l_∞, c and c_0 be respectively the Banach spaces of bounded, convergent and null sequences $x = (x_k)$, normed by $||x|| = \sup_k |x_k|$, where $k \in \mathbf{N}$, the set of positive integers.

Let $\Lambda = (\lambda_n)$ be a non decreasing sequence of positive reals tending to infinity and $\lambda_1 = 1$ and $\lambda_{n+1} \leq \lambda_n + 1$. The generalized de la Vallee-Poussin means is defined by

$$t_n(x) = \frac{1}{\lambda_n} \sum_{k \in I_n} x_k,$$

where $I_n = [n - \lambda_n + 1, n]$. A sequence $x = (x_k)$ is said to be (V, λ)-summable to a number l if $t_n(x) \to l$ as $n \to \infty$ (see[5]). If $\lambda_n = n$, (V, λ)-summability and strong (V, λ)-summability are reduced to $(C, 1)$-summability and $[C, 1]$-summability, respectively.

In [4], Kizmaz defined the sequence spaces

$$X(\Delta) = \left\{ x = (x_k) : (\Delta x_k) \in X \right\}$$

for $X = l_\infty,\ c$ or c_0, where $\Delta x = (\Delta x_k) = (x_k - x_{k+1})$ for all $k \in \mathbf{N}$.
Et and Colak [2] generalized the above sequence spaces to the sequence spaces

$$X(\Delta^m) = \left\{ x = (x_k) : (\Delta^m x_k) \in X \right\}$$

for $X = l_\infty,\ c$ or c_0, where $m \in \mathbf{N}$, $\Delta^0 x = (x_k)$, $\Delta x = (x_k - x_{k+1})$,

$$\Delta^m x = (\Delta^m x_k) = (\Delta^{m-1} x_k - \Delta^{m-1} x_{k+1}) \quad \text{for all} \quad k \in \mathbf{N}.$$

The generalized difference operator has the following binomial representation,

$$\Delta^m x_k = \sum_{v=0}^{m} (-1)^v \binom{m}{v} x_{k+v}$$

for all $k \in \mathbf{N}$.

The following inequality will be used throughout the paper. If $0 \leq p_k \leq \sup p_k = H$, $D = \max(1, 2^{H-1})$ then

$$|a_k + b_k|^{p_k} \leq D\{|a_k|^{p_k} + |b_k|^{p_k}\} \tag{1.1}$$

for all k and $a_k, b_k \in \mathbf{C}$. Also $|a|^{p_k} \leq \max(1, |a|^H)$ for all $a \in \mathbf{C}$.

Let E be a Banach space, we define $w(E)$ to be the vector space of all E-valued sequences that is

$$w(E) = \{x = (x_k) : x_k \in E\}.$$

Let $F = (f_k)$ be a sequence of modulus functions and $p = (p_k)$ be a bounded sequence of positive real numbers. Then we define the following sequence spaces :

$$[V,\lambda,F,p]_1(\Delta^m,E,u) = \Big\{x \in w(E) : \lim_n \frac{1}{\lambda_n} \sum_{k \in I_n} \Big[f_k\Big(||\Delta^m u_k x_k - Le||\Big)\Big]^{p_k} = 0, \text{ for some } L\Big\},$$

$$[V,\lambda,F,p]_0(\Delta^m,E,u) = \Big\{x \in w(E) : \lim_n \frac{1}{\lambda_n} \sum_{k \in I_n} \Big[f_k\Big(||\Delta^m u_k x_k||\Big)\Big]^{p_k} = 0\Big\}$$

and

$$[V,\lambda,F,p]_\infty(\Delta^m,E,u) = \Big\{x \in w(E) : \sup_n \frac{1}{\lambda_n} \sum_{k \in I_n} \Big[f_k\Big(||\Delta^m u_k x_k||\Big)\Big]^{p_k} < \infty\Big\},$$

where $e = (1,1,1,\cdots)$.
If $u = e$ and $f_k = f$, then these spaces reduce to those which were studied by Et, M., Altin, Y. and Altinok, H. [3].

For $f_k(x) = x$, we have

$$[V,\lambda,p]_1(\Delta^m,E,u) = \Big\{x \in w(E) : \lim_n \frac{1}{\lambda_n} \sum_{k \in I_n} \Big[||\Delta^m u_k x_k - Le||\Big]^{p_k} = 0, \text{ for some } L\Big\},$$

$$[V,\lambda,p]_0(\Delta^m,E,u) = \Big\{x \in w(E) : \lim_n \frac{1}{\lambda_n} \sum_{k \in I_n} \Big[||\Delta^m u_k x_k||\Big]^{p_k} = 0\Big\}$$

and

$$[V,\lambda,p]_\infty(\Delta^m,E,u) = \Big\{x \in w(E) : \sup_n \frac{1}{\lambda_n} \sum_{k \in I_n} \Big[||\Delta^m u_k x_k||\Big]^{p_k} < \infty\Big\}.$$

For $p_k = 1$, we have

$$[V,\lambda,F]_1(\Delta^m,E,u) = \Big\{x \in w(E) : \lim_n \frac{1}{\lambda_n} \sum_{k\in I_n} \Big[f_k\Big(||\Delta^m u_k x_k - Le||\Big)\Big] = 0, \text{ for some } L\Big\},$$

$$[V,\lambda,F]_0(\Delta^m,E,u) = \Big\{x \in w(E) : \lim_n \frac{1}{\lambda_n} \sum_{k\in I_n} \Big[f_k\Big(||\Delta^m u_k x_k||\Big)\Big] = 0\Big\}$$

and

$$[V,\lambda,F]_\infty(\Delta^m,E,u) = \Big\{x \in w(E) : \sup_n \frac{1}{\lambda_n} \sum_{k\in I_n} \Big[f_k\Big(||\Delta^m u_k x_k||\Big)\Big] < \infty\Big\}.$$

For $f_k(x) = x$ and $p_k = 1$ for all $k \in \mathbf{N}$, we have

$$[V,\lambda]_1(\Delta^m,E,u) = \Big\{x \in w(E) : \lim_n \frac{1}{\lambda_n} \sum_{k\in I_n} \Big[||\Delta^m u_k x_k - Le||\Big] = 0, \text{ for some } L\Big\},$$

$$[V,\lambda]_0(\Delta^m,E,u) = \Big\{x \in w(E) : \lim_n \frac{1}{\lambda_n} \sum_{k\in I_n} \Big[||\Delta^m u_k x_k||\Big] = 0\Big\}$$

and

$$[V,\lambda]_\infty(\Delta^m,E,u) = \Big\{x \in w(E) : \sup_n \frac{1}{\lambda_n} \sum_{k\in I_n} \Big[||\Delta^m u_k x_k||\Big] < \infty\Big\}.$$

Throughout this paper, X will denote any one of the notations $0, 1$ or ∞ .

In this paper we study some topological properties and inclusion relations between above defined sequence spaces.

2. Main Results

Theorem 2.1 *Let $F = (f_k)$ be a sequence of modulus functions and $p = (p_k)$ be a bounded sequence of positive real numbers. Then the sequence spaces $[V,\lambda,F,p]_1(\Delta^m,E,u)$, $[V,\lambda,F,p]_0(\Delta^m,E,u)$ and $[V,\lambda,F,p]_\infty(\Delta^m,E,u)$ are linear spaces.*

Proof. Let $x, y \in [V, \lambda, F, p]_0(\Delta^m, E, u)$ and $\alpha, \beta \in \mathbf{C}$. Then there exist positive number M_α and N_β such that $|\alpha| \leq M_\alpha$ and $|\beta| \leq N_\beta$. Since f_k is subadditive and Δ^m is linear, we have

$$\frac{1}{\lambda_n} \sum_{k \in I_n} \Big[f_k\Big(||\Delta^m(\alpha u_k x_k + \beta u_k y_k)|| \Big) \Big]^{p_k}$$

$$\begin{aligned}
&\leq \frac{1}{\lambda_n} \sum_{k \in I_n} \Big[f_k(|\alpha| \, ||\Delta^m u_k x_k||) + f_k(|\beta| \, ||\Delta^m u_k y_k) \Big]^{p_k} \\
&\leq D(M_\alpha)^H \frac{1}{\lambda_n} \sum_{k \in I_n} \Big[f_k(||\Delta^m u_k x_k) \Big]^{p_k} + D(N_\beta)^H \frac{1}{\lambda_n} \sum_{k \in I_n} \Big[f_k(||\Delta^m u_k y_k||) \Big]^{p_k} \\
&\qquad \to \ 0 \ \text{ as } \ n \to \infty.
\end{aligned}$$

This proves that $[V, \lambda, F, p]_0(\Delta^m, E, u)$ is a linear space. Similarly we can prove that $[V, \lambda, F, p]_1(\Delta^m, E, u)$ and $[V, \lambda, F, p]_\infty(\Delta^m, E, u)$ are linear spaces in view of the above proof.

Theorem 2.2 *Let $F = (f_k)$ be a sequence of modulus functions. Then $[V, \lambda, F, p]_0(\Delta^m, E, u) \subset [V, \lambda, F, p]_1(\Delta^m, E, u) \subset [V, \lambda, F, p]_\infty(\Delta^m, E, u)$.*

Proof. The first inclusion is obvious. For the second inclusion, let $x \in [V, \lambda, F, p]_1(\Delta^m, E, u)$. Then by definition, we have

$$\frac{1}{\lambda_n} \sum_{k \in I_n} \Big[f_k(||\Delta^m u_k x_k||) \Big]^{p_k}$$

$$\begin{aligned}
&= \frac{1}{\lambda_n} \sum_{k \in I_n} \Big[f_k(||\Delta^m u_k x_k - Le + Le) \Big]^{p_k} \\
&\leq D \frac{1}{\lambda_n} \sum_{k \in I_n} \Big[f_k(||\Delta^m u_k x_k - Le||) \Big]^{p_k} + D \frac{1}{\lambda_n} \sum_{k \in I_n} \Big[f_k(||Le||) \Big]^{p_k}.
\end{aligned}$$

Now, there exists a positive number A such that $||Le|| \leq A$. Hence we have

$$\frac{1}{\lambda_n} \sum_{k \in I_n} \Big[f_k(||\Delta^m u_k x_k||)]^{p_k} \leq \frac{D}{\lambda_n} \sum_{k \in I_n} \Big[f_k(||\Delta^m u_k x_k - Le||) \Big]^{p_k} +$$

$$\frac{D}{\lambda_n}\Big[A]^H \sum_{k\in I_n}[f_k(1)]^H.$$

Since $x \in [V,\lambda,F,p]_1(\Delta^m,E,u)$ we have $x \in [V,\lambda,F,p]_\infty(\Delta^m,E,u)$. Therefore,

$$[V,\lambda,F,p]_1(\Delta^m,E,u) \subset [V,\lambda,F,p]_\infty(\Delta^m,E,u).$$

This completes the proof.

Theorem 2.3 *Let $F=(f_k)$ be a sequence of modulus functions and $p=(p_k)$ be a bounded sequence of positive real numbers. Then $[V,\lambda,F,p]_0(\Delta^m,E,u)$ is a paranormed space with*

$$g_\Delta(x)=\sup_n\Big(\frac{1}{\lambda_n}\sum_{k\in I_n}\Big[f_k(||\Delta^m u_k x_k||)\Big]^{p_k}\Big)^{\frac{1}{K}}$$

where $K=\max(1,\sup p_k)$.

Proof. Clearly $g_\Delta(x)=g_\Delta(-x)$. It is trivial that $\Delta^m u_k x_k=0$ for $x=0$. Since $f(0)=0$, we get $g_\Delta(x)=0$ for $x=0$. Since $\frac{p_k}{K}\le 1$, using the Minkowski's inequality, for each n, we have

$$\begin{aligned}
&\Big(\tfrac{1}{\lambda_n}\sum_{k\in I_n}\Big[f_k(||\Delta^m u_k x_k+\Delta^m u_k y_k||)\Big]^{p_k}\Big)^{\frac{1}{K}}\\
\le\ &\Big(\frac{1}{\lambda_n}\sum_{k\in I_n}\Big[f_k(||\Delta^m u_k x_k)+f_k(||\Delta^m u_k y_k||)\Big]^{p_k}\Big)^{\frac{1}{K}}\\
\le\ &\Big(\frac{1}{\lambda_n}\sum_{k\in I_n}\Big[f_k(||\Delta^m u_k x_k||)\Big]^{p_k}\Big)^{\frac{1}{K}}+\Big(\frac{1}{\lambda_n}\sum_{k\in I_n}\Big[f_k(||\Delta^m u_k y_k||)\Big]^{p_k}\Big)^{\frac{1}{K}}.
\end{aligned}$$

Hence $g_\Delta(x)$ is subadditive. For, the continuity of multiplication, let us take any complex number α. By definition, we have

$$\begin{aligned}
g_\Delta(\alpha x) &= \sup_n\Big(\frac{1}{\lambda_n}\sum_{k\in I_n}\Big[f_k(||\Delta^m \alpha u_k x_k||)\Big]^{p_k}\Big)^{\frac{1}{K}}\\
&\le C_\alpha^{H/K} g_\Delta(x),
\end{aligned}$$

where C_α is a positive integer such that $|\alpha| \leq C_\alpha$. Now, let $\alpha \to 0$ for any fixed x with $g_\Delta(x) \neq 0$. By definition for $|\alpha| < 1$, we have

$$\frac{1}{\lambda_n} \sum_{k \in I_n} \Big[f_k(||\alpha \Delta^m u_k x_k||) \Big]^{p_k} < \epsilon \quad \text{for} \;\; n > n_0(\epsilon) \tag{2.2}$$

Also, for $1 \leq n \leq n_0$, taking α small enough, since f_k is continuous, we have

$$\frac{1}{\lambda_n} \sum_{k \in I_n} \Big[f_k(||\alpha \Delta^m u_k x_k||) \Big]^{p_k} < \epsilon. \tag{2.3}$$

Now, eqn. (2.2) and (2.3) together imply that

$$g_\Delta(\alpha x) \to 0 \;\; \text{as} \;\; \alpha \to 0.$$

Theorem 2.4 *Let $F = (f_k)$ be a sequence of modulus functions and $m \geq 1$, then the inclusion*

$$[V, \lambda, F]_X(\Delta^{m-1}, E, u) \subset [V, \lambda, F]_X(\Delta^m, E, u)$$

is strict. In general

$$[V, \lambda, F]_X(\Delta^i, E, u) \subset [V, \lambda, F]_X(\Delta^m, E, u)$$

for all $i = 1, 2, \cdots, m-1$ and the inclusion is strict.

Proof. Let $x \in [V, \lambda, F]_\infty(\Delta^{m-1}, E, u)$. Then we have

$$\sup_n \frac{1}{\lambda_n} \sum_{k \in I_n} \Big[f_k(||\Delta^{m-1} u_k x_k||) \Big] < \infty.$$

By definition, we have

$$\frac{1}{\lambda_n} \sum_{k \in I_n} \Big[f_k(||\Delta^m u_k x_k||) \Big] = \frac{1}{\lambda_n} \sum_{k \in I_n} \Big[f_k(||\Delta^{m-1} u_k x_k||) \Big] +$$

$$\frac{1}{\lambda_n} \sum_{k \in I_n} \Big[f_k(||\Delta^{m-1} u_{k+1} x_{k+1}||) \Big] \leq \infty.$$

Thus $[V, \lambda, F]_\infty(\Delta^{m-1}, E, u) \subset [V, \lambda, F]_\infty(\Delta^m, E, u)$.

Proceeding in this way, we have

$$[V,\lambda,F]_\infty(\Delta^i,E,u) \subset [V,\lambda,F]_\infty(\Delta^m,E,u)$$

for all $i = 1, 2, \cdots, m-1$. Let $E = \mathbf{C}$ and $\lambda_n = n$ for each $n \in \mathbf{N}$. Then the sequence $x = (x^m) \in [V,\lambda,F]_\infty(\Delta^m,E,u)$ but does not belong to $[V,\lambda,F]_\infty(\Delta^{m-1},E,u)$ for $f_k(x) = x$.

Similarly, we can prove for the case $[V,\lambda,F]_0(\Delta^m,E,u)$ and $[V,\lambda,F]_1(\Delta^m,E,u)$ in view of the above proof.

Corollary 2.5 *Let $F = (f_k)$ be a sequence of modulus functions. Then*

$$[V,\lambda,F,p]_1(\Delta^{m-1},E,u) \subset [V,\lambda,F]_0(\Delta^m,E,u).$$

Theorem 2.5 *Let $F = (f_k)$, $F' = (f'_k)$ and $F'' = (f''_k)$ are sequence of modulus functions. Then we have*
(i) $[V,\lambda,F',p]_X(\Delta^m,E,u) \subset [V,\lambda,F\circ F',p]_X(\Delta^m,E,u)$,
(ii)$[V,\lambda,F',p]_X(\Delta^m,E,u) \cap [V,\lambda,F'',p]_X(\Delta^m,E,u) \subset [V,\lambda,F+F',p]_X(\Delta^m,E,u)$.

Proof. (i) Let $\epsilon > 0$ and choose δ with $0 < \delta < 1$ such that $f(t) < \epsilon$ for $0 \le t \le \delta$. Write $y_k = f'_k(||\Delta^m u_k x_k||)$ and consider

$$\sum_{k\in I_n}[f_k(y_k)]^{p_k} = \sum_1 [f_k(y_k)]^{p_k} + \sum_2 [f_k(y_k)]^{p_k},$$

where the first summation is over $y_k \le \delta$ and second summation is over $y_k \ge \delta$. Since f_k is continuous, we have

$$\sum_1 [f_k(y_k)]^{p_k} < \lambda_n \epsilon^H \tag{2.4}$$

and for $y_k > \delta$, we use the fact that

$$y_k < \frac{y_k}{\delta} \le 1 + \frac{y_k}{\delta}.$$

By the definition, we have for $y_k > \delta$,

$$f_k(y_k) < 2 f_k(1)\frac{y_k}{\delta}.$$

Hence

$$\frac{1}{\lambda_n}\sum_{2}\Big[f_k(y_k)\Big]^{p_k} \leq \max\Big(1,(2f_k(1)\delta^{-1})^H\Big)\frac{1}{\lambda_n}\sum_{k\in I_n} y_k. \tag{2.5}$$

From eqn. (2.4) and (2.5), we have

$$[V,\lambda,F,p]_0(\Delta^m,E,u) \subset [V,\lambda,F\circ F',p]_0(\Delta^m,E,u).$$

This completes the proof of (i).

The proof of (ii) follows from the following inequality:

$$\Big[(f'_k+f''_k)(||\Delta^m u_k x_k||)\Big]^{p_k} \leq D\Big[f'_k(||\Delta^m u_k x_k||)\Big]^{p_k} + D\Big[f''_k(||\Delta^m u_k x_k||)\Big]^{p_k}.$$

Corollary 2.6 *Let $F=(f_k)$ be a sequence of modulus functions. Then*

$$[V,\lambda,p]_X(\Delta^m,E,u) \subset [V,\lambda,F,p]_X(\Delta^m,E,u).$$

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