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Schauder basis in a locally $K-$ convex space and perfect sequence spaces

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Abstract

In this work, we are dealing with the natural topology in a perfect sequence space and the transfert of topologies of a locally $K-$ convex space E with a Schauder basis $(e_i)_i$ to such Space. We are also interested with the compatible topologies on E for which the basis$(e_i)_i$ is equicontinuous, and the weak basis problem. Finally, we give some applications to barrelled Spaces and $G-$Spaces.

Keywords : *non archimedean analysis, locally $K-$ convex spaces, Schauder basis, the weak basis theorem, compatible topologies, perfect sequence spaces, $K-$ barrelled spaces and G- spaces.*

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1. Introduction

The perfect sequence spaces on a field K have been widely studied by several authors, either in the classical case ($K = \mathbb{R}$ *or* $K = \mathbb{C}$) Garling [18] and [19], *Köthe* [23], ..., or in the case of K is a non-archimedean valued field Monna [25], Dorleyn [14], De Grande-De Kimpe [8], ...The importance of the sequence spaces lies on the fact that each space which is locally convex and having a Schauder basis is isomorphic to the sequence space. Thus, instead of studying the spaces that are locally convex and having a Schauder basis one only has to study the sequence spaces.

In this work we are going to establish a way of transforming topologies between this space and a perfect sequence space Λ, where

$\Lambda = \left\{ (\lambda_i)_i \in \omega : \sum_{i=1}^{\infty} \lambda_i e_i \; converges \; in \, (E, \tau) \right\}$ and $(e_i)_i$ is a Schauder basis of a locally $K-$convex space (E, τ) in question (K is a non-archimedean valuated field complete with a non trivial valuation). This study will allow us to solve the following problem:

(1) if $(e_i)_i$ is a Schauder basis of a locally $K-$ convex space (E, τ) ($lKcs$), determine the compatible topologies on E for which $(e_i)_i$ is an equicontinuous Schauder basis.

This problem was studied by many Mathematcians, in particular by De Grande-De Kimpe [10]. It is also proved in ([22], 3.2. see also [32], 2.1 and [12], 2.1) that in a $lKcs$ (E, τ) there exists the finest locally $K-$convex topology ν of countable type compatible with τ. The existence of this topology was also proved in ([21], proposition 2, p. 153). Thus, we are going to make, in the case of $lKcs$ (E, τ) such that $E_\sigma = (E, \sigma(E, E'))$ and $E'_\sigma = (E', \sigma(E', E))$ are sequentially complete, this topology in relation with the original topology of E, by distinguishing the following three cases: K is local, K is spherically complete and K is not spherically complete; which will give us a complete solution of this problem, we'll give a characterization of polar toplogies for which the weak basis problem is true in the case when K is not spherically complete. We should remind that the problem of the weak basis was formulated by several ways and that's one of them ([21], p. 150)

(2) Is every weak Schauder basis for E a Schauder basis for E ?

We shall say that for a $lKcs$ E the weak basis theorem holds if every weak Schauder basis in E is a Schauder basis .

In archimedean analysis, the weak basis theorem was first given for Banach spaces in 1932 by Banach ([3], p. 238) and extended to $(F)-$spaces by Bessaga and Pełczynski [5] $(a\,(F) - \textit{space is a complete, metrizable}$ topological vector space . McArthur [24] proved an analogue for bases of subspaces in Frechet spaces. Arsove and Edwards [1] proved that the answer is positive if E is a barrelled space. Singer shows by an example ([33], p. 153) that a weak basis need not be Schauder basis. Dubinsky and Retherford [15] observed that the answer is negative in general. Bennet and Cooper [4] proved it for strict $(LF)-$ spaces and Floret [17] for sequentially retractive $(LF)-$ spaces. M. De wilde [13] obtained a rather general result for bornological, sequentially complete and webbed spaces. W J. Stiles ([35], corollary 4.5, p. 413) showed that the theorem fails in non locally convex spaces l^p $(0 \prec p \prec 1)$. N. J. Kalton [20] gave a class of spaces for which this theorem is true. Joel H. Shapiro ([29], theorem 1, p. 1294) gave the following generalization of stiles'result

The weak basis theorem fails in every locally bounded non locally convex $(F)-$space which has a weak basis ; he also gave a wide class of space which the weak basis theorem fails and proved the same for the space H^p ($H^p :=$ the linear space of functions f analytic in the open unit disc $|z| \prec 1$ such that $\|f\|_p^p = \sup\limits_{0\leq r\prec 1} \int_0^{2\pi} \left| f\left(re^{it}\right)\right|^p dt \prec \infty$). Efimova [16] proved the weak basis theorem for regular inductive limits of a sequence of normed barrelled spaces. M. Valdivia has shown the result for metrizable barrelled spaces. J. Orihuela [27] gave a result which showed the linking between the weak basis theorem and the closed graph theorem. In [10] N. De Grande-De Kimpe solved completely the weak basis problem for locally convex space (lcs) E having a $\sigma\,(E',E)-$ sequentially complete topological dual E'.

In $n.a$ analysis, the weak basis problem has the following simple solution [21]: For a $lKcs$ E with a weak Schauder basis the weak basis theorem holds iff E is an Orlicz-Pettis-space (a space, where every weakly convergent sequence is convergent). We know that, if K is spherically complete then every $lKcs$ E is an $OP-$space; so the weak basis theorem holds in this case. If K is not spherically complete J. Kąkol and T.Gilsdorf ([21], corollary 6, p. 155) proved that the weak basis theorem holds if E is a polarly barrelled polar space (a locally $K-$convex space is called polarly barrelled if every closed, polar and absorbing absolutely $K-$convex subset of E is a zero-neighbourhood), they provided a wide class of non-polar spaces E with a weak Schauder basis which is a basic sequence in the original topology of E ([21], example 7, p. 155 and 156). Finally J. Kąkol and T.Gilsdorf remark that they do not know if the following result is true: Let E be a Banach space with a weak Schauder basis; then, E is a polar space iff every weak Schauder basis in E is a basic sequence ([21], remark 13, p. 160). This conjecture was set by P. Garcia and Schikhof ([28], p. 233) in the form of the following question: Does there exist a polar Banach space which is not $OP-$space, and does not contain l^{∞} ? . The answer of this question is negative; in the other words, the conjecture is true (see proposition 33).

In §. 2 we'll give general results that are related to $lKcs$, to polar toplogies of $\mathcal{A}-$convergence and to space of sequences. In §. 3 we study the perfect sequence spaces over K, we give a characterization of the natural topology noted Na with the sets absolutely $K-$convex, weakly bounded and compactoid if K is spherically complete, with the sets absolutely K-convex and compact if K is local and with the sets absolutely $K-$convex weakly bounded and $\Lambda-$closed (Λ is a perfect sequence space on K) when K is not spherically complete. We are interested in §. 4 in the study of transfer of topologies between a perfect sequence spaces and a $lKcs$ (E,τ) that has a Schauder basis, using the two following algebraic isomorphisms

$\Phi : E \longrightarrow \Lambda \quad x \quad \longmapsto (\lambda_i)_i$ and $\Psi : E' \longrightarrow \Delta \quad f \quad \longmapsto (\mu_i)_i$ for every $x = \sum_{i=1}^{\infty} \lambda_i e_i$ and $f = \sum_{i=1}^{\infty} \mu_i f_i$; where $(f_i)_i$ is the weak Schauder basis of E' associated to the Schauder basis $(e_i)_i$ of E ([9], lemma 3, p. 402) and Λ and Δ are two sequence spaces which we'll define like in [9]. This study will allow us to solve the problem (1) by distinguishing the three cases K is local, K is spherically complete and K is not spherically complete. Some results that are related to problem (2) are given in the §. 5 by considering a polar $lKcs$ (E,τ) which has a weak Schauder basis $(e_i)_i$ and as E_σ and E'_σ are sequentially complete, we characterize the finest compatible topology on E for which $(e_i)_i$ is a Schauder basis; this basis is necessary equicontinuous. Then, we give a necessary and sufficient condition which the topology τ must fulfill so as to admit $(e_i)_i$ as Schauder basis in the case when K is non spherically complete. Then we deduce a new characterization of $OP-$spaces.

Finally, in §. 6 we give applications to G-spaces and to $K-$barrelled spaces. We show that the result established by N. De Grande- De Kimpe in [10] in the classical case, for the barrelled spaces, is also true in the non archimedian case. For a $G-$space (E,τ) ; we show that τ is the only topology on E compatible with the duality $\langle E, E' \rangle$ for which $(e_i)_i$ is an equicontinuous Schauder basis.

2. Preliminaries

1. Throughout K is a non-archimedean (n.a) non trivially valued complete field with the valuation $|.|$, and the valuation ring is $B(0,1) = \{\lambda \in K : |\lambda| \leq 1\}$.

2. Let E be a $K-$vector space. A subset A of E is absolutely $K-$convex if it is $B(0,1)-$ module. For a set $X \subset E$ its absolutely $K-$convex hull $\Gamma(X)$ is the smallest absolutely $K-$convex set containing X.

3. A topology on a vector space E over K is said to be locally $K-$convex if there exists in E a fundamental system of zero-neighbourhoods consisting of absolutely $K-$convex subsets of E.

In this paper the letter E will always stand for Hausdorff locally $K-$convex space over a field K.

4. A subset A of E is called compactoid if for every zero-neighbourhood U in E, there exists a finite set $F \subset E$ such that $A \subset \Gamma(F) + U$.

5. A subset A of E is called c-compact if every convex filter on A has a cluster point on A.

- An absolutely $K-$convex subset of a locally $K-$convex space E is called K-closed if for every $x \in E$ the set $\{|\lambda| \,/\, \lambda \in K : \lambda x \in A\}$ is closed in $|K|$; the $K-$closed hull of A is the smallest subset of E which is K-closed and contains A, it is denoted by $K_c(A)$.

6. A sequence $(e_i)_i$ is a Schauder basis of E if every $x \in E$ can be written uniquely as $x = \sum\limits_{i=1}^{\infty} \lambda_i e_i$ where the coefficient functionals $f_n : x = \sum\limits_{i=1}^{\infty} \lambda_i e_i \longmapsto \lambda_n$ are continuous.

- The sequence $(f_n)_n$ is called the weak Schauder basis associated to basis $(e_i)_i$.

- For every $n \geq 1$, let p_n the map $x = \sum\limits_{i=1}^{\infty} \lambda_i e_i \longmapsto \lambda_n e_n$; the Schauder basis $(e_i)_i$ is called equicontinuous if the sequence $(p_n)_n$ is equicontinuous on E, this is equivalent to the equicontinuity of the sequence $(S_n)_n$ where $S_n : x = \sum\limits_{i=1}^{\infty} \lambda_i e_i \longmapsto \sum\limits_{i=1}^{n} \lambda_i e_i$, for every $n \geq 1$.

7. Let $\langle , \rangle$ be a duality between E and F where E and F are two vectors spaces over K (see [2] for general results);

- If A is a subset of E, the polar of A is a subset of F defined by:
$A^{\circ} = \{y \in F \,/\, |\langle x, y\rangle| \leq 1 \; for \; all \; x \in A\}$.

We define also the polar of a subset B of F in the same way.

- The weak topology $\sigma(E,F)$ on E is noted simply σ and $\{A^{\circ} \ / \ A \in \mathcal{F}\}$ is a zero-neighbourhood base, where $\mathcal{F}$ is the set of finite subset of F.

- A subset A of F is said to be $E-$ closed if for every $y \in F \setminus A$, there exists $x \in E$ such that $|\langle x,y\rangle| \succ 1$ and $|\langle x,A\rangle| \leq 1$; the $E-$closed hull $E_c(A)$ of A is the smallest $E-$closed subset of F containing A.

Proposition 1. *Let A be an absolutely $K-$convex subset of F, then A is $E-$ closed, if and only if, A is $K-$closed and $\sigma(F,E)-$ closed.*

Proof. By [2], theorem 4.2, p. 233, proposition 2.5, p. 224 and corollary 4.3, p. 233. ∎

8. Let $\mathcal{A}$ be a family of $\sigma(F,E)-$bounded subsets of F such that

(a) $\mathcal{A}$ is directed by inclusion,

(b) $F = \bigcup\limits_{A \in \mathcal{A}} A$,

(c) there exists $\lambda_0 \in K, |\lambda_0| > 1$, such that $\lambda_0 A \in A$, $for\ all\ A \in \mathcal{A}$.

A topology τ on E is called polar topology of $\mathcal{A}-$ convergence, if τ has a fundamental system of zero-neighbourhoods consisting of $\{A^{\circ}/\ A \in \mathcal{A}\}$

- A vector topology τ on E is called polar topology if there exists a family $\mathcal{A}$ of $\sigma(F,E)-$bounded subsets of F which has the properties (a), (b) and (c), such that τ is a polar topology of $\mathcal{A}-$convergence.

- If τ is a polar topology of $\mathcal{A}-$convergence on E, it is determined by the family of n.a seminorms $(p_A)_{A\in\mathcal{A}}$, where $p_A(x) = \sup\{|\langle x,y\rangle| \ / \ y \in A\}$([10], p. 277).

- If $\mathcal{A}$ is the family of all subsets of F that are:

1. Absolutely $K-$convex, weakly bounded and weakly c-compacts, we have the c-compact topology $\tau_c(E,F) = \tau_c$,

2. Absolutely convex and $\sigma(F,E)-$compact, we have the Mackey topology $\tau_m(E,F) = \tau_m$,

3. $\sigma(F,E)-$bounded and $E-$closed, we have the $E-$ closed topology $\tau_e(E,F) = \tau_e$.

9. A locally $K-$convex topology τ on E is called compatible with the duality $\langle E,F\rangle$ or $(E,F)-$ compatible if, F is isomorphic to the topological dual of E provided with the topology τ. $\sigma(E,F)$ is the smallest of $(E,F)-$ compatible topology.

- A sequence $(e_n,f_n)_n$ of $E \times F$ is called biorthogonal if $\langle e_n,f_n\rangle = \delta_{nm}$, for all n,m where δ_{nm} is the Kronecker delta.

10. The space of all sequences in K is denoted by ω, it is provided with the product topology τ_ω. A linear subspace of ω is called a sequence space.

φ, c_0 and l^{∞} are respectively, the space of all sequences in K with only finitely many non-zero terms, the space of the sequences in K converging to zero and the space of the bounded sequences in K.

- for all $n \geq 1$, $e^n = (\delta_{nm})_m$.

- Let $A \subset \omega$, the $\beta-$dual of A is the subset A^{β} of ω defined as

$$A^{\beta} = \left\{ \lambda = (\lambda_i) \in \omega \; / \lim_i \lambda_i \alpha_i = 0, \; for \; all \; \alpha = (\alpha_i)_i \in A \right\}.$$

- A is called $\beta-$perfect (or perfect) if $A = A^{\beta\beta}$.

- A is solid if whenever $(a_n)_n \in A$ *and* $(\lambda_n)_n \in \omega$ such that $|\lambda_n| \leq 1$ for each n, then $(\lambda_n a_n)_n \in A$. The spaces ω, φ and c_0 are solids.

- The smallest solid subset of ω containing A is called the solid hull of A, it is denoted by $\hat{A}$; and we have

$$\hat{A} = \left\{ (\lambda_n a_n)_n \, / \, (a_n)_n \in A \; and \, (\lambda_n)_n \in \omega : |\lambda_n| \leq 1 \; for \; all \; n \geq 1 \right\}.$$

- Let X be a sequence space in K; $A \subset X$ is called solid in X if $\hat{A} \cap X = A$. $\hat{A} \cap X$ is called the solid hull of A in X.

- A topology on a vector space X is called solid if there exists in X a fundamental system of zero neighbourhoods consisting of solids subsets in X.

11. A G-space is a locally $K-$convex space (E, τ) such that E' is $\sigma(E', E)-$ sequentially complete and $\tau = \tau_c$ $(resp. \tau_e, \tau_m)$ if K is spherically complete, (resp. not spherically complete, local). In the last case, we find the notion of G-space given and studied by N. De Grande-De Kimpe in [11] in the classical case ($K = \mathbb{R}$ *or* $K = \mathbb{C}$).

3. The natural topology in a perfect sequence spaces

Let Λ be a sequence space over K containing φ, we consider the duality $\left\langle \Lambda, \Lambda^{\beta} \right\rangle$ defined by: $((\lambda_n)_n, (\mu_n)_n) \longmapsto \langle (\lambda_n)_n, (\mu_n)_n \rangle = \sum_{n=1}^{\infty} \lambda_n \mu_n$ for every $(\lambda_n)_n \in \Lambda$ and $(\mu_n)_n \in \Lambda^{\beta}$.

For every $\mu = (\mu_n)_n \in \Lambda^{\beta}$, let $\hat{p}_{\mu}$ the n.a seminorm defined as $\hat{p}_{\mu}(\lambda) = \sup_n |\lambda_n \mu_n|$, For every $\lambda = (\lambda_n)_n \in \Lambda$.

We call the locally $K-$convex topology on Λ determined by the family of seminorms $(\hat{p}_{\mu})_{\mu \in \Lambda^{\beta}}$ the natural topology; it will be denoted by Na.

Remark 1. *The weak topology σ on Λ is weaker than the natural topology Na.*

Proposition 2. *If Λ is perfect, then it is weakly sequentially complete.*

Proof. $([25]\,,\ 5.2,\ p.\ 1550)\,.$ ∎

Proposition 3. *If Λ is perfect then every $\sigma-$bounded subset of Λ is τ_b-bounded, where τ_b is the strong topology $\tau_b\left(\Lambda,\Lambda^\beta\right)$ on Λ.*

Proof. $([8]\,,\ proposition\ 8,\ p.\ 476)\,.$ ∎

Corollary 1. *If Λ is perfect, all polars topologies on Λ yield the same bounded sets.*

Lemma 1. *The solid hull of a finite subset of Λ^β is $\sigma-$bounded.*

Proof. Obvious. ∎

Lemma 2. *Let A be a $\sigma-$bounded and solid subset of Λ^β, then the polar of A in the duality $\left\langle\Lambda,\Lambda^\beta\right\rangle$ is given by $A^\circ=\{\lambda\in\Lambda\ /\hat{p}_\mu(\lambda)\leq 1,\ for\ all\ \mu\in A\}\,.$*

Proof. $([8]\,,\ proposition\ 1,\ p.\ 472)\,.$ ∎

Proposition 4. *The natural topology on Λ is a polar topology.*

Proof. Obvious. ∎

Remark 2. *The natural topology is a solid topology; in fact it is the coarsest of the polar and solid topologies on Λ.*

Proposition 5. *If Λ is perfect, the natural topology Na is compatible with the duality $\left\langle\Lambda,\Lambda^\beta\right\rangle$.*

Proof. The Na topology is polar and for every $\mu\in\Lambda^\beta$, $\left(\{\hat{\mu}\}\right)^\circ=\left[K_c\left(\overline{\{\hat{\mu}\}}^{\sigma\left(\Lambda^\beta,\Lambda\right)}\right)\right]^\circ$ [2], corollary 4.3, p. 233. Then, if we take $\mathcal{A}=\left\{K_c\left(\overline{\hat{A}}^{\sigma\left(\Lambda^\beta,\Lambda\right)}\right)\ /\ A\subset\Lambda^\beta \text{ and A is finite}\right\}$ so Na is a polar topology of $\mathcal{A}-$convergence, where $\mathcal{A}$ is formed by a $\sigma\left(\Lambda^\beta,\Lambda\right)-$ bounded and $\Lambda-$closed subsets of Λ^β (proposition 1). Then by [2], theorem 4.3, p. 233 the natural topology Na is compatible. ∎

Remark 3. *For every $\mu\in\Lambda^\beta$, $\overline{\Gamma\left(\{\hat{\mu}\}\right)}$ is weakly-c-compact if K is spherically complete and weakly compact if K is local.*

Proposition 6. *If Λ is perfect, then it is complete under any polar solid topology.*

Proof. Let τ be a solid polar topology of $\mathcal{A}-$convergence on Λ; we consider $(\lambda^i)_{i\in I}$ as a Cauchy-net in (Λ, τ).

Let $A \in \mathcal{A}$, there exists $i_0 \in I$ such that $\lambda^i - \lambda^j \in A^\circ$, *for all* $i,\ j \geq i_0$; so we have (1) $\sup\limits_{\alpha=(\alpha_n)_n\in A} \sup\limits_n \left|\alpha_n \left(\lambda_n^i - \lambda_n^j\right)\right| \leq 1$, for all $i, j \geq i_0$ (*lemma* 2).

Then for every n, $(\lambda_n^i)_{i\in I}$ is a Cauchy-net in K, so there exists $\lambda_n \in K$ such that $\lambda_n = \lim\limits_i \lambda_n^i$. Therefore, from (1) we obtain:

(2) $\sup\limits_{\alpha=(\alpha_n)_n\in A} \sup\limits_n \left|\alpha_n \left(\lambda_n^i - \lambda_n\right)\right| \leq 1$, for all $i \geq i_0$.

Let $\alpha = (\alpha_n)_n \in \Lambda^\beta$, there exists $A \in \mathcal{A}$ such that $\alpha \in A$ and we have for all $n \geq 1, |\alpha_n \lambda_n| \leq \max\left(|\lambda_n^{i_0}\alpha_n|, |(\lambda_n^{i_0} - \lambda_n)\,\alpha_n|\right)$.

Hence $\lambda = (\lambda_n)_n \in \Lambda^{\beta\beta} = \Lambda$ (Λ *is perfect*) and by (2) we have $\lambda = \lim\limits_i \lambda^i$ in (Λ, τ). ∎

Proposition 7. *Let A be a subset of Λ ; if A is $Na-$bounded then $\hat{A}$ is $Na-$bounded.*

Proof. Obvious. ∎

Proposition 8. *Suppose that Λ is perfect and let τ be a polar topology on Λ and A be a subset of Λ. If A is $\tau-$bounded, then $\hat{A}$ is $\tau-$bounded.*

Proof. A is $\tau-$bounded $\Rightarrow$ A is $Na-$bounded (corollary 1 and proposition 4) $\Rightarrow$ $\hat{A}$ is $Na-$bounded (proposition 7) $\Rightarrow$ $\hat{A}$ is $\tau-$bounded (corollary 1 and proposition 4). ∎

Corollary 2. *τ_b is a solid topology.*

Proof. Λ^β is perfect, then for every $A \subset \Lambda^\beta$, A is $\sigma-$bounded $\Longleftrightarrow$ $\hat{A}$ is $\sigma-$bounded. ∎

Lemma 3. *Let E and F be a locally $K-$convex spaces and A a compactoid subset of E. If $(f_n)_{n\geq 1}$ is an equicontinuous sequence of linear mappings from E to F pointwise converging to a mapping f, then $(f_n)_{n\geq 1}$ converges to f uniformly on A.*

Proof. ([8], *proposition* 13, *p.* 477). ∎

Remark 4. *The lemma before is true if we replace compactoid by pre-compact (if K is local) or bounded and c-compact (if K is spherically complete).*

Proposition 9. *Let A be a compactoid subset of (Λ, Na). Then for every $\alpha = (\alpha_n)_n \in \Lambda^\beta, \lim_k |\alpha_k| \sup_{\lambda \in A} |\lambda_k| = 0$.*

Proof. Let $\alpha = (\alpha_n)_n \in \Lambda^\beta$; for every $n \in I\!N$, we consider $\alpha^n = \alpha_n e^n$ with $e^n = (\delta_{nm})_m$. Then for every $\mu = (\mu_n)_n \in \Lambda$, $\hat{p}_\mu(\alpha^n) = |\mu_n \alpha_n| \overset{n \to \infty}{\longrightarrow} 0$, so $\lim_{n\to\infty} \alpha^n = 0$ in $\left(\Lambda^\beta, Na\right)$. On the other hand $(\alpha^n)_n$ is $Na-$equicontinuous ([8], proposition 3, p. 474); then according to lemma before $(\alpha^n)_n$ converges to 0 uniformly on A. ∎

Remark 5. *Suppose that K is spherically complete and let τ be a locally $K-$convex topology compatible with the duality $\left\langle \Lambda, \Lambda^\beta \right\rangle$ on Λ and A an absolutely $K-$convex bounded and c-compact subset of Λ in (Λ, τ). Then for every $\alpha \in \Lambda^\beta$, $\lim_k |\alpha_k| \sup_{\lambda\in A} |\lambda_k| = 0$.*

Proof. Remark 4, proposition 5, [36] theorem 4.21 and [7] proposition 3. ∎

Proposition 10. *The sequence $(e^n)_n$ is a Schauder basis of (Λ, Na).*

Proof. Let $\lambda = (\lambda_n)_n \in \Lambda$, then for every $\mu = (\mu_n)_n \in \Lambda^\beta, \hat{p}_\mu(\lambda_i e^i) = |\lambda_i \mu_i| \overset{i\to\infty}{\longrightarrow} 0$. Therefore $\sum_i \lambda_i e^i$ converges in (Λ, Na) and so every element $\lambda = (\lambda_n)_n \in \Lambda$ can be written uniquely as $\lambda = \sum_{n=1}^{\infty} \lambda_n e^n$. On the other hand, for every $n \in I\!N$, $e^n \in \Lambda^\beta$ and we have $\hat{p}_{e^n}(\lambda) = |\lambda_n|$ *for all* $\lambda = (\lambda_n)_n \in \Lambda$, hence the maps $\lambda = \sum_{i=1}^{\infty} \lambda_i e^i \longmapsto \lambda_n$ is $Na-$continuous. ∎

Theorem 1. *Suppose that K is spherically complete and Λ is perfect. A subset A of Λ is compactoid in Λ_{Na} if, and only if, it is a subset of the solid hull of a singleton of Λ.*

Proof. $\Longrightarrow$] Let A be a compactoid subset of Λ_{Na}.

Let $\varrho > 1$ and $\lambda = (\lambda_n)_n \in \omega$ such that $|\lambda_n| = \varrho^n$ for all $n \geq 1$.

A is compactoid in Λ_{Na}, so it is $Na-$bounded, and therefore $\sup\limits_{(\alpha_i)_i\in A} |\alpha_i| < +\infty$, for all $i \geq 1$, there exists $n_i \geq 1$ such that $\varrho^{n_i-1} \leq \sup\limits_{(\alpha_i)\in A} |\alpha_n| \leq \varrho^{n_i}$.

Let $\mu = (\mu_i)_i$ the element of ω given by $\mu_i = \lambda_{n_i}$ for all $i \geq 1$, then for every $\alpha = (\alpha_i)_i \in \Lambda^\beta$ we have: for all $i \geq 1$,

$$|\mu_i\alpha_i| \leq |\alpha_i| \left(\varrho^{n_i} - \sup_{(\gamma_i)\in A} |\gamma_i|\right) + |\alpha_i| \sup_{(\gamma_i)\in A} |\gamma_i| \leq \varrho$$

$|\alpha_i| \sup\limits_{(\gamma_i)\in A} |\gamma_i|$.

Now, $\lim\limits_i |\alpha_i| \sup\limits_{(\gamma_i)_i\in A} |\gamma_i| = 0$ (*proposition* 9), hence $\lim\limits_i \mu_i\alpha_i = 0$. Then $\mu \in \Lambda^{\beta\beta} = \Lambda^{\cdot}$ On the other hand, if $\alpha = (\alpha_i)_i \in A$, we have $|\alpha_i| \leq |\mu_i|$, for every $i \geq 1$, hence $\alpha \in \{\hat{\mu}\}$.

$\Longleftarrow$] It suffices to prove that $\{\hat{\lambda}\}$ is compactoid in Λ_{Na}, for every $\lambda \in \Lambda$.

Let $\lambda = (\lambda_n)_n$ an element of Λ, then for every $\alpha = (\alpha_n)_n$ in Λ^β, there exists $n_0 \in \mathbb{N}$ such that $|\lambda_n\alpha_n| \leq 1$ *for all* $n > n_0$. We put $\lambda^i = \lambda_i\ e^i$, *for all* i , $1 \leq i \leq n_0$.

If $\mu = (\mu_i\lambda_i)_i$ is an element of $\{\hat{\lambda}\}$, $\mu = \sum\limits_{i=1}^{\infty} \mu_i\lambda_i e^i = \sum\limits_{i=1}^{n_0} \mu_i\lambda^i + \sum\limits_{i>n_0} \mu_i\lambda_i e^i$.

Now, $\hat{p}_\alpha \left(\sum\limits_{i>n_0} \mu_i\lambda_i e^i\right) = \sup\limits_{i>n_0} |\mu_i\lambda_i\alpha_i| \leq 1$

So $\mu \in \Gamma(\lambda^1, \lambda^2, ..., \lambda^{n_0}) + B_{\hat{p}_\alpha}(0,1)$, where $B_{\hat{p}_\alpha}(0,1) = \{\lambda \ / \ \hat{p}_\alpha(\lambda) \leq 1\}$.

Then $\{\hat{\lambda}\} \subset \Gamma(\lambda^1, \lambda^2, ..., \lambda^{n_0}) + B_{\hat{p}_\alpha}(0,1)$. ∎

Remarks 1. *1. N. De Grande-De Kimpe gave an analogue proposition of theorem in which she characterize the weakly c-compact subsets ([8], proposition 15, p. 478). This proposition is in fact true for all compatible topologies with the duality $\left\langle \Lambda, \Lambda^\beta \right\rangle$ in particular for the natural topology Na.*

2. $K_c\left(\overline{\{\hat{\mu}\}}^{\sigma(\Lambda,\Lambda^\beta)}\right)$ *is compactoid for every field K.*

Consequently the solid hull of a bounded subset which is absolutely $K-$ convex and c-compact of Λ is also bounded and c-compact for every compatible topology.

Corollary 3. *i. For every $\lambda \in \Lambda^\beta, \{\hat{\lambda}\}$ is compactoid in $\left(\Lambda^\beta, Na\right)$ for every field K.*

ii. Suppose that Λ is perfect and K is spherically complete and let A be a subset of Λ. Then A is compactoid in Λ_{Na} if, and only if, there exists a sequence $(\alpha^n)_n$ converging to zero in Λ_{Na} such that $A \subset \overline{\Gamma(\alpha^1, \alpha^2, ..., \alpha^n, ...)}$.

Proof. i. Λ^β is perfect; it suffices to use the theorem 1.

ii. Let $\lambda = (\lambda_n)_n \in \Lambda$ such that $A \subset \{\hat{\lambda}\}$ $(theorem\ 1)$. We put $\alpha^n = \lambda_n e^n$ for all $n \geq 1$, then for every $\mu = (\mu_n)_n \in \Lambda^\beta$ we have $\hat{p}_\mu(\alpha^n) = |\mu_n \lambda_n| \overset{n\to\infty}{\longrightarrow} 0$, so the sequence $(\alpha^n)_n$ is converging to zero in Λ_{Na}. On the other hand for every $a = (a_n)_n \in A$, $a \in \{\hat{\lambda}\}$, therefore there exists $(\mu_n)_n \in \omega$ such that $|\mu_n| \leq 1$ for all n and $a = (\mu_n \lambda_n)_n$; so $a = \sum_{n=1}^{\infty} \mu_n \alpha^n$(proposition 10). Then $a \in \overline{\Gamma(\alpha^1, \alpha^2, ..., \alpha^n, ...)}$.

Conversely, suppose that $A \subset \overline{\Gamma(\alpha^1, \alpha^2, ..., \alpha^n, ...)}$ where $(\alpha^n)_n$ converges to zero in Λ_{Na}. Let U be a zero-neighbourhood in Λ_{Na}, then there is $n_0 \in I\!N$ such that for all $n > n_0$, $\alpha^n \in U$. So $\Gamma(\alpha^1, \alpha^2, ..., \alpha^n, ...) \subset \Gamma(\alpha^1, \alpha^2, ..., \alpha^{n_0}) + U$, (we can choose U absolutely $K-$convex and open). Then $\overline{\Gamma(\alpha^1, \alpha^2, ..., \alpha^n, ...)} \subset \Gamma(\alpha^1, \alpha^2, ..., \alpha^{n_0}) + U$, because $\Gamma(\alpha^1, \alpha^2, ..., \alpha^{n_0}) + U$ is $Na-$ closed.

Characterization of the natural topology a- K is spherically complete

Theorem 2. *If Λ is perfect, then the natural topology on Λ is the polar topology of $\mathcal{A}-$convergence, where $\mathcal{A}$ is the family of all compactoid subsets of $\left(\Lambda^\beta, Na\right)$.*

Proof. Let $\mathcal{A}$ be the family of all compactoid subsets of $\left(\Lambda^\beta, Na\right)$ then for every $A \in \mathcal{A}$, A is $\sigma-$ *bounded and* $\mathcal{A}$ satisfies the conditions $(a), (b)$ *and* (c) of 8. Let τ be the polar topology of $\mathcal{A}-$ convergence on Λ; then $\tau = Na$, (theorem 1) . ∎

Theorem 3. *If Λ is perfect, then the natural topology on Λ is the polar topology of $\mathcal{A}-$convergence where $\mathcal{A}$ is the family of all absolutely $K-$convex, bounded and c-compact subsets of $\left(\Lambda^\beta, Na\right)$.*

Proof. The same as theorem 2 and apply remarks 1. ∎

Remark 6. *If K is spherically complete and Λ is perfect then $Na = \tau_c$; where τ_c is the c-compact topology on Λ. And for every topology τ on Λ, τ is compatible with the duality $\left\langle \Lambda, \Lambda^\beta \right\rangle$ if, and only if, $\sigma \leq \tau \leq Na$.*

b- K is local

If K is local, then [31], proposition 1 and [7], proposition 2, p. 177 induce that all results before still hold when the word absolutely K-convex, bounded and c-compact (or compactoid) is replaced by absolutely $K-$convex and compact; and the characterization of the natural topology became:

Theorem 4. *If Λ is perfect, then the natural topology on Λ is the polar topology of $\mathcal{A}-$convergence where $\mathcal{A}$ is the family of all absolutely $K-$ convex and compact subsets of $\left(\Lambda^{\beta}, Na\right)$.*

c- K is not spherically complete

Theorem 5. *If Λ is perfect, then the natural topology on Λ is the polar topology of $\mathcal{A}-$convergence where*
$\mathcal{A}=\left\{K_c\left(\overline{\widetilde{A}}^{\sigma(\Lambda^{\beta},\Lambda)}\right) / \ A \subset \Lambda^{\beta} \text{ and } A \text{ is finite}\right\}.$

Proof. By proposition 5. ∎

4. Locally $K-$convex spaces with a Schauder basis and perfect sequence spaces

Let (E,τ) be a locally $K-$convex space where τ is a polar topology of $\mathcal{A}-$convergence, $(e_i)_i$ be a Schauder basis of (E,τ) and $(f_i)_i$ the associated weak Schauder basis. If $S_n(x) = \sum_{i=1}^{n} \lambda_i e_i$ and $T_n(f) = \sum_{i=1}^{n} \mu_i f_i$ for all $x \in E$, all $f \in E^{'}$ (the topological dual of E) and all $n \geq 1, \langle S_n(x), f\rangle = \langle x, T_n(f)\rangle$, *for all* $n \geq 1$, $x \in E$ *and* $f \in E^{'}$. For every $A \subset E^{'}$, we put $\widetilde{A} = \{T_n(a) \ / \ n \in \mathbb{N};\ a \in A\}$ with $T_0 = id_{E'}$ and for every $A \subset E$ we put $S(A) = \{x \in A / S_n(x) \in A,\ for\ all\ n \geq 1\}$. We define also $\widetilde{A}$ for $A \subset E$ and $S(A)$ for $A \subset E^{'}$. N. De Grande-De Kimpe has defined the topology $\widetilde{\tau}$ of $\widetilde{\mathcal{A}}-$convergence where $\widetilde{\mathcal{A}} = \left\{\widetilde{A} / \ A \in \mathcal{A}\right\}$, and she gave a characterization of this topology ([10], proposition 1.2, p. 278). We enhance this result in theorem 6, p. 19.

Remarks 2. *1.* $\widetilde{p}_A(x) = \sup_n p_A\left(\sum_{i=1}^{n} \langle x, f_i\rangle e_i\right)$ *for all* $A \in \mathcal{A}$ *for all* $x \in E$; *in the case where* $F = E^{'}$, $p_{\widetilde{A}}(x) = \widetilde{p}_A(x) = \sup_n p_A(S_n(x)) = \sup_n p_A(f_n(x).x_n)$ *([10], proposition 1.1, p. 278).*

2. The $\widetilde{\sigma}-$ topology associated to the weak topology $\sigma = \sigma\left(E, E^{'}\right)$ on E is defined by the family of seminorms n.a $(p_f)_{f\in E'}$, where $p_f(x) = \sup_n |\langle S_n(x), f\rangle|$, for every $x \in E$ and $f \in E^{'}$; and we have $\widetilde{\sigma} \leq \widetilde{\tau}$.

Example 1. *Let Λ be a perfect sequence space over K. The topology $\widetilde{\sigma} = \widetilde{\sigma}\left(\Lambda, \Lambda^{\beta}\right)$ associated to $\sigma = \sigma\left(\Lambda, \Lambda^{\beta}\right)$ is defined by the family of seminorms n.a $(p_\mu)_{\mu\in\Lambda^\beta}$, where $p_\mu(\lambda) = \sup_n \left|\sum_{i=1}^{n} \lambda_i\mu_i\right| = \sup_n |\lambda_n\mu_n|$, for every $\lambda = (\lambda_i)_i \in \Lambda$ and $\mu = (\mu_i)_i \in \Lambda^{\beta}$. Then the topology $\widetilde{\sigma}$ is exactly the natural topology studied in §. 3.*

We Consider the two linear mappings $\Phi : E \longrightarrow \Lambda,\ \ x = \sum_{i=1}^{\infty} \lambda_i e_i \longmapsto (\lambda_i)_i$ and $\Psi : E^{'} \longrightarrow \Delta,\ \ f = \sum_{i=1}^{\infty} \mu_i f_i \longmapsto (\mu_i)_i$; where Λ and Δ are the sequence spaces defined as $\Lambda = \left\{ (\lambda_i)_i \in \omega \ / \ \sum_{i=1}^{\infty} \lambda_i e_i \textit{ converges in } (E, \tau) \right\}$ and
$\Delta = \left\{ (\mu_i)_i \in \omega \ / \ \sum_{i=1}^{\infty} \mu_i f_i \textit{ converges in } E^{'}_{\sigma} \right\}$. Φ *and* Ψ are algebraic isomorphisms.

Proposition 11. $\Lambda \subset \Delta^{\beta}$ *and* $\Delta \subset \Lambda^{\beta}$.

Proposition 12. *i.* Φ *is* $\left(\sigma\left(E, E^{'}\right), \sigma(\Lambda, \Delta)\right) -$ *continuous;*
ii. Ψ *is* $\left(\sigma\left(E^{'}, E\right), \sigma(\Delta, \Lambda)\right) -$ *continuous.*

Proof. i. Let $\mu \in \Delta$, we consider $V = \{\lambda = (\lambda_i)_i \in \Lambda / |\langle \lambda, \mu\rangle| \leq 1\}$. We put $f = \psi^{-1}(\mu)$, then $f \in E^{'}$. We Consider $U = \{\ x \in E \ / |\langle x, f\rangle| \leq 1\}$; U is a zero neighbourhood in $\left(E, \sigma\left(E, E^{'}\right)\right)$, and we have
$\Phi(U) = \{\lambda = \Phi(x) \in E / |\langle x, f\rangle| \leq 1\} \ = V$.

If $x = \sum_{i=1}^{\infty} \lambda_i e_i$ *and* $\mu = (\mu_i)_i$, then $\Phi(x) = (\lambda_i)_i$ and $f = \sum_{i=1}^{\infty} \mu_i f_i$. Therefore $\langle x, f\rangle = \sum_{i=1}^{\infty} \lambda_i\mu_i = \langle \Phi(x), \mu\rangle$.

ii. Same proof as for i. ∎

Proposition 13. *If Φ^* and Ψ^* are the algebraic adjoints of Φ and Ψ respectively, then $\Phi^* = \Psi^{-1}$ and $\Psi^* = \Phi^{-1}$.*

Proof. Φ^* take his values in $E^{'}$ ([30], p. 128). For every $x \in E$ and $\mu \in \Delta$ we have $\langle x, \Phi^*(\mu)\rangle = \langle \Phi(x), \mu\rangle = \sum_{i=1}^{\infty} \lambda_i \mu_i$, where $x = \sum_{i=1}^{\infty} \lambda_i e_i$ and $\mu = (\mu_i)_i$. So $\langle x, \Phi^*(\mu)\rangle = \langle x, \Psi^{-1}(\mu)\rangle$. Then $\Phi^* = \Psi^{-1}$.

The same for $\Psi^* = \Phi^{-1}$. ∎

Proposition 14. a. *For every $A \subset E, (\Phi(A))^{\circ} = \Psi(A^{\circ})$;*

b. *For every $B \subset E^{'}, (\Psi(B))^{\circ} = \Phi(B^{\circ})$.*

Proof. a. Let $A \subset E$, then $(\Phi(A))^{\circ} = (\Phi^*)^{-1}(A^{\circ})$ [[2], proposition 2.8, p. 225].

Now $\Phi^* = \Psi^{-1}$ (proposition 13), so $(\Phi(A))^{\circ} = (\Psi^{-1})^{-1}(A^{\circ}) = \Psi(A^{\circ})$.

b. The same proof. ∎

The topology τ_{Φ} defined on Λ by Φ has a zero-neighbourhood base consisting of the family $(\Phi(A^{\circ}))_{A\in\mathcal{A}}$ ([6], II. 29), τ_{Φ} is a polar topology of $\Psi(\mathcal{A})$ $-$convergence, where $\Psi(\mathcal{A}) = \{\Psi(A) \,/\, A \in \mathcal{A}\}$ (*proposition* 14).

Examples 1. *1. If we consider the space E_{σ}, then the topology σ_{Φ} has a zero-neighbourhood base the set $\left\{(\psi(A))^{\circ} \,/\, A \subset E^{'} \text{ and } A \text{ is finite}\right\}$.*

For $A = \left\{(f^i)_{1\leq i\leq n} \,/\, f^i \in E^{'}\right\}$, put $f^i = \sum_{j=1}^{\infty} \mu_j^i\, f_j$, for every i, $1 \leq i \leq n$; $(\Psi(A))^{\circ} = \left(\left\{\left(\mu_j^i\right)_{j\geq 1}, \ 1 \leq i \leq n\right\}\right)^{\circ}$. Hence σ_{Φ} is exactly the weak topology.

2. Let τ be a polar topology of $\mathcal{A}-$convergence on E and $\widetilde{\tau}$ the associated polar topology, then $\widetilde{\tau}_{\Phi} = (\widetilde{\tau})_{\Phi}$ has $\left[\left(\Psi\left(\widetilde{A}\right)\right)^{\circ}\right]_{A\in\mathcal{A}}$ as a zero-neighbourhood base. For every $A \in \mathcal{A}$, $\Psi\left(\widetilde{A}\right) = \widetilde{\Psi(A)}$, then $\widetilde{\tau}_{\Phi}$ is defined by the family of n.a seminorms $\left(\widetilde{p}_{\Psi(A)}\right)_{A\in\mathcal{A}}$, where :

$$\widetilde{p}_{\Psi(A)}\left((\lambda_i)_{i\geq 1}\right) = \sup_n \ p_{\psi(A)}(\lambda_1, ..., \lambda_n, 0, ...) = \sup_n \ \sup_{\mu=(\mu_i)\in A} \left|\sum_{i=1}^{n} \lambda_i \mu_i\right| = \sup_{\mu=(\mu_i)\in A} \widetilde{p}_{\mu}\left((\lambda_i)_{i\geq 1}\right).$$

3. The direct image topology of $\widetilde{\sigma}$ *with* Φ *on* Λ, *noted* $\widetilde{\sigma}_{\Phi}$, *is defined with the family of semi-norms n.a* $\left(\widetilde{p}_{\Psi(f)}\right)_{f\in E'}$, *where* $\widetilde{p}_{\Psi(f)}\left((\lambda_i)_i\right) = \widetilde{p}_{\mu}\left((\lambda_i)_i\right),\ (\lambda_i)_i \in \Lambda$ *and* $\mu = \Psi(f)$. *Then* $\widetilde{\sigma}_{\Phi}$ *is exactly the natural topology on* Λ.

Some properties of the topology $\widetilde{\sigma}$

Lemma 4. *i. If* E_σ *is sequentially complete then* $\Lambda = \Delta^{\beta}$;
ii. If E'_σ *is sequentially complete then* $\Delta = \Lambda^{\beta}$;
iii. If E_σ *and* E'_σ *are sequentially complete then* Λ *is perfect.*

Proof. i. $\Lambda \subset \Delta^{\beta}$ (proposition 11).

Let $\lambda = (\lambda_i)_i$ an element of Δ^{β}, so $\lambda = \lim_{n\to\infty} \sum_{i=1}^{n} \lambda_i e^i$ in $\left(\Delta^{\beta}\sigma\left(\Delta^{\beta}\Delta^{\beta\beta}\right)\right) = \Delta^{\beta}_{\sigma}$ where $e^i = (\delta_{ij})_j$ *for all* $i \geq 1$, (Δ^{β} is perfect). Then $(e^i)_i$ is a Schauder basis of Δ^{β}_{σ} (propositions 5, 10 and remark 1), so $\lambda = \lim_{n\to\infty} \sum_{i=1}^{n} \lambda_i e^i$ in $\left(\Delta^{\beta}, \sigma\left(\Delta^{\beta}, \Delta\right)\right)$. $\left(\sum_{i=1}^{n} \lambda_i e^i\right)_n$ is a Cauchy-sequence in $(\Lambda, \sigma(\Lambda, \Delta))$ which is sequentially-complete (examples 1. 1), then $\lambda = \lim_{n\to\infty} \sum_{i=1}^{n} \lambda_i e^i$ in $(\Lambda, \sigma(\Lambda, \Delta))$ and so $\lambda \in \Lambda$.

ii. $\Delta \subset \Lambda^{\beta}$ (proposition 11). Let $\lambda = (\lambda_i)_i \in \Lambda^{\beta}$, for every $x \in E$, there exists $\alpha = (\alpha_i)_i \in \Lambda$ such that $x = \sum_{i=1}^{\infty} \alpha_i e_i$. for all $n \geq 1$, $\lambda_n f_n(x) = \lambda_n \alpha_n \overset{n\to\infty}{\longrightarrow} 0$; then $\sum_{j\geq 1} \lambda_j f_j$ is convergent in E'_σ; and so $\lambda = (\lambda_i)_i \in \Delta$. ∎

Proposition 15. *i. If* E_σ *is sequentially complete then* $E_{\widetilde{\sigma}}$ *is complete;*
ii. If E'_σ *is sequentially complete then* $E'_{\widetilde{\sigma}}$ *is complete;*
where $E_{\widetilde{\sigma}} = \left(E, \widetilde{\sigma}\left(E, E'\right)\right)$ *and* $E'_{\widetilde{\sigma}} = \left(E', \widetilde{\sigma}\left(E', E\right)\right)$.

Proof. i. Suppose that $E_\sigma$is sequentially complete, then by lemma 4 $\Lambda = \Delta^{\beta}$ and $\left(\Lambda, \widetilde{\sigma}\left(\Lambda, \Delta^{\beta\beta}\right)\right)$ is complete (proposition 6 and remark 2), then $(\Lambda, \widetilde{\sigma}(\Lambda, \Delta))$ is complete and $E_{\widetilde{\sigma}}$ is also complete (examples 1. 3).
ii. Same proof as for i. ∎

Proposition 16. *If* E_σ *is sequentially complete and has an equicontinuous Schauder basis, then* E *is isomorphic to a closed subspace of some power of* K.

Proof. Suppose that E_σ is sequentially complete and admits an equicontinuous Schauder basis, then the topologies $\sigma\left(E,E^{'}\right)$and $\widetilde{\sigma}\left(E,E^{'}\right)$ coincides on E, then by proposition 15 $\left(E,\sigma\left(E,E^{'}\right)\right)$ is complete; and the proposition follows ([7], proposition 7, p. 179). ∎

Proposition 17. *If E_σ and E'_σ are sequentially complete and E has a weak Schauder basis $(e_i)_{i\geq 1}$, then $\widetilde{\sigma}$ is the smallest compatible topology on E for which $(e_i)_i$ is an equicontinuous Schauder basis.*

Proof. E'_σ is sequentially complete, then $\Delta=\Lambda^\beta$ (lemma 3.1) and so $\left(\Lambda,\widetilde{\sigma}\left(\Lambda,\Lambda^\beta\right)\right)'=\Delta$, since$\left(\Lambda,\widetilde{\sigma}\left(\Lambda,\Lambda^\beta\right)\right)'=\Lambda^\beta$. Consequently $(E,\widetilde{\sigma}(E,E'))'$ $=E'$ (example 1. 3) and the result follows from propositions 15 and [10], proposition 1.2, p. 278. ∎

Compatibility of the $\widetilde{\tau}-$topology

We establish the compatibility and the completeness of $\widetilde{\tau}$ by distinguishing the three cases: K is local, K is spherically complete and K is non spherically complete.

a. K **is local**

Lemma 5. *Let E be a topological vector space with an equicontinuous Schauder basis $(e_i)_i$; then for every $A\subset E$ the following are equivalent:*

a. A is precompact;

b. (i). for all $i\geq 1$, $p_i(A)$ is precompact and (ii). $\left(\sum\limits_{i=1}^{n}p_i\right)_n$ converges uniformly on A. Where the p_n are defined in §. 2.

Proof. a $\Longrightarrow$ b For every $i\geq 1$ $p_i(A)$ is precompact (p_i is continuous). On the other hand, the sequence of linear mappings $\left(\sum\limits_{i=1}^{n}p_i\right)_n$ is equicontinuous and converging pointwise to a mapping id_E and A is compactoid, then $\left(\sum\limits_{i=1}^{n}p_i\right)_n$ converges uniformly on A (§. 3, lemma 3).

b $\Longrightarrow$ a Let U be a zero-neighbourhood, then there exists V a neighbourhood of zero and $n_0\in I\!N^*$ such that $V+V\subset U$ and $\sum\limits_{n_0<i}p_i(a)\in V$, *for all* $a\in A$. On the other hand $B=\sum\limits_{i=1}^{n_0}p_i(A)$ is precompact, then

there exist $b_1, b_2, ..., b_p \in E$ such that $B \subset \bigcup_{i=1}^{p} (b_i + V)$. Then $A \subset B + V \subset \bigcup_{i=1}^{p} (b_i + V) + V \subset \bigcup_{i=1}^{p} (b_i + U)$. ∎

Lemma 6. *For every $n \geq 1$, the mapping $p_n : E'_{\sigma} \longrightarrow E'_{\widetilde{\sigma}}$, $\sum_{i=1}^{\infty} \mu_i f_i \longmapsto \mu_n f_n$, is continuous; where $(f_i)_i$ is the weak Schauder basis asssociated to $(e_i)_i$.*

Proof. Let $n \geq 1$ and $x \in E$, then for every $f = \sum_{i=1}^{\infty} \mu_i f_i = \sum_{i=1}^{\infty} \langle e_i, f \rangle f_i$ we have: $\widetilde{p}_x (p_n (f)) = \sup_m |\langle x, T_m (p_n (f)) \rangle| = |\langle x, p_n (f) \rangle| = |\langle x, f_n \rangle| \, |\langle x_n, f \rangle|$.

Take $y = \langle x, f_n \rangle x_n$, then $y \in E$ and we have $p_y (f) = \widetilde{p}_x (p_n (f))$. ∎

Remark 7. *Lemma 6 is true for every K.*

Lemma 7. *Let $A \in \mathcal{A}$, then the statements* **a.** *and* **b.** *are equivalente*

a. *A is precompact in $E'_{\widetilde{\sigma}}$;*

b. **(i)**. *A is precompact in E'_{σ} and* **(ii)**. *for all $x \in E, \lim_n \widetilde{p}_A (x - S_n (x)) = 0$.*

Proof. We consider $A \in \mathcal{A}$ such that A is precompact in $E'_{\widetilde{\sigma}}$, then A is precompact in E'_{σ} $(\sigma \leq \widetilde{\sigma})$. On the other hand , for every $x \in E$, we have :

$$\widetilde{p}_A (x - S_n (x)) = \sup_{k \in \mathbb{N}} p_A (S_k (x - S_n (x))) = \sup_{f \in A} \widetilde{p}_x (f - T_n (f)).$$

Since $(e_i)_i$ is a Schauder basis of (E, τ), the sequence $(f_i)_i$ is an equicontinuous Schauder basis of $E'_{\widetilde{\sigma}}$ [9], lemma 3, p. 402 and [10], proposition 1.2, p. 278.

Furthermore A is precompact in $E'_{\widetilde{\sigma}}$, so $(T_n)_n$ converges to $id_{E'}$ uniformly on A in $E'_{\widetilde{\sigma}}$ (§. 3, lemma 3); then $\lim_{n\to\infty} \sup_{f \in A} \widetilde{p}_x (f - T_n (f)) = 0$ *for all* $x \in E$, and so $\lim_{n\to\infty} \widetilde{p}_A (x - S_n (x)) = 0$.

Conversely A is precompact in $E'_{\sigma} \Longrightarrow$ *for all* $i \geq 1$ $p_i (A)$ is precompact in $E'_{\widetilde{\sigma}}$ (lemma 6). On the other hand we have *for all* $x \in E$ and *all* $n \geq 1$ $\widetilde{p}_A (x - S_n (x)) = \sup_{f \in A} \widetilde{p}_x (f - T_n (f)) = \sup_{f \in A} \widetilde{p}_x \left(f - \sum_{i=1}^{n} p_i (f) \right)$.

So $\lim_{n\to\infty} \widetilde{p}_x \left(f - \sum_{i=1}^{n} p_i (f) \right) = 0$ for all $x \in E$, this means that $\left(\sum_{i=1}^{n} p_i \right)_n$

converges uniformly to $id_{E'}$ on A in $E'_{\widetilde{\sigma}}$; then by lemma 5, A is precompact in $E'_{\widetilde{\sigma}}$. ∎

Lemma 8. *If E' is $\sigma(E', E)$ −sequentially complete, then for every $A \in \mathcal{A}$, the following are equivalent:*

a. *A is $\widetilde{\sigma}$−relatively compact;*

b. **(i)**. *A is relatively compact in E'_{σ} and* **(ii)**. *for all $x \in E$, $\lim\limits_{n\to\infty} \widetilde{p}_A(x - S_n(x)) = 0$.*

Proof. Suppose that A is $\widetilde{\sigma}$−relatively compact in $E'_{\widetilde{\sigma}}$; $\overline{A}^{\widetilde{\sigma}}$ is compact in $E'_{\sigma}\,(\sigma \leq \widetilde{\sigma})$. Since $A \subset \overline{A}^{\widetilde{\sigma}}$ and $\overline{A}^{\widetilde{\sigma}}$ is closed in E'_{σ} then $\overline{A}^{\sigma} \subset \overline{A}^{\widetilde{\sigma}}$ and so $\overline{A}^{\sigma}$ is compact in E'_{σ}. Furthermore by lemma 7 we have **(ii)**.

Conversely, take A such that **(i)** and **(ii)** of **b** holds, then A and so $\overline{A}^{\widetilde{\sigma}}$ are precompacts in $E'_{\widetilde{\sigma}}$ (lemma 7). Consequently $\overline{A}^{\widetilde{\sigma}}$ is compact in $E'_{\widetilde{\sigma}}$ ($E'_{\widetilde{\sigma}}$ is complete: proposition 15.ii). ∎

Proposition 18. *Let $A \in \mathcal{A}$.*

1. $\widetilde{A}$ is $\widetilde{\sigma}\left(E', E\right)$ − precompact;

2. If E' is $\sigma\left(E', E\right)$ − sequentially complete, then

i. $\widetilde{A}$ is $\widetilde{\sigma}\left(E', E\right)$ − relatively compact;

ii. $\Gamma\left(\widetilde{A}\right)$ is $\sigma\left(E', E\right)$ − relatively compact.

Proof. Let $A \in \mathcal{A}$.

1. $\widetilde{A}$ is $\sigma\left(E', E\right)$ − bounded [10], lemma 1.2, p. 277, then it is $\sigma\left(E', E\right)$ − relatively compact ([2], proposition 2.3, p. 223) and so $\widetilde{A}$ is precompact in $\left(E', \sigma\left(E', E\right)\right)$. On the other hand for every $x \in E$ $\lim\limits_{n\to\infty} \widetilde{p}_A(x - S_n(x)) = 0$ ($(e_i)_i$ is a Schauder basis of $(E, \widetilde{\tau})$). Therefore, by lemma 7 and remarks 2, $\widetilde{A}$ is precompact in $E'_{\widetilde{\sigma}}$.

2. $\widetilde{A}$ is $\sigma(E', E)$ −relatively compact and $\lim\limits_{n\to\infty} \widetilde{p}_A(x - S_n(x)) = 0$ for every $x \in E$, then $\widetilde{A}$ is $\widetilde{\sigma}\left(E', E\right)$ − relatively compact (lemma 8).

3. $\widetilde{A}$ is $\widetilde{\sigma}\left(E', E\right)$ −relatively compact, so $B = \overline{\Gamma\left(\widetilde{A}\right)}^{\widetilde{\sigma}}$ is $\widetilde{\sigma}\left(E', E\right)$ −compact because B is a closed in a complete space $E'_{\widetilde{\sigma}}$ (proposition 15 and [30], p. 26). Hence $\Gamma\left(\widetilde{A}\right)$ is $\sigma\left(E', E\right)$ − relatively compact in E'_{σ} (lemma 8). ∎

Proposition 19. *If (E,τ) has a Schauder basis and $E^{'}$ is $\sigma\left(E^{'},E\right)-$ sequentially complete, then $\widetilde{\tau}$ is compatible with the duality $\left\langle E,E^{'}\right\rangle$.*

Proof. We have $\sigma \leq \widetilde{\tau}$. On the other hand, $\widetilde{\tau}$ is generated by the family $\left(\overline{\Gamma\left(\widetilde{A}\right)}^{\sigma\left(E^{'},E\right)}\right)_{A\in\mathcal{A}}$ ([2], §. 3, proposition 3.4, p. 228) and $\overline{\Gamma\left(\widetilde{A}\right)}^{\sigma\left(E^{'},E\right)}$ is $\sigma\left(E^{'},E\right)-$ compact for every $A\in\mathcal{A}$, so $\widetilde{\tau}\leq\tau_m$, where τ_m is the Mackey topology on E. ∎

b. K **is spherically complete**

Lemma 9. *Let E be a topological $K-$vector space with an equicontinuous Schauder basis $(e_i)_i$; then for every $A\subset E$ the statements* **a** *and* **b** *are equivalente*

a. *A is compactoid;*

b. **(i)**. *for all $i\geq 1$ $p_i(A)$ is compactoid and* **(ii)**. $\left(\sum_{i=1}^{n}p_i\right)_n$ *converges uniformly on A.*

Proof. Suppose that A is compactoid; then for every $i\geq 1$ $p_i(A)$ is compactoid. On the other hand, $(e_i)_i$ is an equicontinuous Schauder basis, so $\left(\sum_{i=1}^{n}p_i\right)_n$ converges pointwise to the mapping id_E ; since A is compactoid, this convergence is uniform on A (§. 3, lemma 3).

Conversely let U and V are two zero-neigbourhoods such that $V+V\subset U$, then the convergence of $\left(\sum_{i=1}^{n}p_i\right)_n$ on A implies the existence of $n_0\in I\!N$ such that $\sum_{i=n_0+1}^{\infty}p_i(x)\in V$ for all $x\in A$. On the other hand, **(i)** of lemma induces the existance of $x_1,...,x_n\in E$ such that $\sum_{i=1}^{n_0}p_i(A)\subset V+\Gamma(B)$, where $B=\{x_1,...,x_n\}$ $\left(\sum_{i=1}^{n_0}p_i(A)\ \textit{is compactoid}\right)$.

Then for every $x\in A$, $x=\sum_{i=1}^{n_0}\lambda_ie_i+\sum_{i=n_0+1}^{\infty}\lambda_ie_i$

$$= \sum_{i=1}^{n_0} p_i(x) + \sum_{i=n_0+1}^{\infty} p_i(x) \in V + \Gamma(B) + V,$$

so $A \subset U + \Gamma(B)$. ∎

Lemma 10. *Let $A \in \mathcal{A}$, then the following are equivalent*

a. *A is compactoid in $E'_{\widetilde{\sigma}}$;*

b. **(i)**. *A is compactoid in E'_σ and* **(ii)**. *for all $x \in E$ $\lim_{n\to\infty} \widetilde{p}_A(x - S_n(x)) = 0$.*

Proof. Same proof as for lemma 7 using lemma 9 and remark 7. ∎

Proposition 20. *Let τ be a polar topology of $\mathcal{A}-$ convergence on E and $(e_i)_i$ be a Schauder basis of (E,τ), then for every $A \in \mathcal{A}$*

i. *$\widetilde{A}$ is $\widetilde{\sigma}-$compactoid;*

ii. *If E' is $\sigma\left(E', E\right) -$ sequentially complete then*

a. *$\widetilde{A}$ is $\widetilde{\sigma}-$ relatively-c-compact;*

b. *$\Gamma\left(\widetilde{A}\right)$ is $\sigma-$ relatively-c-compact.*

Proof. **i.** Let $A \in \mathcal{A}$, then $\widetilde{A}$ is $\sigma-$bounded [10], lemma 1.2, p. 277 and, since K is spherically complete, $\widetilde{A}$ is compactoid in E'_σ ([31], proposition 18.ii, p. 145). On the other hand for all $x \in E$ $\lim_{n\to\infty} \widetilde{p}_A(x - S_n(x)) = 0$ since $(e_i)_i$ is a Schauder basis of $(E, \widetilde{\tau})$. Then $\widetilde{A}$ is copmactoid in $E'_{\widetilde{\sigma}}$ (lemma 10).

ii. Let $A \in \mathcal{A}$; then

a. $\overline{\widetilde{A}}^{\widetilde{\sigma}}$ is compactoid in $E'_{\widetilde{\sigma}}$ (by **i**) $\Longrightarrow$ $E'_{\widetilde{\sigma}}$ is complete, because E' is $\sigma\left(E', E\right) -$ sequentially complete (proposition 15), then $\overline{\widetilde{A}}^{\widetilde{\sigma}}$ is also complete and so it is c-compact in $E'_{\widetilde{\sigma}}$ [31], theorem 9, p. 141.

b. $B = \overline{\Gamma\left(\widetilde{A}\right)}^{\widetilde{\sigma}}$ is c-compact in $E'_{\widetilde{\sigma}}$, then it is $\sigma\left(E', E\right) - c-$compact and $\sigma\left(E', E\right)-$closed $(\sigma \leq \widetilde{\sigma})$, therefore $\overline{\Gamma\left(\widetilde{A}\right)}^{\sigma\left(E', E\right)}$ is $\sigma\left(E', E\right) - c-$compact.

Proposition 21. *If (E, τ) has a Schauder basis and E' is $\sigma\left(E', E\right) -$ sequentially complete, then $\widetilde{\tau}$ is compatible with the duality $\left\langle E, E' \right\rangle$.*

Proof. The topology $\widetilde{\tau}$ is a polar topology of $\mathcal{B}-$ convergence; where $\mathcal{B}=\left(\overline{\Gamma\left(\widetilde{A}\right)}^{\sigma\left(E',E\right)}\right)_{A\in\mathcal{A}}$ and $\mathcal{A}$ is the family which defines the topology τ; for every $A\in\mathcal{A}$, $\overline{\Gamma\left(\widetilde{A}\right)}^{\sigma\left(E',E\right)}$ is $\sigma\left(E',E\right)$-c-compact in E'_{σ} (proposition 20), then $\widetilde{\tau}$ is compatible with the duality $\left\langle E,E'\right\rangle$ [2], theorem 4.4, p. 234. ∎

c. K is not spherically complete

K is not spherically complete $\Longrightarrow$ K is dense $\Longrightarrow$For every absolutely $K-$ convex A in E', $K_c\ (A)=\bigcap\limits_{|\lambda|\succ1}\lambda\ A\Longrightarrow$ $for\ all\,|\mu|\succ1\ \mu K_c\ (A)=K_c\,(\mu A)$. Then we have the following proposition :

Proposition 22. *If (E,τ) has a Schauder basis, then $\widetilde{\tau}$ is compatible.*

Proof. Let $\mathcal{A}$ be a family which defines the topology τ, such that for all $A\in\mathcal{A}$ A is absolutely $K-$ convex; then $K_c\left(\overline{\widetilde{A}}^{\sigma\left(E',E\right)}\right)^{\circ}=\left(\widetilde{A}\right)^{\circ}$ $for\ all$ A$\in\mathcal{A}$ [2], corollary 4.3, p. 233, so if we take $\beta=\left(K_c\left(\overline{\widetilde{A}}^{\sigma\left(E',E\right)}\right)\right)_{A\in\mathcal{A}}$, then β verify the conditions (a), (b) and (c) of 8. Therefore $\widetilde{\tau}$ is a polar topology of $\beta-$ convergence and its elements are $E-$ closed. Then $\widetilde{\tau}$ is compatible [2], theorem 4.3, p. 233. ∎

Completeness of the topology $\widetilde{\tau}$

Proposition 23. *Let (E,τ) be a locally $K-$convex space and $(e_i)_i$ be a Schauder basis of (E,τ). If E and E' are weakly-sequentially complete, then $(E,\widetilde{\tau})$ is complete.*

Proof. The space $\left(E,\widetilde{\sigma}\left(E,E'\right)\right)$ is complete (proposition 15), then by remarks 2 and ([2], theorem 3.2, p. 230)$(E,\widetilde{\tau})$ is complete. ∎

The following theorem is a consequence for previous results

Theorem 6. *Let (E,τ) be a locally $K-$ convex space with a Schauder basis $(e_i)_i$ such that E_σ and E'_σ are sequentially complete, then $\widetilde{\tau}$ is complete and it is the coarsest compatible topology on E finer than τ for which $(e_i)_i$ is an equicontinuous Schauder basis.*

5. The weak basis Problem

Throughout this section we shall assume that (E,τ) has a weak Schauder basis $(e_i)_i$ and the spaces E'_σ and E_σ are sequentially complete. We then characterize the finest$\left(E,E'\right)$−compatible topology on E for which $(e_i)_i$ is a Schauder basis; according to theorem 6, $(e_i)_i$ is equicontinuous for that topology. We shall distinguish three cases: K is local, K is spherically complete or K is not spherically complete.

a. K is local

Let $\mathcal{B}=\left\{B\subset E'/B=\widetilde{B}\ and\ B\ is\ \widetilde{\sigma}-precompact\right\}$; it is obviously that $\mathcal{B}$ is not empty and verifies the properties $(a),(b)$ *and* (c) of 8. Let $\mathcal{U}$ be the polar topology of $\mathcal{B}-$convergence on E; we have the following propositions

Proposition 24. *$\mathcal{U}$ is compatible with the duality $\left\langle E,E'\right\rangle$ and $(e_i)_i$ is an equicontinuous Schauder basis of $(E,\mathcal{U})$.*

Proof. E' is $\sigma\left(E',E\right)-$ sequentially complete $\Longrightarrow$ *for all* $B\in\mathcal{B}$, $\overline{\Gamma(B)}^{\sigma\left(E',E\right)}$ is $\sigma\left(E',E\right)-$ compact (proposition 18) $\Longrightarrow\mathcal{U}$ is compatible with the duality $\left\langle E,E'\right\rangle$ [2], theorem 4.5, p. 235.

We'll prove that $(e_i)_i$ is a Schauder basis of $(E,\mathcal{U})$. $(e_i)_i$ is a weak Schauder basis $\Longrightarrow(f_i)_i$ is a Schauder basis of $\left(E',\sigma\left(E',E\right)\right)$, then $(f_i)_{i\geq1}$ is an equicontinuous Schauder basis of $\left(E',\widetilde{\sigma}\left(E',E\right)\right)$; therefore $(T_n)_n$ is equicontinuous in$\left(E',\widetilde{\sigma}\left(E',E\right)\right)$ and converges pointwise to the mapping $id_{E'}$, then the convergence is uniformly on every $B\in\mathcal{B}$, this means that for all $x\in E$ $\lim\limits_n\sup\limits_{f\in B}\widetilde{p}_x\left(f-\sum\limits_{i=1}^{n}p_i(f)\right)=0.$

For every $x\in E$ and for every $B\in\mathcal{B}$ we have $\sup\limits_{f\in B}\widetilde{p}_x\left(f-\sum\limits_{i=1}^{n}p_i(f)\right)=\widetilde{p}_B(x-S_n(x))$ (lemma 7). Then $\lim\limits_n\widetilde{p}_B(x-S_n(x))=0$ for all $x\in E$ and for all $B\in\mathcal{B}$, and so $(S_n(x))_n$ converges to x in $(E,\mathcal{U})$ for every $x\in E$.

Moreover the associated sequence $(f_i)_i$ of $(e_i)_i$ verifies for all $i\geq1$ $f_i\in(E,\mathcal{U})'$ (U is compatible). ∎

Proposition 25. *$\mathcal{U}$ is the finest compatible topology on E for which $(e_i)_i$ is a Schauder basis.*

Proof. Let τ be a compatible topology on E such that $(e_i)_i$ be a Schauder basis of (E,τ). Then $B = \overline{\Gamma\left(\widetilde{A}\right)}^{\sigma\left(E^{'},E\right)}$ is a $\sigma\left(E^{'},E\right)-$ compact of $E^{'}$ (proposition 18) and since $B=\widetilde{B}$, B° is a zero-neighbourhood in $(E,\mathcal{U})$; now $B^{\circ}=\left(\widetilde{A}\right)^{\circ}$, for every $A\in\mathcal{A}$, where $\mathcal{A}$ is a family which defines the topology τ, then $\widetilde{\tau}\leq\mathcal{U}$ and so $\tau\leq\mathcal{U}$. ∎

Proposition 26. *A weak Schauder basis $(e_i)_i$ is a Schauder basis for a polar compatible topology on E if, and only if, for every $A\in\mathcal{A}$, $\widetilde{A}$ is $\widetilde{\sigma}\left(E^{'},E\right)-$ relatively compact.*

Proof. $\Longrightarrow$] By proposition 18.

$\Longleftarrow$] Suppose that for every $A\in\mathcal{A}$, $\widetilde{A}$ is $\widetilde{\sigma}\left(E^{'},E\right)$ $-$relatively compact, then $\overline{\Gamma\left(\widetilde{A}\right)}^{\sigma\left(E^{'},E\right)}$ is $\sigma\left(E^{'},E\right)-$ compact (proposition 18), so $\tau\leq U$ and $(e_i)_i$ is a Schauder basis of (E,τ). ∎

Corollary 4. *A weak Schauder basis $(e_i)_i$ is a Schauder basis for a polar compatible topology τ on E if, and only if, for every $A\in\mathcal{A}$, $\widetilde{A}$ is $\widetilde{\sigma}\left(E^{'},E\right)-$ relatively compact, where $\mathcal{A}$ is a family that define the topology τ.*

Proof. τ_m is compatible, then it suffices to use the proposition 26. ∎

Remark 8. *If τ is a polar topology of $\mathcal{A}-$ convergence on E having a weak Schauder basis $(e_i)_i$, then for every $A\in\mathcal{A}$, $\widetilde{A}$ is $\widetilde{\sigma}\left(E^{'},E\right)-$ relatively compact.*

Proof. E is an $OP-$ space, so $(e_i)_i$ is a Schauder basis of (E,τ), then according to proposition 18 we have the conclusion. ∎

b. K is spherically complete

Let $\mathcal{N}=\left\{N\subset E^{'}/\ N=\widetilde{N}\ and\ N\ is\ \widetilde{\sigma}\left(E^{'},E\right)-compactoid\right\}$; it is obviously that $\mathcal{N}$ is not empty and verifies the properties $(a),(b)$ *and* (c) of **8**. Let $\mathcal{V}$ the polar topology of $\mathcal{N}-$convergence on E, then we have the following propositions

Proposition 27. *The topology $\mathcal{V}$ is compatible with the duality $\left\langle E,E^{'}\right\rangle$ and $(e_i)_i$ is an equicontinuous Schauder basis of $(E,\mathcal{V})$.*

Proof. Same proof as for proposition 24 using the proposition 20 and [2], theorem 4.4, p. 234. ∎

Proposition 28. *$\mathcal{V}$ is the finest compatible topology on E for which $(e_i)_i$ is a Schauder basis.*

Proof. Same proof as for proposition 25 using the proposition 20. ∎

Proposition 29. *A weak Schauder basis $(e_i)_i$ is a Schauder basis for a polar compatible topology τ on E if, and only if, for every $A \in \mathcal{A}$, $\widetilde{A}$ is $\widetilde{\sigma}\left(E^{'}, E\right)-$relatively c-compact, where $\mathcal{A}$ is a family that defines the topology τ.*

Proof. Same proof as for proposition 26 using the proposition 20. ∎

Remark 9. *if τ is a polar topology of $\mathcal{A}-$convergence on E that having a weak Schauder basis $(e_i)_i$, then for every $A \in \mathcal{A}$, $\widetilde{A}$ is $\widetilde{\sigma}\left(E^{'}, E\right)-$ relatively c-compact.*

Corollary 5. *A weak Schauder basis $(e_i)_i$ is a Schauder basis for a topology τ_c on E if, and only if, for every absolutely $K-$convex, $\sigma\left(E^{'}, E\right)-$bounded and $\sigma\left(E^{'}, E\right)-c-$compact A of $E^{'}$, $\widetilde{A}$ is $\widetilde{\sigma}\left(E^{'}, E\right)-$relatively c-compact.*

Proof. It is sufficient to take $\tau = \tau_c$ in proposition 29. ∎

c. K is not spherically complete

Let $\mathcal{M}$ the family of all $M \subset E^{'}$ such that $M = \widetilde{M}$, M is $\sigma\left(E^{'}, E\right)-$bounded, $E-$ closed and $(T_n)_n$ converges uniformly on M *in* $E^{'}_{\widetilde{\sigma}}$, where $E^{'}_{\widetilde{\sigma}} = \left(E^{'}, \widetilde{\sigma}\left(E^{'}, E\right)\right)$. Let ϑ be the polar topology of $\mathcal{M}-$convergence.

Theorem 7. *ϑ is the finest compatible topology on E for which $(e_i)_i$ is an equicontinuous Schauder basis.*

Proof. ϑ is compatible [2], theorem 4.3, p. 233.

Let $M \in \mathcal{M}$, then $(T_n)_n$ converges uniformly on M in $E^{'}_{\widetilde{\sigma}} \Longrightarrow \lim\limits_{n} \sup\limits_{f \in M} \widetilde{p}_x\left(f - T_n(f)\right) = 0$, so $\lim\limits_{n} p_M\left(x - S_n(x)\right) = 0$. Then $(e_i)_i$ is a Schauder basis of ϑ.

Let τ be a polar and compatible topology of $\mathcal{A}-$ convergence such that $(e_i)_i$ be an equicontinuous Schauder basis, then $\tau = \widetilde{\tau}$. Therefore

$\mathcal{A} = \left\{ A \subset E' / \ A = \widetilde{A} \ and \ A \ is \ \sigma \left(E', E \right) - bounded \ and \ E - closed \right\}.$

Let $A \in \mathcal{A}$, so for every $x \in E$ $\lim\limits_n p_A (x - S_n (x)) = 0$, then $\lim\limits_n \sup_{f \in A} \widetilde{p}_x (f - T_n (f)) = 0$, therefore $(T_n)_n$ converges uniformly on A in $E'_{\widetilde{\sigma}}$. Then $\tau \leq \vartheta$. ∎

Theorem 8. *Let τ be a polar topology of $\mathcal{A}-$ convergence on E such that E_σ and E'_σ are sequentially complete and $(e_i)_i$ be a weak Schauder basis of E, then $(e_i)_i$ is a Schauder basis of (E, τ) if, and only if, for all $A \in \mathcal{A}$ the sequence $(T_n)_n$ converges uniformly on A in $E'_{\widetilde{\sigma}}$.*

Proof. If $(e_i)_i$ is a Schauder basis of (E, τ), then $(e_i)_i$ is an equicontinuous Schauder basis of $\widetilde{\tau}$; therefore for all $A \in \mathcal{A}$ the sequence $(T_n)_{n \geq 1}$ converges uniformly on $\widetilde{A}$ in $E'_{\widetilde{\sigma}}$, so for all $A \in \mathcal{A}$, $(T_n)_n$ converges uniformly on A in $E'_{\widetilde{\sigma}}$.

Conversely, let $A \in \mathcal{A}$, then for all $x \in E$ $\widetilde{p}_A (x - S_n (x)) = \sup\limits_{f \in A} \widetilde{p}_x (f - T_n (f))$.

But $\lim\limits_n \sup\limits_{f \in A} \widetilde{p}_x (f - T_n (f)) = 0$, then $\lim\limits_n \widetilde{p}_A (x - S_n (x)) = 0$ and so $\lim\limits_n p_A (x - S_n (x)) = 0$. ∎

Theorem 9. *Under the conditions of theorem 8, E is an $OP-$ space if, and only if, for all $A \in \mathcal{A}$ the sequence $(T_n)_n$ converges uniformly on A in $E'_{\widetilde{\sigma}}$.*

Proof. ([21], proposition 1) and theorem 8. ∎

Proposition 30. *Let τ be a polar topology of $\mathcal{A}-$ convergence on E and $(e_i)_i$ be a weak Schauder basis of E, then if every $\widetilde{\sigma}-$ equicontinuous sequence of E' that converging pointwise to zero converges uniformly on every $A \in \mathcal{A}$ in $E'_{\widetilde{\sigma}}$, then $(e_i)_i$ is a Schauder basis of (E, τ).*

Proof. $(f_i)_i$ is an equicontinuous Schauder basis of $\left(E', \widetilde{\sigma} \left(E', E \right) \right) = E'_{\widetilde{\sigma}} \Longrightarrow (T_n)_n$ is an equicontinuous sequence of $E'_{\widetilde{\sigma}} \Longrightarrow$ the sequence $\left(id_{E'} - T_n \right)_n$ is pointwise converging to zero in $E'_{\widetilde{\sigma}}$ and this convergence is uniformly on every $A \in \mathcal{A}$ in $E'_{\widetilde{\sigma}} \Longrightarrow \lim\limits_n \sup\limits_{f \in A} \widetilde{p}_x (f - T_n (f)) = 0$ for all $x \in E \Longrightarrow \lim\limits_n \widetilde{p}_A (x - S_n (x)) = 0$ for all $x \in E \Longrightarrow \lim\limits_n p_A (x - S_n (x)) = 0$ for all $x \in E$. ∎

6. Application to barrelled spaces and G-spaces

Barrelled spaces

a. K is spherically complete

Proposition 31. *If (E, τ) is a barrelled locally $K-$ convexe space which is $\sigma\left(E, E'\right)-$ sequentially complete and having a weak Schauder basis, then (E, τ) is complete and every weak Schauder basis is an equicontinuous Schauder basis of (E, τ).*

Lemma 11. *If (E, τ) is barrelled and having a Schauder basis then $\tau = \widetilde{\tau}$.*

Proof. Let $A \in \mathcal{A}$, where $\mathcal{A}$ is a family that defines the topology τ; $\left(\widetilde{A}\right)^{\circ}$ is a barrel, so it is a zero-neighbourhood in (E, τ), then $\widetilde{\tau} \leq \tau$. ∎

Proof of proposition Let $(e_i)_i$ be a weak Schauder basis of E, then $(e_i)_i$ is a Schauder basis of (E, τ) (E *is an* $OP-space$) $\Longrightarrow (e_i)_i$ is an equicontinuous Schauder basis of $\widetilde{\tau} \Longrightarrow (e_i)_i$ is an equicontinuous Schauder basis of (E, τ) $(\tau = \widetilde{\tau})$. ∎

b. K is not spherically complete

Proposition 32. *Every weak Schauder basis in a polarly barrelled polar locally $K-$convex space is an orthogonal basic sequence.*

Proof. ([21], corollary 6, p. 155). ∎

Proposition 33. *Let E be a Banach space with a weak Schauder basis; then E is a polar space if and only if, every weak Schauder basis in E is a basic sequence.*

Proof. For the sufficient condition, one only has to use theorem 3.2 $(\alpha) \Longrightarrow (\beta)$ of [28]. The necessary condition is a particular case of proposition 32. ∎

G-spaces

Proposition 34. *If (E, τ) is a weakly-sequentially complete G-space that having a Schauder basis, then (E, τ) is complete and this basis is equicontinuous.*

Proof. By theorem 6. ∎

Proposition 35. *If (E,τ) is a G-space with a Schauder basis $(e_i)_i$, there is no strictly finer locally $K-$convex topology on E for which $(e_i)_i$ is still a Schauder basis.*

Proof. By proposition 19 if K is local, proposition 21 if K is spherically complete and by proposition 22 if K is not spherically complete. ∎

Proposition 36. *Suppose that K is spherically complete; if (E,τ) is a weakly-sequentially complete G-space that having a weak Schauder basis $(e_i)_i$, then (E,τ) is complete and $(e_i)_i$ is an equicontinuous Schauder basis of (E,τ).*

Proof. Let $(e_i)_i$ be a weak Schauder basis of E, then E is an $OP-$space $\Longrightarrow (e_i)_i$ is a Schauder basis of (E,τ). So the proposition is an immediate consequence of proposition 34. ∎

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