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Accretive operators and Banach Alaoglu theorem in Linear 2-normed spaces

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Abstract

In this paper we introduce the concept of accretive operator in linear 2-normed spaces, focusing on the relationships and the various aspects of accretive, m-accretive and maximal accretive operators. We prove the analogous of Banach-Alaoglu theorem in linear 2- normed spaces, obtaining an equivalent definition for accretive operators in linear 2-normed spaces.

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1. Introduction

The concept of 2- metric spaces, linear 2- normed spaces and 2-inner product spaces, introduced by S. Gahler in 1963, paved the way for a number of authors like, A. White, Y. J. Cho, R. Freese, C. R. Diminnie, for working on possible applications of Metric geometry, Functional Analysis and Topology as a new tool. A systematic presentation of the recent results related to the Geometry of linear 2-normed spaces as well as an extensive list of the related references can be found in the book [1]. In [4] S. Gahler introduced the following definition of linear 2-normed spaces.

2. Preliminaries

Definition 2.1 (3). *Let X be a real linear space of dimension greater than 1 and $\|.,.\|$ be a real valued function on $X \times X$ satisfying the properties,*

A1: $\|x, y\| = 0$ iff x and y are linearly dependent

A2: $\|x, y\| = \|y, x\|$

A3: $\|\alpha x, y\| = |\alpha| \|y, x\|$

A4: $\|x + y, z\| \leq \|x, z\| + \|y, z\|$

for every $x, y, z \in X$ and $\alpha \in R$

then the function $\|.,.\|$ is called a 2-norm on X. The pair $(X, \|.,.\|)$ is called a linear 2- normed space.

Some of the basic properties of 2-norms, they are non-negative and $\|x, y + \alpha x\| = \|x, y\|$ for all x and y in X and for every α in R.

The most standard example for a linear 2-normed space is $X = R^2$ equipped with the following 2-norm,

$$\|x_1, x_2\| = abs \det \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix} \text{ where } x_i = (x_{i1}, x_{i2}) \text{ for } i = 1, 2$$

Every linear 2-normed space is a locally convex TVS. In fact, for a fixed $b \in X$, $P_b(x) = \|x, b\|$ is a semi norm, where $x \in X$ and the family $\{P_b; b \in X\}$ of semi norms generates a locally convex topology on X.

Definition 2.2 (3). *Let $(X, \|.,.\|)$ be a linear 2-normed space, then a map $T : X \times X \to R$ is called a 2- linear functional on X whenever for every $x_1, x_2, y_1, y_2 \in X$ and $\alpha, \beta \in R$*

(i) $T(x_1 + x_2, y_1 + y_2) = T(x_1, y_1) + T(x_1, y_2) + T(x_2, y_1) + T(x_2, y_2)$

(ii) $T(\alpha x_1, \beta y_1 = \alpha\beta T(x_1, y_1)$

hold.

A 2-linear functional $T : X \times X \to R$ is said to be bounded if there exists a real number $M > 0$ such that $|T(x, y)| \leq M\|x, y\|$ for all x, y in X.
The norm of the 2-linear functional $T : X \times X \to R$ is defined for all x, y in X by

$$\|T\| = \inf\{M > 0; |T(x, y)| \leq M\|x, y\|\}.$$

It can be seen that

$$\begin{aligned}\|T\| &= sup\,\{|T(x,y)|\,; \|x, .y\| \leq 1\}\\ &= sup\,\{|T(x,y)|\,; \|x, y\| = 1\}\\ &= sup\left\{\tfrac{|T(x,y)|}{\|x,y\|}; \|x, y\| \neq 0\right\}\end{aligned}$$

Definition 2.3 (2). *Let $(X, \|., .\|)$ be a linear 2- normed space, E be a subset of X then the sequentially closure of E is $\overline{E} = \{x \in X : x_n \subset E / x_n \to x\}$. We say, E is sequentially closed if $E = \overline{E}$.*

Definition 2.4 (3). *Let X_z^* be the set of all bounded linear 2- functional on $X \times V\langle z\rangle$ then the duality map is defined by $I(x, z) = \{F \in X_z^*; F(x, z) = \|x, z\|^2$ and $\|F\| = \|x, z\|\}$*

3. Main Results

Let $(X, \|., .\|)$ be a linear 2- normed space and $A : D(A) \subset X \to X$ be an operator with domain $D(A) = \{x \in X; Ax \neq 0\}$ and range $R(A) = \cup\{Ax; x \in D(A)\}$. We may identify A with its graph and the closure of A with the closure of its graph.

Definition 3.1. *: An operator $A : D(A) \subset X \to X$ is said to be accretive if, for every $z \in D(A)$*

$$\|x - y, z\| \leq \|(x - y) + \lambda(Ax - Ay), z\| \text{ for all } x, y \in D(A) \text{ and } \lambda > 0.$$

Throughout this article $[x, y] \in A$ means $x, y \in X$ such that $y = Ax$.

Definition 3.2. *: An operator $A : D(A) \subset X \to X$ is said to be m-accretive if $R(I + \lambda A) = X$ for $\lambda > 0$.*

An operator $A : D(A) \subset X \to X$ and $B : D(B) \subset X \to X$ be two operators then B is said to be an extension of A if $D(A) \subset D(B)$ and $Ax = Bx$ for every $x \in D(A)$, denote it by $A \subset B$.

Definition 3.3. *: An operator $A : D(A) \subset X \to X$ is said to be a maximal accretive operator in X if A is an accretive operator in X and for every accretive operator B of X with $A \subset B$ then $A = B$.*

Theorem 3.4. *If A is an m-accretive operator in X then A is a maximal accretive operator.*

Proof: Let B be an accretive operator with $A \subset B$. Let $\lambda > 0$ and $[x, y] \in B$.

Since A is m- accretive we have $x+\lambda y \in R(I+\lambda A)$ implies there exists $[x_1, y_1] \in A$ such that $x + \lambda y = x_1 + \lambda y_1$

Since B is accretive and $[x_1, y_1] \in B$ we have for every $z \in X$,

$$\begin{aligned} \|x - x_1, z\| &\leq \|(x - x_1) + \lambda(Bx - Bx_1), z\| \\ &= \|(x - x_1) + \lambda(y - y_1), z\| \\ &= \|(x + \lambda y) - (x_1 + \lambda y_1), z\| = \|0, z\| \text{ for every } z \in X \\ &= 0 \end{aligned}$$

implies $x - x_1 = 0$ and $x = x_1$

Therefore $y = y_1$ implies $[x, y] \in A$. So A = B.

Hence A is a maximal accretive operator.

Lemma 3.5. *Let A be an accretive operator in X and let $(u, v) \in X \times X$ then A is maximal accretive in X iff for every $[x, y] \in A$ and $z \in X$ and $\lambda > 0$ one has $\|x - u, z\| \leq \|(x - u) + \lambda(y - v), z\|$ implies $[u, v] \in A$.*

Proof:

Let A be a maximal accretive operator in X. Put $T = A \cup [u, v]$

Suppose $\|x - u, z\| \leq \|(x - u) + \lambda(y - v), z\|$ for every $[x, y] \in A, z \in X$ and $\lambda > 0$

then T is accretive in X and $A \subset T$ implies $[u, v] \in A$

Conversely, suppose that if A is accretive operator in X and

$\|x-u, z\| \leq \|(x-u)+\lambda(y-v), z\|$ for every $[x, y] \in A, z \in X$ and $\lambda > 0$ implies $[u, v] \in A$

Let B be accretive in X with $A \subset B$ and $[x_1, y_1] \in B$

Since B is accretive in X, for every $[x, y] \in A, z \in X$ and $\lambda > 0$ one has

$\|x - x_1, z\| \leq \|(x - x_1) + \lambda(Bx - Bx_1), z\| = \|(x - x_1) + \lambda(y - y_1), z\|$ which

implies $[x_1, y_1] \in A$. Therefore $B \subset A$. So $A = B$.

Hence A is maximal accretive in X.

Theorem 3.6. *If A is an accretive operator in X then there exists a maximal accretive operator containing A.*

Proof:

Let $B = \{B;$ B is accretive in X and $A \subset B\}$ then $(B, \subset)$ is a partially ordered set.

Let T be a totally ordered set with $T \subset B$ then by Zorn's lemma there exists a maximal element in B, is a maximal accretive operator containing A.

Theorem 3.7. *Let A be an accretive operator in X then the closure $\overline{A}$ of A is accretive.*

Proof:

Let $[x_1, y_1], [x_2, y_2] \in \overline{A}$ then there exists sequences $\{[x_n, y_n]\}$, $\{[x_m, y_m]\}$ in A such that $x_n \to x_1; y_n \to y_1; x_m \to x_2; y_m \to y_2$ and $\lambda > 0$.

Since A is accretive in X one has

$$\begin{aligned}\|x_n - x_m, z\| &\leq \|(x_n - x_m) + \lambda(Ax_n - Ax_m), z\| \text{for every } z \in X \\ &= \|(x_n - x_m) + \lambda(y_n - y_m), z\| \text{ for every } z \in X\end{aligned}$$

as $n \to \infty$, $\|x_1 - x_2, z\| \leq \|(x_1 - x_2) + \lambda(y_1 - y_2), z\|$for every $z \in X$

implies $\overline{A}$ is accretive in X.

Theorem 3.8. *Let A be a maximal accretive operator in X then A is sequentially closed.*

Proof: For all $x_n, y_n \in D(A)$, Let $\{[x_n, y_n]\}$ in A such that $x_n \to u, y_n \to v$ and $\lambda > 0$

Since A is accretive in X and $[x, y] \in A$ implies $\|x - x_n, z\| \leq \|(x - x_n) + \lambda(y - y_n), z\|$for every $z \in X$

as $n \to \infty$ we have $\|x - u, z\| \leq \|(x - u) + \lambda(y - v), z\|$ for every $z \in X$

Therefore, by Lemma 3.6 $[u, v] \in A$. Hence A is sequentially closed.

Corollary 3.9. *If A is an m-maximal accretive operator in X then A is sequentially closed.*

Proof: We have an m-accretive operator A in X is a maximal accretive operator in X. Hence by Theorem 3.8, A is sequentially closed.

Next we prove analogous of Banach Alaoglu theorem in linear 2- normed spaces.

Theorem 3.10. *Let X be a linear 2- normed space then the closed unit ball of X_z^* is weak* compact, i.e. $B = \{f \in X_z^*; \|f\| \leq 1\}$ is compact for the weak* topology.*

Proof:

If $f \in B$ then $|f(x,z)| \leq \|f\|\|x,z\|$ for every $x, z \in X$

Let $D_{x,z} = \{\lambda \in R; |\lambda| \leq \|x,z\|\}$ be a closed interval then it is compact.

We have $f(x,z) \in D_{x,z}$ for every $x, z \in X$. Take $D = \Pi_{x \in X} D_{x,z}$ for every $z \in X$. Equip product topology on D then, by Tychnoff's theorem D is compact.

Consider the canonical projection $\Pi_{x,z} : D \to D_{x,z}$

Equip B with the relative topology induced by weak* topology. So it is enough to prove that B is homeomorphic with a closed subset C of D.

Define $T : B \to D$ as follows:

If $f \in B$ then $f(x,z) \in D_{x,z}$for every $x, z \in X$

So, define $Tf = (f(x,z))_{x,z \in X}$of D has the property that $(x,z)^{th}$ coordinate is a 2-linear functional of index (x,z).

Construct the set C of all $(\lambda_{x,z})_{x,z \in X} \in$ in D such that

$\lambda_{(x_1+x_2,z_1+z_2)} = \lambda_{x_1,z_1} + \lambda_{x_1,z_2} + \lambda_{x_2,z_1} + \lambda_{x_2,z_2}$

$\lambda_{\alpha x_1,\beta z_1} = \alpha\beta\lambda_{x_1,z_1}$ for every $x_1, x_2, z_1, z_2 \in X$ and $\alpha, \beta \; inR$

We have $T(B) \subset C$

If $\lambda_{x,z} \in C$ for $x, z \in X$

Define $f : X \times X \to R$ by $f(x,z) = \lambda_{x,z}$ is a 2-linear functional on X.

Also $|f(x,z)| = |\lambda_{x,z}| \leq \|x,z\|$ implies $\|f\| \leq 1$. Therefore $f \in B$.

And $Tf = f(x,z)_{x,z \in X} = (\lambda_{x,z})_{x,z \in X}$. So $C \subset T(B)$. Therefore $T(B) = C$.

Next we have to prove that,

(i) T is one-to-one

(ii) C is a closed subset of D

(iii) T is bicontinuous (ie; homeomorphiism) from B onto $T(B) = C$

For,

(i) Let $f, g \in B$ with $Tf = Tg$ then $f(x,z) = g(x,z)$ for every $x, z \in X$ implies $f = g$. So T is one-to-one.

(ii) For $x_1, x_2, z_1, z_2 \in X$, Define $\phi : D \to R$ by $\phi(\lambda_{x,z}) = \lambda_{(x_1+x_2,z_1+z_2)} - \lambda_{x_1,z_1} - \lambda_{x_1,z_2} - \lambda_{x_2,z_1} - \lambda_{x_2,z_2}$

Take $u = \lambda_{x,z}$ then we have $\phi(u) = \pi_{(x_1+x_2,z_1+z_2)}(u) - \pi_{x_1,z_1}(u) - \pi_{x_1,z_2}(u) - \pi_{x_2,z_1}(u) - \pi_{x_2,z_2}(u)$

Since π is continuous we have ϕ is continuous.

Define $\phi^{-1}[0] = \{\lambda_{x,z} \in D : \lambda_{(x_1+x_2,z_1+z_2)} = \lambda_{x_1,z_1} + \lambda_{x_1,z_2} + \lambda_{x_2,z_1} + \lambda_{x_2,z_2}\}$. Then $\phi^{-1}[0]$ is closed in D. Denote this closed set by $C_{(x_1,x_2,z_1,z_2)}$.

Similarly, for fixed $v_1, v_2 \in X$ and $\alpha, \beta \in R$ the set $\{(\lambda_{x,z})_{x,z \in X}; \lambda_{\alpha x_1,\beta z_1} = \alpha\beta\lambda_{x_1,z_1}\}$ is closed in D. Denote it by $C_{(v_1,v_2,\alpha,\beta)}$.

Therefore, $C = (\cap C_{(x_1,x_2,z_1,z_2)}) \cap (\cap C_{(v_1,v_2,\alpha,\beta)})$ where $x_1, x_2, z_1, z_2, v_1, v_2$ varies over X and α, β varies over R. Hence C is closed in D.

(iii) In the view of (i) T maps bijectively onto $T(B) = C$

Consider a sub basic neighbourhood of f_0 for the relative weak* topology on B of the form:

Let $e \in X$ then $V = \{f \in B; \|f(x_0, y_0) - f_0(x_0, y_0), e\| \leq \varepsilon\}$

Therefore, $T(V) = \{[f(x,y)]_{x,z} \in X; f \in V\}$

$=\{[f(x,y)]_{x,z\in X}; f \in B$ with $\|f(x_0,y_0) - f_0(x_0,y_0), e\| \leq \varepsilon\}$

$=\{[f(x,y)]_{x,z\in X}; f \in B$ with $\|\pi_{(x_0,y_0)}(Tf) - \pi_{(x_0,y_0)}(Tf_0), e\| \leq \varepsilon\}$

is a sub basic neighbourhood of Tf_0 for the relative topology induced on $T(B) = C$ by the product topology on D. So,T is bicontinuous from B onto $T(B) = C$

Theorem 3.11. *Let X be a linear 2-normed space and $x, y \in X$ then for every $z \in X$, $\|x, z\| \leq \|x + \alpha y, z\|$ for every $\alpha > 0$ iff there is $F \in I(x, z)$ such that $Re((y, z), F) \geq 0$ ["Re" means "real part of"]*

Proof:

If x = 0 then the result holds true.

Assume that $x \neq 0$. Suppose $Re((y, z), F) \geq 0$ for some $F \in I(x, z)$ then

$\|x, z\|^2 = F(x, z) = Re(F(x, z)) \leq Re(F(x + \alpha y)) \leq \|F\|\|x + \alpha y, z\|$ for $\alpha > 0$

Since, $\|F\| = \|x, z\|$ we have $\|x, z\| \leq \|x + \alpha y, z\|$ for $\alpha > 0$

Conversely, suppose that $\|x, z\| \leq \|x + \alpha y, z\|$ for $\alpha > 0$

For each $\alpha > 0$ let $F_\alpha \in I(x + \alpha y, z)$ and $g_\alpha = \frac{F_\alpha}{\|F_\alpha\|}$ then $\|g_\alpha\| = 1$

Then,

$\|x, z\| \leq \|x + \alpha y, z\| = g_\alpha(x + \alpha y, z) = Re[g_\alpha(x, z)] + \alpha Re[g_\alpha(y, z)] \leq \|x, z\| + \alpha Re[g_\alpha(y, z)]$

implies $lim\{inf_{(\alpha\downarrow 0)} Re[(x, z), g_\alpha]\} \geq \|x, z\|$ and $Re[(y, z), g_\alpha] \geq 0$

By the above theorem, the closed unit ball of X_z^* is weak* compact then the net $\{g_\alpha\}$ has a cluster point g with $\|g\| = 1$.

$Re[(x, z), g_\alpha \geq \|x, z\|$ and $Re[(y, z), g_\alpha] \geq 0$ implies $Re[\frac{(x,z)}{\|x,z\|}, g_\alpha] \geq 1$

implies $\|g\| = 1$ and $g(x, z) = \|x, z\|$

Take $F = \|x, z\|g$ then $F(x, z) = \|x, z\|g(x, z) = \|x, z\|^2$. Therefore, $F \in I(x, z)$ and $Re[(y, z), F] \geq 0$

Remark 3.12. *From the above theorem we get "A is an accretive operator in a linear 2-normed space X iff for every $u, v \in D(A)$ there exists $f \in I(u - v, z)$ such that $Re[f(Au - Av, z)] \geq 0$".*

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