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On certain isotopic maps of central loops

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Abstract

It is shown that the Holomorph of a C-loop is a C-loop if each element of the automorphism group of the loops is left nuclear. Condition under which an element of the Bryant-Schneider group of a C-loop will form an automorphism is established. It is proved that elements of the Bryant-Schneider group of a C-loop can be expressed a product of pseudo-automorphisms and right translations of elements of the nucleus of the loop. The Bryant-Schneider group of a C-loop is also shown to be a kind of generalized holomorph of the loop.

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1. Introduction

Central loops(C-loops) are loops which satisfy one of the identities called "Central identity" as named by F. Fenyves [9], [10]. Closely related to the central identity are left central(LC) and right central (RC) identities. The expressions for the mentioned identities are as follows;

$$(yx \cdot x)z = y(x \cdot xz) \quad \text{central identity} \tag{1.1}$$

$$\begin{gathered} i.\ xx \cdot yz = (x \cdot xy)z \ \equiv \\ ii.\ (x \cdot xy)z = x(x \cdot yz) \ \equiv \\ iii.\ (xx \cdot y)z = x(x \cdot yz) \\ \text{LC- identities} \end{gathered} \tag{1.2}$$

$$\begin{gathered} i.\ yz \cdot xx \quad = \quad y(zx\ .\ x) \ \equiv \\ ii.\ (yz \cdot x)x = y(zx \cdot x) \equiv \\ iii.\ (yz \cdot x)x = y(z \cdot xx) \\ \text{RC- identities} \end{gathered} \tag{1.3}$$

Recently Phillips and Vojtechovsky [20], found out that in addition to the identities above, LC and RC identity can also be defined respectively by,

$$(y \cdot xx)z = y(x \cdot xz) \quad \textit{and} \quad (yx \cdot x)z = y(xx \cdot z) \tag{1.4}$$

C-loops are one of the least studied loops. Few publications that have considered C-loops include Fenyves [9], [10], Phillips and Vojtechovsky [18] [20] [19], Chein [5]. The difficulty in studying them is as a result of the nature of their identities when compared with other Bol-Moufang identities(the element occurring twice on both sides has no other element separating it from itself).

2. Preliminaries

Theorem 2.1. *([10], [20]) Let $(L, \cdot)$ be an LC-loop(RC-loop). Then:*

1. *$(L, \cdot)$ is a left (right) alternative loop,*
2. *$(L, \cdot)$ is a left (right) inverse property loop,*
3. *$(L, \cdot)$ is a left (right) nuclear square loop,*
4. *$(L, \cdot)$ is a left (right) power alternative loop,*

5. $(L, \cdot)$ *is a middle square loop,*

6. $(L, \cdot)$ *is power associative loop.*

Definition 2.1. *A triple* (α, β, γ) *of bijections is called an isotopism of loop* $(L, \cdot)$ *onto a loop* $(H, \circ)$ *provided* $x\alpha \circ y\beta = (x \cdot y)\gamma\ \forall\ x, y \in L$. $(H, \circ)$ *is called an isotope of* $(L, \cdot)$. *The loops* $(L, \cdot)$ *and* $(H, \circ)$ *are said to be isotopic to each other.*

Definition 2.2. *Let* α *and* β *be a permutation of L and let* ι *denotes the identity map on L. Then* (α, β, ι) *is a principal isotopism of a loop* $(L, \cdot)$ *onto a loop* $(L, \circ)$ *which imply that* (α, β, ι) *is an isotopism of* $(L, \cdot)$ *onto* $(L, \circ)$.

Definition 2.3. *An isotopism of* $(L, \cdot)$ *onto* $(L, \cdot)$ *is called an autotopism of* $(L, \cdot)$. *The group of autotopisms of L is denoted by A(L).*

Remark 2.1. *The components of isotopism are usually denoted by lower case Greek letters. However, we shall denote the components of autotopism by capital letters, thus if* $T = (U, V, W)$ *is an autotopism of a loop* $(L, \cdot)$, *then*

$$xU \cdot yV = (xy)W, \forall\ x, y \in L.$$

The set of all autotopism of a loop is a group with the inverse of $T\ T^{-1} = (U, V, W)^{-1} = (U^{-1}, V^{-1}, W^{-1})$. *The identity element of the group being* (I, I, I) *where* I *is the identity map of L. If* $T = (U, U, U)$, *then T is called the automorphism* $(L, \cdot)$

Definition 2.4. *If* $\langle U, V, W\rangle$ *is autotopism of an inverse property loop* $(L, .)$ *then* $\langle W, JVJ, U\rangle$ *and* $\langle JUJ, W, V\rangle$ *are autotopism of L. Moreover if* $\langle U, V, W\rangle = \langle S, SR_c, SR_c\rangle$ *the S is called a pseudoautomorphism of L with companion c. The set of all pseudoautomorphisms of L is denoted by* $PS(L, .)$.

Definition 2.5. *Let* $(L, \cdot)$ *be an inverse property loop with the nucleus denoted by N. Then an automorphism* α $of(L, \cdot)$ *is left nuclear* iff $a\alpha{\cdot}a^{-1} \in N$ *for all* $a \in L$.

Definition 2.6. *Let* $(L, .)$ *be a loop and* $BS(L, .)$ *be the set of all permutations* θ *of* Q *such that*

$$< \theta R_g^{-1}, \theta L_f^{-1}, \theta >$$

is an autotopism of $(L, .)$ *for some* $f, g \in L$, *then* $BS(L, .)$ *is called the Bryant-Schneider group of the loop.*

Definition 2.7. *Let $(L,\cdot)$ be a loop, $A(L)$ a group of automorphisms of loop $(L,\cdot)$ and let H $H = A(L)\times L$ and define*

$$(\alpha,x)\ o\ (\beta,y) \ = \ (\alpha\beta,\ x\beta\cdot y)$$

$\forall\ (\alpha,x),(\beta,y)\in H$. Then the loop (H,o) is called the A(L)-holomorph of $(L,\cdot)$ or simply holomorphy of $(L,.)$.

3. Holomorphy

Theorem 3.1. *Let $(L,\cdot)$ be a an LC-loop and A(L) be a group of automorphism of $(L,\cdot)$. Then the A(L)-holomorph (H,o) of $(L,\cdot)$ is an LC-loop if and only if*

(3.1) $$(x\alpha\cdot xy)z = x\alpha(x\cdot yz)$$

$\forall\ x,y,z\in L$ and $\forall\ \alpha\in A(L)$.

Proof.
Suppose A(L)-holomorph (H,o) of $(L,\cdot)$ is an LC-loop we have

(3.2) $$\{(\alpha,x)o[(\alpha,x)o(\beta,y)]\}o(\gamma,z) \ = \ (\alpha,x)o\{(\alpha,x)o[(\beta,y)o(\gamma,z)]\}$$

$\forall\ x,y,z\in L$ and $\forall\ \alpha,\beta,\gamma\in A(L)$. Thus

$$\{(\alpha,x)o(\alpha\beta,x\beta\ .\ y)\}o(\gamma,z) \ = \ (\alpha,x)o\{(\alpha,x)o(\beta\gamma,y\gamma\cdot z)\}$$

$$\{\alpha\cdot\alpha\beta,x\alpha\beta\cdot(x\beta\cdot y)\}o(\gamma,z) \ = \ (\alpha,x)o\{(\alpha\cdot\beta\gamma,x\beta\gamma\cdot(y\gamma\cdot z))\}$$

$$\{(\alpha\cdot\alpha\beta)\gamma,[x\alpha\beta\cdot(x\beta\cdot$$

$$\text{y})]\gamma\cdot z\} \ = \ \{\alpha(\alpha\cdot\beta\gamma),x\alpha\cdot\beta\gamma\cdot x\beta\gamma(y\gamma\cdot z)\}\forall\ x,y,z\in L \text{ and } \forall\ \alpha,\beta,\gamma\in$$

$A(L)$. Therefore

$$\{x\alpha\beta\cdot(x\beta\cdot y)\}\gamma\cdot z \ = \ x\alpha\cdot\beta\gamma.x\beta\gamma(y\gamma\cdot z)$$

$\forall\ x,y,z\in L$ and $\forall\ \alpha,\beta,\gamma\in A(L)$.

Therefore,

$$\{x\alpha\cdot\beta\gamma\cdot(x\beta\gamma\cdot y\gamma)\}\cdot z = x\alpha\cdot\beta\gamma\cdot x\beta\gamma\cdot(y\gamma\cdot z)$$

putting $\phi \ = \ \beta\gamma$, gives

$$\{x\alpha\phi\cdot(x\phi\cdot y\gamma)\}z \ = \ x\alpha\phi\cdot x\phi(y\gamma\cdot z)$$

hence

$$\{x\alpha \cdot (x \cdot y\gamma\phi^{-1})\} \cdot z\phi^{-1} \;=\; \{x\alpha \cdot x(y\gamma\phi^{-1} \cdot z\phi^{-1}\}$$

$\forall\ x, y, z \in L$ and $\forall\ \alpha, \phi, \gamma \in A(L)$. If we put $\overline{y} = y\gamma\phi^{-1}$ and $\overline{z} = z\phi^{-1}$, we obtain

$$(x\alpha \cdot x\overline{y})\overline{z} = x\alpha \cdot (x \;\cdot \overline{y}\ \overline{z})$$

And replacing $\overline{y}$ and $\overline{z}$ by y and z respectively we have

$$(x\alpha \cdot xy)z \;=\; x\alpha(x \cdot yz)$$

$\forall\ x, y, z \in L$ and $\forall\ \alpha\ \in A(L)$, which is equation (3.1).

The converse is obtained by reversing the process.

Corollary 3.1. *Let $(L, \cdot)$ be a loop, and $A(L)$ be the group of all automorphism of L, then L is an LC-loop if*

$$B \;=\; \langle L_x L_{x\alpha}, I, L_x L_{x\alpha}\rangle \tag{3.3}$$

is an autotopism of L, $\forall\ x, y, z\ \in L$ and $\forall\ \alpha \in A(L)$

Proof. This is a consequence of (3.1)

Theorem 3.2. *Let $(L, \cdot)$ be a loop and $A(L)$ be a group of automorphism of $(L, \cdot)$. Then the $A(L)$-holomorph (H,o) of $(L, \cdot)$ is an RC-loop if and only if*

$$y((z \cdot x\alpha)x) = \;(yz \cdot x\alpha)x \tag{3.4}$$

$\forall\ x, y, z \in L$ and $\forall\ \alpha \in A(L)$.

Proof.

The procedure for the proof is like that of Theorem 3.1 above hence it is omitted.

Corollary 3.2. *Let $(L, \cdot)$ be any loop and $A(L)$ be the group of all automorphisms of L, then L is an RC-loop if and only if*

$$B \;=\; \langle I, R_{x\alpha}R_x, R_{x\alpha}R_x\rangle \tag{3.5}$$

is an autotopism of L, for all $x, y, z\ \in L$ and all $\alpha \in A(L)$

Proof.
From 3.4)

$$y((z \cdot x\alpha)x) \ = \ (yz \cdot x\alpha)x$$
$$\Rightarrow \ y \cdot zR_{x\alpha}R_x = yzR_{x\alpha}R_x$$

$\forall\ x, y, z \in L$ *and* $\forall\ \alpha \in A(L)$.

$$\Rightarrow \ \langle I, R_{x\alpha}R_x, R_{x\alpha}R_x \rangle$$

is an autotopism of $(L, \cdot)$ $\forall\ x \in L$ *and* $\forall\ \alpha \in A(L)$.

Conversely, suppose (3.5) hold, then $\forall\ y, z \in L$ we have

$$yI \cdot zR_{x\alpha}R_x \ = \ yzR_{x\alpha}R_x$$
$$y((z \cdot x\alpha)x) \ = \ yz(x\alpha \cdot x)$$

$\forall\ x, y, z \in L$ and $\forall\ \alpha \in A(L)$.

Theorem 3.3. *Let $(L, \cdot)$ be a loop and $A(L)$ be a group of automorphism of $(L, \cdot)$. Then the $A(L)$-holomorph (H,o) of (L,.) is a C-loop if and only if*

$$(y \cdot x\alpha)x \cdot z \ = \ y(x\alpha \cdot xz) \tag{3.6}$$

$\forall\ x, y, z \in L$ and $\forall\ \alpha \in A(L)$.

Proof.
The procedure for the proof is like that of theorem 3.1 hence it is omitted.

Corollary 3.3. *Let (L,.) be a loop and $A(L)$ be the group of all automorphisms of L, then L is a C-loop if and only if*

$$B \ = \ \langle R_{x\alpha}R_x, L_{x\alpha}^{-1}L_{x^{-1}}, I \rangle \tag{3.7}$$

is an autotopism of L, for all $x, y, z\ \in$L and all $\alpha\ \in$$A(L)$

Proof. From (3.6)

$$(y \cdot x\alpha)x \cdot z \ = \ y(x\alpha \cdot xz)$$
$$\Rightarrow \ yR_{x\alpha}R_x \cdot z = y \cdot zL_xL_{x\alpha}$$

$\forall\ x, y, z \in L$ *and* $\forall\ \alpha \in A(L)$.
substituting $\overline{z} = zL_xL_{x\alpha}$ we have

$$yR_{x\alpha}R_x \cdot \overline{z}L_{(x\alpha)^{-1}}L_{x^{-1}} = y\overline{z}$$

$\forall\ x, y, \overline{z} \in L$ *and* $\forall\ \alpha \in A(L)$.

$$\Rightarrow\ \langle R_{x\alpha} R_x, L_{(x\alpha)^{-1}} L_{x^{-1}}, I\rangle$$

is an autotopism of $(L, \cdot)$ $\forall\ x \in L$ *and* $\forall\ \alpha \in A(L)$.

Conversely, suppose equation (3.7) is an autotopism of $(L, \cdot)$, therefore '$\forall\ y, z \in L$ we have

$$yR_{x\alpha}R_x \cdot zL_{x\alpha}^{-1}L_{x^{-1}} \;=\; yz \cdot I$$

$$yR_{x\alpha}R_x \cdot \overline{z} \;=\; y \cdot \overline{z}L_x L_{x\alpha} I$$

$$(y \cdot x\alpha)\overline{z} \;=\; y(x\alpha \cdot x\overline{z})$$

$\forall\, x, y, \overline{z} \in L$ *and* $\forall\ \alpha \in A(L)$ hence $(L, \cdot)$ is a C-loop.

3.1. Nuclear Automorphism

Theorem 3.4. *Let $(L, \cdot)$ be a loop and $A(L)$ be a group of automorphism of $(L, \cdot)$. Then the $A(L)$-holomorph (H,o) of $(L, \cdot)$ is a C-loop iff $(L, \cdot)$ is a C-loop and each $\alpha \in A(L)$ is a left nuclear automorphism of $(L, \cdot)$.*

Proof. Suppose (H, o) is a C-loop. Since $(L, \cdot)$ is isomorphic to a subloop of (H, o), it follows that $(L, \cdot)$ must be a C-loop. From Theorem (3.1), equation (3.1) holds $\forall\ x, y, z \in L$ *and* $\forall\ \alpha \in A(L)$. Furthermore, by Theorem (3.1) and Corollary (3.3),

$$A(x) = \langle R_x^2, L_x^{-2}, I\rangle \text{ and } B(x) = \langle R_x R_{x\alpha}, L_x^{-1} L_{x\alpha}^{-1}, I\rangle$$

are autotopisms of $(L, \cdot), \forall\ x \in L$ and $\forall\ \alpha \in A(L)$. Therefore by Theorem (3.1)and we have
$A_\lambda(x) = \langle L_x^{-2}, I, L_x^{-2}\rangle$, $A_\mu^{-1}(x) = \langle I, R_x^2, R_x^2\rangle$,
$B_\lambda^{-1}(x) = \langle L_{x\alpha} L_x, I, L_{x\alpha} L_x\rangle$ *and* $B_\mu(x) = \langle I, R_x R_{x\alpha}, R_x R_{x\alpha}\rangle$
are also autotopisms of $(L, \cdot), \forall\ x \in L$ and $\forall\ \alpha \in A(L)$. If these are combined we have

$$A_\lambda(x) B_\lambda^{-1}(x) = \langle L_x^{-2}, I, L_x^{-2}\rangle\langle L_x L_{x\alpha}, I, L_x L_{x\alpha}\rangle$$

$$A_\lambda(x) B_\lambda^{-1}(x) = \langle L_x^{-1} L_{x\alpha}, I, L_x^{-1} L_{x\alpha}\rangle \tag{3.8}$$

and

$$B_\mu(x) A_\mu^{-1}(x) = \langle I, R_{x\alpha} R_x, R_{x\alpha} R_x\rangle\langle I, R_x^{-2}, R_x^{-2}\rangle$$

$$(3.9) \qquad B_\mu(x)A_\mu^{-1}(x) = \langle I, R_{x\alpha}R_x^{-1}, R_{x\alpha})R_x^{-1}\rangle$$

as autotopisms of $(L,\cdot), \forall\ x \in L$ and $\forall\ \alpha \in A(L)$. Now if we apply (3.8) and (3.9) to $1\cdot b$ and $a\cdot 1$ respectively, we have

$$1L_x^{-1}L_{x\alpha}\cdot b = (1\cdot b)L_x^{-1}L_{x\alpha}$$

$$(x\alpha\cdot x^{-1})b = bL(x)^{-1}L_{x\alpha}$$

$$bL_{x\alpha}L_x^{-1} = bL_x^{-1}L_{x\alpha}$$

and

$$a\cdot 1R_{x\alpha}R_x^{-1} = (a\cdot 1)R_{x\alpha}R_x^{-1}$$

$$a(x\alpha\cdot x^{-1}) = aR_{x\alpha}R_x^{-1}$$

$$aR_{x\alpha\cdot x^{-1}} = aR_{x\alpha}R_x^{-1}$$

and respectively we have

$$(3.10) \qquad L_{x\alpha\cdot x^{-1}} = L_x^{-1}L_{x\alpha}$$

$$(3.11) \qquad R_{x\alpha\cdot x^{-1}} = R_{x\alpha}R_x^{-1}$$

$\forall\ x \in L$ *and* $\forall\ \alpha \in A(L)$. If we put equations(3.10) and (3.11) into equations(3.8) and (3.9) respectively, we have

$$A_\lambda(x)B_\lambda^{-1}(x) = \langle L_{x\alpha\cdot x^{-1}}, I, L_{x\alpha\cdot x^{-1}}\rangle$$

and

$$B_\mu(x)A_\mu^{-1}(x) = \langle I, R_{x\alpha\cdot x^{-1}}, R_{x\alpha\cdot x^{-1}}\rangle$$

$\forall\ x \in L$ *and* $\forall\ \alpha \in A(L)$. These therefore imply that $x\alpha\cdot x^{-1} \in N_\lambda(L)$ and $x\alpha\cdot x^{-1} \in N_\rho(L)$. Consequently, $x\alpha\cdot x^{-1} \in N(L)$ since $(L,\cdot)$ is an inverse property loop. Hence $\alpha \in A(L)$, is left nuclear.

Conversely, suppose $(L,\cdot)$ is a C-loop and each $\alpha \in A(L)$ is left nuclear. Then for each $\alpha \in A(L)$ and each $x \in L$ the element $x\alpha\cdot x^{-1} \in N_\mu(L)$, thus

$$x\alpha\cdot y = ((x\alpha\cdot x^{-1})x)y$$

$$x\alpha\cdot y = (x\alpha\cdot x^{-1})xy$$

$\forall\ y \in L$

$$yL_{x\alpha} = yL_xL_{x\alpha\cdot x^{-1}} \ \Rightarrow\ L_x^{-1}L_{x\alpha} = L_{x\alpha\cdot x^{-1}}$$

$\forall\ x \in L$ *and* $\forall\ \alpha \in A(L)$. But for $\forall\ x \in L$ *and* $\forall\ \alpha \in A(L)$, we know that $x\alpha \cdot x^{-1} \in N_\lambda(L)$. Hence,

$$C = \langle L_{\alpha \cdot x^{-1}}, I, L_{x\alpha \cdot x^{-1}} \rangle = \langle L_x^{-1} L_{x\alpha}, I, L_x^{-1} L_{x\alpha} \rangle$$

is an autotopism of $(L, \cdot), \forall\ x \in L$ and $\forall\ \alpha \in A(L)$. But again , $A = \langle L_x^2, I, L_x^2 \rangle$ is an autotopism of $(L, \cdot), \forall\ x \in L$. Therefore,

$$AC = \langle L_x L_{x\alpha}, I, L_x L_{x\alpha} \rangle$$

is an autotopism of $(L, \cdot), \forall\ x \in L$ and $\forall\ \alpha \in A(L)$. So also is $(AC)_\lambda^{-1} = \langle R_{x\alpha} R_x, L_{x\alpha}^{-1} L_x^{-1}, I \rangle$. Therefore of $yz, \forall\ y, z \in L$, we have

$$yR_{x\alpha} R_x \cdot z L_{x\alpha}^{-1} L_x^{-1} = yz$$

if we put $\overline{z} = z L_{x\alpha}^{-1} L_x^{-1}$, in this we have

$$yR_{x\alpha} R_x \cdot \overline{z} = y \cdot \overline{z} L_x L_{x\alpha}$$

$$((y \cdot x\alpha)x)\overline{z} = y(x\alpha \cdot x\overline{z})$$

$\forall\ x, y, \overline{z} \in L$ and $\forall\ \alpha \in A(L)$. Replacing $\overline{z}$ by z, $\forall\ x, y, z \in L$ and $\forall\ \alpha \in A(L)$ and we have a central identity. Hence, (H, o) is a C-loop.

Theorem 3.5. *The set $S(L)$ of all left nuclear automorphism of an C-loop $(L, \cdot)$, is a normal subgroup of the automorphism group of $(L, \cdot)$.*

Proof. $S(L) \neq \emptyset$, from the Theorem 3.4 it was shown that

$$L_{u\alpha \cdot u^{-1}} = L_u^{-1} L_{u\alpha}$$

$\forall\ u \in L$ and $\forall\ \alpha \in S(L)$ (since for an inverse property loop L, $L_{u^{-1}} = L_u^{-1}$ $\forall u \in L$). Then $u\alpha \cdot u^{-1} \in N_\lambda(L, \cdot), \forall\ u \in L$ and $\forall\ \alpha \in S(L)$. It follows then that

$$A(\alpha, u) = \langle L_{u\alpha \cdot u^{-1}}, I, L_{u\alpha \cdot u^{-1}} \rangle = \langle L_u^{-1} L_{u\alpha}, I, L_u^{-1} L_{u\alpha} \rangle$$

$\forall\ u \in L$ and $forall\ \alpha \in L$. Hence if $\alpha, \beta \in S(L)$, we have

$$A(\alpha, u) A(\beta, u\alpha) = \langle L_u^{-1} L_{u\alpha}, I, L_u^{-1} L_{u\alpha} \rangle \langle L_{u\alpha}^{-1} L_{u\alpha\beta}, I, L_{u\alpha}^{-1} L_{u\alpha\beta} \rangle$$

$$A(\alpha, u) A(\beta, u\alpha) \ = \ \langle L_u^{-1} L_{u\alpha\beta}, I, L_u^{-1} L_{u\alpha\beta} \rangle \tag{3.12}$$

is an autotopism of $(L,\cdot), \forall\ u \in L$. Therefore $\forall\ y \in L$ we have

$$\begin{aligned} 1L_u^{-1}L_{u\alpha\beta}\cdot y &= (1\cdot y)L_u^{-1}L_{u\alpha\beta} \\ (u\alpha\beta\cdot u^{-1})\cdot y &= yL_u^{-1}L_{u\alpha\beta} \\ yL_{u\alpha\beta\cdot u^{-1}} &= yL_u^{-1}L_{u\alpha\beta} \\ \\ \Rightarrow L_{u\alpha\beta\cdot u^{-1}} &= L_{u^{-1}L_{u\alpha\beta}} \end{aligned}$$

(3.13)

Thus, (3.13) into (3.12) gives

$$A(\alpha,u)A(\beta,u\alpha) = \langle L_{u\alpha\beta.u^{-1}}, I, L_{u\alpha\beta.u^{-1}}\rangle \tag{3.14}$$

From equation (3.14), $u\alpha\beta.u^{-1} \in N_\lambda(L,\cdot), \forall u \in L$, hence $u\alpha\beta.u^{-1} \in N$, for all $u \in L$ and so $\alpha\beta \in S(L)$, since $(L,\cdot)$ is an inverse property loop.

If $\alpha \in S(L)$, then $A(\alpha,u)$ is an autotopism of $(L,\cdot)\forall\ u \in L$, so also is $A(\alpha,u\alpha^{-1})^{-1}\ \forall\ u \in L$, i.e

$$\begin{aligned} A(\alpha,u\alpha^{-1})^{-1} &= \langle L_{u\alpha^{-1}}^{-1}L_{u\alpha^{-1}.\alpha}, I, L_{u\alpha^{-1}}^{-1}L_{\alpha^{-1}.\alpha}\rangle^{-1} \\ &= \langle L_{u\alpha^{-1}}^{-1}L_u, I, L_{u\alpha^{-1}}^{-1}L_u\rangle^{-1} \\ &= \langle L_u^{-1}L_{u\alpha^{-1}}, I, L_u^{-1}L_{u\alpha}^{-1}\rangle \\ &= \langle L(u\alpha^{-1}.u^{-1}), I, L(u\alpha^{-1}.u^{-1})\rangle \end{aligned}$$

Hence it follows that $\alpha^{-1} \in S(L)$. Thus $S(L)$ is a subgroup of the automorphism group of $(L,\cdot)$.
Let $\alpha \in S(L)$, then $u\alpha\cdot\alpha^{-1} \in N_\lambda(L,\cdot), \forall\ u \in L$ and

$$(u\alpha.u^{-1})xy = (u\alpha.u^{-1})x.y$$

$\forall\ u,x,y \in L$, if γ is an automorphism of $(L,\cdot)$, then we have

$$\{u\alpha\gamma\cdot(u\gamma)^{-1})\}(x\gamma\cdot y\gamma) = \{u\alpha\gamma\cdot(u\gamma)^{-1}\}x\gamma\cdot y\gamma$$

$\forall\ u,x,y \in L$, and if we replace u by $u\gamma^{-1}$ in the last expression, we have

$$(u\gamma^{-1}\alpha\gamma\cdot u^{-1})(x\gamma\cdot y\gamma) = (u\gamma^{-1}\alpha\gamma\cdot u^{-1})x\gamma\cdot y\gamma$$

Thus, $u\gamma^{-1}\alpha\gamma\cdot u^{-1} \in N_\lambda(L,\cdot)$ and since L is an inverse property loop, the three nuclei coincide, then $u\gamma^{-1}\alpha\gamma\cdot u^{-1} \in N(L,\cdot)$ for all $u \in L$ and all automorphism γ of $(L,\cdot)$. Hence $\gamma^{-1}\alpha\gamma \in S(L)$ for all $\alpha \in S(L)$ and all automorphism γ of $(L,\cdot)$. So $S(L)$ is indeed normal in the automorphism group of $A(L)$ of $(L,.)$.

4. Bryant-Schneider group

Theorem 4.1. *Let $(L,\cdot)$ be a C-loop, an element θ of the Bryant-Schneider group of L is an automorphism of L provided*

$$< \theta R_{g^{-1}}, \theta L_{f^{-1}}, \theta >$$

is an autotopism of $(L,.)$ if f and g are elements of the nucleus of $(L,.)$.

Proof : Let $(L,.)$ is a C-loop then

$$< R_{y^{-1}} R_{y^{-1}}, L_y L_y, I >$$

is an autotopism for all $x \in L$. $\theta \in BS(L,.)$ imply that $< \theta R_{g^{-1}}, \theta L_{f^{-1}}, \theta >$ is also an autotopism for some $g, f \in (L,.)$

Hence $< \theta R_{g^{-1}}, \theta L_{f^{-1}}, \theta >< R_{y^{-1}} R_{y^{-1}}, L_y L_y, I >=$ $< \theta R_{g^{-1}} R_{y^{-1}} R_{y^{-1}}, \theta L_{f^{-1}} L_y L_y, \theta >$ is an autotopism of for all $y \in L$ and some $g, f \in L$. Since $(L,.)$ is an alternative property loop, then

$$R_{y^{-1}} R_{y^{-1}} = R_{(y^{-1})^2} = R_{(y^2)^{-1}}$$

and $L_y L_y = L_{y^2}$ therefore $< \theta R_{g^{-1}} R_{y^{-1}} R_{y^{-1}}, \theta L_{f^{-1}} L_y L_y, \theta >=$ $< \theta R_{g^{-1}} R_{(y^2)^{-1}}, \theta L_{f^{-1}} L_{y^2}, \theta >$. If $g = (y^2)^{-1}$ and $f = y^2$ we obtain $< \theta, \theta, \theta >$ Hence θ is an automorphism of $(L,.)$. $g = (y^2)^{-1}$ and $f = y^2$ implies that $f = g^{-1} = y^2$. Then it follows that f and g are elements of $N(L,.)$ the nucleus of $(L,.)$ since the square of every element $y \in L$ belongs to $N(L,.)$.

Theorem 4.2. *Let $(L,.)$ be a C-loop and let $\theta \in S(L,.)$ (the symmetric group of L). Then $\theta \in BS(L,.)$ if there is a unique $\alpha \in P(L,.)$ (the set pseudo-automorphisms of $(L,.)$) and a unique $f \in N(L,.)$ such that $\theta = \alpha R_f (\alpha = \theta R_f^{-1})$.*

Proof :
Let $(L,.)$ be a C-loop then

$$A =< R_{x^{-1}} R_{x^{-1}}, L_x L_x, I >$$

an autotopism of $(L,.)$ for all $x \in L$.
$B =< I, R_{x^2}, R_{x^2} >=< R_{x^2}, \rho R_{x^2} \rho, I >$ is also an autotopism for all $x \in L$.
Therefore by Bruck[4]

$$BA =< R_{x^2}, \rho R_{x^2} \rho, I >< R_{x^{-1}} R_{x^{-1}}, L_x L_x, I >=< I, \rho R_{x^2} \rho L_x L_x, I >$$

is an autotopism for all $x \in L$. $\theta \in BS(L,.)$ implies that

$$C =< \theta R_{f^{-1}}, \theta L_{g^{-1}}, \theta >$$

is an autotopism for some $f, g \in L$

$$CBA = \theta R_{f^{-1}}, \theta L_{g^{-1}}, \theta >< I, \rho R_{x^2}\rho L_x L_x, I >=$$
$$< \theta R_{f^{-1}}, \theta L_{g^{-1}}\rho R_{x^2}\rho L_x L_x, \theta >$$

which implies that $< \alpha, \theta L_{g^{-1}}\rho R_{x^2}\rho L_x L_x, \alpha R_f >$ is autotopism of for some $f, g \in Q$ and all $x \in L$. Now if

$$< \alpha, \theta L_{g^{-1}}\rho R_{x^2}\rho L_x L_x, \alpha R_f >$$

is an autotopism we have $s\alpha.t\beta = (s.t)\alpha R_f$ for all $s, t \in L$ where $\beta = \theta L_{g^{-1}}\rho R_{x^2}\rho L_x L_x$. If s is set to be e in the last autotopism and noting that $e\alpha = e\theta R_{e\theta} = e$ we get $\beta = \alpha R_f$ therefore $< \alpha, \alpha R_f, \alpha R_f >$ is an autotopism of $(L,.)$ for some $f \in L$ hence α is a pseudo-automorphism with companion f. $\theta = \alpha R_f$ implies that the elements of the Bryant-Schneider group of a C-loop $(L,.)$ can be expresses in terms of pseudo-automorphisms $P(L,.)$ and right translations of elements of the nucleus of $(L,.)$. To show uniqueness, let $\alpha_1 R_{x_1} = \alpha_2 R_{x_2}$ where $\alpha_1, \alpha_2 \in P(L,.)$ and $x_1, x_2 \in N(L,.)$. Then $\alpha_2^{-1}\alpha_1 = R_{x_2}R_{x_1}^{-1}$ which implies that $e\alpha_2^{-1}\alpha_1 = eR_{x_2}R_{x_1}^{-1}$. Then we observe that $e = x_2 x_1^{-1}$ and therefore $x_1 = x_2$. It the follows that $\alpha_1 = \alpha_2$.

Remark 4.1. *Robinson[21] considered the Bryant-Schneider group of a Bol loop and found out that they can be expressed as a product of pseudo-automorphisms and right translations. Theorem 2.2 above shows that the Bryant-Schneider group of a C-loop can also be expressed in the same way. This further emphasis the fact that C-loops are analogous to Moufang loops since Moufang loops satisfies the Bol identities(right and left).*

Theorem 4.3. *Let $(L,.)$ be a C-loop . If $x, y \in Q$, let $\odot$ be a binary operation defined on the pseudo-automorphism $PS(L,.)$ by*

$$\alpha \odot \beta = \alpha R_x \beta R_y R_{(x\beta.y)^{-1}}$$

for all $\alpha\beta \in PS(L,.)$. Let $H = PS(L,.) \times Q$ and for

$$(\alpha, x) \circ (\beta, y) = (\alpha \odot \beta, x\beta.y).$$

Then $(H, \circ)$ a group which is isomorphic to $BS(L,.)$.

Proof :
Let $\alpha, \beta \in PS(L, .)$ and let $x, y \in N(L, .)$ the nucleus of $(L, .)$. Then we know from the immediate preceding theorem that there exist unique $\delta \in PS(L, .)$ and unique $z \in N(L, .)$ such that $\alpha R_x \beta R_y = \delta R_z$. Thus we observe that

$$(u\alpha.x)\beta y = u\delta.z$$

for all $u \in L$. If we set $u = e$ we obtain $x\beta.y = z$. Therefore $\alpha R_x \beta R_y = \delta R_{(x\beta.y)^{-1}}$ and so

$$\delta = \alpha R_x \beta R_y R_{(x\beta.y)^{-1}} = \alpha \odot \beta$$

Hence $\odot$ is a closed binary operation of $PS(L, .)$. It is also obvious now that $(\alpha, x) \mapsto \alpha R_x$ provided $x \in N(L, .)$ gives an isomorphism of $(H, \circ)$ onto the $BS(L, .)$ of a C-loop. Hence the Bryant-Schneider group of a C-loop is a form generalized holomorph of the loop.

Theorem 4.4. *A finite C-loop is isomorphic to all its loop isotopes if*

$$[(L, .) : N(L, .)]^2 = [PS(L, .) : A(L)]$$

where $A(L)$ is the automorphism group of $(L, .)$

Proof :
By Theorem 4.2 it is clear that $\mid BS(L, .) \mid = \mid L \mid\mid PS(L, .) \mid$. By Bryant & Schneider[2] $(L, .)$ is isomorphic to all its loop isotopes if

$$|L|^2 |A(L, .)| = |BS(L, .)||N_\mu(L, .)|$$

But in a C-loop the nuclei coincide hence $\mid N_\mu(L, .) \mid = \mid N(L, .) \mid$. Now by Theorem 4.2 $|BS(L, .)| = |PS(L, .)||N(L, .)|$ and therefore we have

$$|L|^2 |A(L, .)| = |PS(L, .)||N(L, .)|^2$$

which implies that

$$\left[\frac{|L|}{|N(L, .)|}\right]^2 = \frac{|PS(L, .)|}{|A(L, .)|}$$

which is the same as

$$[L : N(L, .)]^2 = [PS(L, .) : A(L, .)]$$

as required.

Corollary 4.1. *Let $(L, .)$ be a C-loop then*

$$[PS(L, .) : A(L, .)] \neq 4$$

Proof :
The proof follows directly from Lemma 2.9 of [20] and Theorem 4.4

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