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The signature in actions of semisimple Lie groups on pseudo-Riemannian manifolds

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Abstract

We study the relationship between the signature of a semisimple Lie group and a pseudoRiemannian manifold on wich the group acts topologically transitively and isometrically. We also provide a description of the bi-invariant pseudo-Riemannian metrics on a semisimple Lie Group over R in terms of the complexification of the Lie algebra associated to the group, and then we utilize it to prove a remark of Gromov.

Keywords : *semisimple Lie groups, bi-invariant metric, local freeness.*

Subjclass : *[2000] Primary: 53C05; Secondary: 53C10.*

1. Introduction

On a semisimple Lie group G the Killing–Cartan form is invariant by automorphisms, and it defines an $Ad(g)$-invariant scalar product on $Lie(G) = \mathbf{g}$. Then the left action of G on itself joint to the Killing–Cartan form of $\mathbf{g}$ provide a pseudo-Riemannian structure on G which is bi-invariant. This permits us to study semisimple Lie groups from the point of view of geometry, i.e. we choose an appropiate pseudo-Riemannian metric and compute the various geometrical objects, such as curvature, and geodesics.

It is known that there is a bijective correspondence between the $Ad(g)$-invariant nondegenerate symmetric bilinear forms on $\mathbf{g}$ and the bi-invariant pseudo-Riemannian metrics on G. Under such correspondence, a bilinear form on $\mathbf{g}$ which is not a multiple of the Killing–Cartan form defines a pseudo-Riemannian metric on G that might be expected to provides a geometry that differs from the one given by the Killing–Cartan form. The first thing we want to prove is the fact that such situation does not occur, i.e. every bi-invariant pseudo-Riemannian metric on a semisimple Lie group is a finite sum of Killling–Cartan forms.

We inquire about the relationship of the pseudo-Riemannian invariants of G and M, respectively, for some bi-invariant pseudo-Riemannian metric on G. In this work, we restrict our attention to the signature, which we will denote with (m_1, m_2) and (n_1, n_2) for M and G, respectively.

The second goal of this work is to obtain an estimate between the signatures of G and M, in the case of $G = G_1 \cdots G_l$ and each G_i is a connected simple Lie group and carries a bi-invariant pseudoRiemannian metric. If we denote $n_0^i = \min\{n_1^i, n_2^i\}$ and $m_0 = \min\{m_1, m_2\}$, then we are going to prove that $n_0^1 + \cdots + n_0^l \leq m_0$.

The organization of this article is as follows. In section 2 we collect some basic results about complexification of a real Lie algebra and invariant bilinear forms on a simple Lie algebra that will needed in the proof of the main theorem on that section. Also we give the classification of the $Ad(g)$-invariant bilinear forms on a semisimple Lie algebra. This is mentioned in [2], but the generalization to semisimple Lie groups is new. As a consequence we give the classification of the bi-invariant pseudo-Riemannian metrics on G. In section 3 we use the results obtained previously to obtain an estimated between the signatures of the metrics on M and G, respectively.

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2. Complexification of a real Lie algebra

Let V be a vector space over R of finite even dimension. A complex structure on V is an R-linear endomorphism J of V such that $J^2 = -I$. When there exist a complex structure J on V, we define V_J as the complex vector space associated to V by the rule: $(a + ib)X = aX + bJX$ for $X \in V$ and $a, b \in R$.

A Lie algebra $\mathbf{g}$ over R is said to have a compatible complex structure J if J is a complex structure on the real vector space $\mathbf{g}$ and in addition $[X, JY] = J[X, Y]$ for $X, Y \in \mathbf{g}$. It is easy to see that $\mathbf{g}_C$ then becomes a complex Lie algebra.

If V is an arbitrary finite dimensional vector space over R, the R-linear map $J : (X, Y) \mapsto (-Y, X)$ is a complex structure on $V \times V$. The complex vector space $(V \times V)_J$ is called the complexification of V and will be denoted by V^C. We write $X + iY$ instead of (X, Y) in V^C.

If $\mathbf{g}$ is a Lie algebra over R, owing to the conventions above, the complex space $\mathbf{g}^C$ consists of all symbols $X + iY$ with $X, Y \in \mathbf{g}$, and it is a complex Lie algebra whose Lie bracket is given by

$$[X + iY, Z + iT] = [X, Y] - [Y, T] + i\Big([Y, Z] + [X, T]\Big).$$

Definition 1. *A real Lie algebra $\mathbf{g}_0$ is called a real form of a complex Lie algebra $\mathbf{g}$ if its complexification $(\mathbf{g}_0)^C$ is isomorphic to $\mathbf{g}$ as a complex Lie algebra.*

For semisimple Lie algebras over C the existence of a real form is proved in [1, Thm. III.6.3]. This real form is also compact, which means that the Killing-Cartan form of $\mathbf{g}$ is strictly negative definite.

If $\mathbf{g}$ is a real Lie algebra and $T\colon \mathbf{g} \to \mathbf{g}$ is a R-linear map, then there is a C-linear map $T^C : \mathbf{g}^C \to \mathbf{g}^C$ defined by $T^C(X + iY) = TX + iTY$.

The complexification of the adjoint representation $ad\colon \mathbf{g} \to \mathbf{gl}(\mathbf{g})$ is the C-linear map $ad^C\colon \mathbf{g}^C \to \mathbf{gl}(\mathbf{g}^C)$ given by

$\mathrm{ad}^C(X + iY)(Z + iW) = [X + iY, Z + iW]$
$= [X, Z] - [Y, W] + i[X, W] + i[Y, Z]$.

A trivial calculation shows that if T and $ad^C(X)$ commute for every $X \in \mathbf{g}$, then T^C y $ad^C(X + iY)$ commute for every $X, Y \in \mathbf{g}$.

For an R-linear map $T\colon \mathbf{g} \to \mathbf{g}$ and a given complex structure J on $\mathbf{g}$, we can consider the C-linear map $T_J\colon \mathbf{g}_J \to \mathbf{g}_J$ defined by $T_J(X) = T(X) - JTJ(X)$. Consider also the adjoint representation $ad\colon \mathbf{g}_J \to \mathbf{gl}(\mathbf{g}_J)$. It is easy to show that if T and $ad(X)$ commute for each $X \in \mathbf{g}$, then T_J commutes with $ad(Z)$ for each $Z \in \mathbf{g}_J$.

The next proposition (for a proof see [1, Thm. X.1.5]) it will be useful later.

Proposition 1. *For a simple Lie algebra* $\mathbf{g}$ *over* R *there are two possibilities:*

1. $\mathbf{g}^C$ *is simple;*
2. $\mathbf{g}^C$ *is nonsimple; in this case* $\mathbf{g}$ *admits a complex structure* J *whose associated complex Lie algebra* $\mathbf{g}_J$ *is simple.*

For the above proposition, the classification of $Ad(G)$-invariant bilinear forms should be separated into two cases. The first case is the following.

Theorem 1. *Let* $\mathbf{g}$ *be a simple Lie algebra over* R *which has simple complexification. Then the* $Ad(G)$*-invariant bilinear forms of* $\mathbf{g}$ *are multiples of the Killing–Cartan form of* $\mathbf{g}$.

Proof. Let D be an $Ad(G)$-invariant bilinear form on $\mathbf{g}$.

Since the Killing–Cartan form B of $\mathbf{g}$ is nondegenerate, there is a linear map $T\colon \mathbf{g} \to \mathbf{g}$ such that $D(X,Y) = B(X,TY)$ for each $X,Y \in \mathbf{g}$. Using the $Ad(G)$-invariance of D we obtain

B(X,TY) = D(X,Y)
= D(Ad(g)X, Ad(g)Y)
= B(Ad(g)X, T Ad(g)Y)
= B(X, Ad(g^{-1})$TAd(g)Y$),

so $TY - Ad(g^{-1})TAd(g)Y = 0$. Therefore $Ad(g)T = TAd(g)$ for each $g \in G$.

We claim that $ad(X)T = Tad(X)$ for each $x \in \mathbf{g}$. Taking $g = \exp(tX)$ we obtain $Ad(\exp(tX))T = TAd(\exp(tX))$, and from the identity $Ad(\exp(X)) = e^{Ad(X)}$ we conclude that $Te^{tad(X)} = e^{tad(X)}T$.

Differentiating the last equation at $t = 0$, we obtain $ad(X)T = Tad(X)$.

From the first remark above we obtain that T^C y $ad^C(X+iY)$ commute for every $X,Y \in \mathbf{g}$. Let $\lambda \in C$ be an eigenvalue of T^C. Since $\mathbf{g}^C$ is simple and ad^C is an irreducible representation, by Schur's Lemma we conclude that $T^C = \lambda I$, then $T^C(X+iY) = \lambda(X+iY)$. If we write $\lambda = \lambda_1 + i\lambda_2$, then it is very easy to conclude that $T = \lambda_1 I$, where $\lambda_1 \in R$, and then $D = \lambda_1 B$. □

Theorem 2. *Let* $\mathbf{g}$ *be a simple Lie algebra over* R *which has nonsimple complexification and let* J *be a fixed complex structure on* $\mathbf{g}$. *Let* D *be an* $Ad(G)$*-invariant bilinear form on* $\mathbf{g}$, *then* $D(X,Y) = aB(X,Y)+bB(X,JY)$ *for some* $a, b \in R$ *and for each* $X, Y \in \mathbf{g}$.

Proof. There is a linear map $T\colon \mathbf{g} \to \mathbf{g}$ such that $D(X,Y) = B(X,TY)$ for each $X, Y \in \mathbf{g}$, where B is the Killing–Cartan form of $\mathbf{g}$.

We shall prove that T satisfies $TJ+JT = \lambda_1 I+\lambda_2 J$ for some $\lambda_1, \lambda_2 \in R$.

By the $Ad(G)$-invariance of D we conclude that T and $Ad(g)$ commute for each $g \in G$. From the second remark above we obtain that T_J and $ad(X)$ commute for each $X \in \mathbf{g}_J$.

Using the irreducibility of the adjoint representation of ad_J, its follow by Schur's Lemma that there is $\lambda = \lambda_2 - i\lambda_1 \in C$ such that $T_J = \lambda I$. From this we obtain $TJ + JT = \lambda_1 I + \lambda_2 J$, where $\lambda_1, \lambda_2 \in R$.

Define $\widehat{T} = T - \frac{1}{2}\lambda_2 I + \frac{1}{2}\lambda_1 J$. Then $\widehat{T}$ and $ad(X)$ commute for each $X \in \mathbf{g}$ and it is easy to check that $\widehat{T}J + J\widehat{T} = 0$.

We can find an orthogonal basis $\{X_1, \dots, X_r\}$ of the complex Lie algebra $\mathbf{g}_J$ such that $\{X_1, \dots, X_r, J(X_1), \dots, J(X_r)\}$ is a basis of $\mathbf{g}$ over R. Let $\mathbf{g}_k$ be the R-linear subspace given by

$$\mathbf{g}_k := \sum_{x\in\Delta} R(iH_\alpha) + \sum_{x\in\Delta} R(X_\alpha - X_\alpha) + \sum_{x\in\Delta} R(i(X_\alpha + X_\alpha)),$$

where Δ is the corresponding set of nonzero roots of $\mathbf{g}_J$, and for each $\alpha \in \Delta$ we select $X_\alpha \in \mathbf{g}^\alpha$ with the properties of [1, Thm. III.5.5]. It follows that B, the Killing-Cartan form of $\mathbf{g}_J$, is strictly negative definite on $\mathbf{g}_k$. Moreover, $\mathbf{g}_J = \mathbf{g}_k \oplus J\mathbf{g}_k$. Normalizing the basis given by $\mathbf{g}_k$ we obtain a new basis of $\mathbf{g}_J$ given by $\{X_1, \dots, X_r\}$ such that $\{X_1, \dots, X_r, J(X_1), \dots, J(X_r)\}$ is a basis of $\mathbf{g}$ over R. It is clear that $B_{\mathbf{g}_k}(Y_i, Y_j) = \delta_{ij}\epsilon_i$, for $Y_i, Y_j \in \mathbf{g}$, where $\epsilon_i = 1$ if $i = 1, \dots, r$ and $\epsilon_i = -1$ if $i = r+1, \dots, 2r$. Using [1, Lm. III.6.1]) we conclude that $B_{\mathbf{g}}(X,Y) = (B(X,Y))(B(X,Y))$ for $X, Y \in \mathbf{g}$.

In this basis the matrix representation of the complex structure J and the Killing-Cartan form of $\mathbf{g}$ are given by

$$J = \begin{pmatrix} 0 & -I_r \\ I_r & 0 \end{pmatrix}, \qquad B_{\mathbf{g}} = \begin{pmatrix} I_r & 0 \\ 0 & I_r \end{pmatrix}.$$

If the matrix representation of T in that basis has the form

$$T = \begin{pmatrix} A_0 & B_0 \\ C_0 & D_0 \end{pmatrix},$$

then we deduce from the identity $TJ + JT = \lambda_1 I + \lambda_2 J$ that

$$T = \begin{pmatrix} A_0 & C_0 + \lambda_1 I \\ C_0 & -A_0 + \lambda_2 I \end{pmatrix}.$$

If we rewrite T in the following form

$$2\mathrm{T} = \begin{pmatrix} 2A_0 - \lambda_2 I_r & 2C_0 + \lambda_1 I_r \\ 2C_0 + \lambda_1 I_r & -2A_0 + \lambda_2 I_2 \end{pmatrix} + \lambda_1 \begin{pmatrix} 0 & I_r \\ -I_r & 0 \end{pmatrix}$$

$$+ \lambda_2 \begin{pmatrix} I_r & 0 \\ 0 & I_r \end{pmatrix},$$

and we denote $2E_0 = 2A_0 - \lambda_2 I_r$ and $2L_0 = 2C_0 + \lambda_1 I_r$, then we have

$$2T = \begin{pmatrix} 2E_0 & 2L_0 \\ 2L_0 & -2E_0 \end{pmatrix} - \lambda_1 J + \lambda_2 I.$$

Define $\widehat{T} = T - \frac{1}{2}\lambda_2 I + \frac{1}{2}\lambda_1 J$. Then $\widehat{T}$ and $ad(X)$ commute for each $X \in \mathbf{g}$ and it is easy to check that $\widehat{T}J + J\widehat{T} = 0$.

We prove now that $\widehat{T} = 0$, and from this the theorem follows.

Let $\mathbf{u}$ be a real form of $\mathbf{g}_J$ (for a proof see [1, Thm. III.6.3]) . Then for the map $\widehat{T}: \mathbf{g} \to \mathbf{g}$ there are R-linear maps $E_0, L_0: \mathbf{u} \to \mathbf{u}$ such that $\widehat{T}(X) = E_0(X) + JL_0(X)$ for each $X \in \mathbf{u}$.

Let X, Y be in $\mathbf{u}$, then $[X,Y] \in \mathbf{u}$ and $\widehat{T}([X,Y]) = E_0[X,Y] + JL_0[X,Y]$. On the other hand, $[\widehat{T}X, Y] = [E_0X, Y] + J[L_0X, Y]$, because $\widehat{T}$ and $ad(X)$ commute for each $X \in \mathbf{g}$. It follows that

$$\widehat{T}([X,Y]) - [\widehat{T}X,Y] = E_0[X,Y] - [E_0X,Y] + JL_0[X,Y] - J([L_0X,Y]).$$

The relationship $\widehat{T} \circ ad(X) = ad(X) \circ \widehat{T}$, for all $X \in \mathbf{g}$, shows that $\widehat{T}([X,Y]) = [\widehat{T}X,Y]$, therefore we conclude that

$$E_0[X,Y] = [E_0X,Y], \qquad L_0[X,Y] = [L_0X,Y]. \tag{2.1}$$

It follows from the above relationship that

$$\begin{aligned} &\widehat{T}([JX,JY]) = -\widehat{T}([JY,JX]) \\ &= -\widehat{T}ad(JY)(JX) \\ &= -ad(JY)\widehat{T}(JX) \\ &= [\widehat{T}(JX), JY] \end{aligned}$$

From this and the direct sum, $\mathbf{g} = \mathbf{u} \oplus J\mathbf{u}$, we can conclude that

$$E_0[X,Y] = -[E_0X,Y], \qquad L_0[X,Y] = -[L_0X,Y]. \tag{2.2}$$

It is easy to conclude from (2.1) and (2.2) that $E_0 = 0 = L_0$ on $[\mathbf{u},\mathbf{u}]$. Since $\mathbf{u}$ is a semisimple Lie algebra we have $[\mathbf{u},\mathbf{u}] = \mathbf{u}$. Hence $\widehat{T} = 0$. Therefore $T = \frac{1}{2}\lambda_2 I + \frac{1}{2}\lambda_1 J$, and

$$D(X,Y) = B(X,TY) = aB(X,Y) + bB(X,JY).$$

□

The classification of the $Ad(G)$-invariant bilinear forms on a simple Lie algebra was done in the previous theorems. The next result gives the classification of the $Ad(G)$-invariant bilinear forms on a semisimple Lie algebra, but before that we will need the following easily proved result.

Lemma 1. *Let $F\colon \mathbf{g}\times\mathbf{g} \to R$ be a symmetric bilinear form that is $Ad(G)$-invariant, then $F([X,W],Y) = F(X,[W,Y])$ for $W,X,Y \in \mathbf{g}$. The converse holds if G is connected.*

Proof. It is sufficent to show that $F([W,X],X) = 0$ for all $W,X \in \mathbf{g}$.
We consider the following function

$$f(s) = F(Ad(\alpha(s))X, Ad(\alpha(s))X),$$

We affirm that f is constant on each one-parameter subgroup α of G.
From this the direct assertion follows. □

Theorem 3. *Let $\mathbf{g} = \mathbf{g}_1 \oplus \mathbf{g}_2 \oplus \cdots \oplus \mathbf{g}_l$ be a Lie algebra, where each $\mathbf{g}_i$ is a simple ideal of $\mathbf{g}$. We shall suppose the following:*

- *The complexification of each $\mathbf{g}_i$, for $i = 1,\ldots,k$ is simple; and*
- *The complexification of each $\mathbf{g}_i$, for $i = k+1,\ldots,l$ is not simple and so there exists a complex structure J_i for each $\mathbf{g}_i$.*

Then every $Ad(G)$-invariant bilinear form D on $\mathbf{g}$ is given by the following
$D = \lambda_1 B_{\mathbf{g}_1} \oplus \cdots \oplus \lambda_k B_{\mathbf{g}_k}$
$\oplus\, (\mu_1^{k+1} B_{\mathbf{g}_{k+1}} + \mu_2^{k+1} B_{\mathbf{g}_{k+1}}^{J_{k+1}}) \oplus \cdots \oplus (\mu_1^{l} B_{\mathbf{g}_l} + \mu_2^{l} B_{\mathbf{g}_l}^{J_l}),$ *where each $B_{\mathbf{g}_i}$ is the Killing–Cartan form on $\mathbf{g}_i$, for $i = 1,\ldots,l$; all λ_i and μ_i^j are real numbers, and $B_{\mathbf{g}_j}^{J_i}(X,Y) = B_{\mathbf{g}_j}(X, J_iY)$.*

Proof. It follows from Lemma 1 that

$$D(X,[Y,Z]) = D([X,Y],Z), \qquad \text{for all } X,Y,Z \in \mathbf{g}.$$

On the other hand, it is easy to show the following properties:

1. $[\mathbf{g}_i, \mathbf{g}_j] = \{0\}$ for all $i \neq j$; and
2. $[\mathbf{g}_i, \mathbf{g}_i] = \mathbf{g}_i$, for all i.

Using the above information we have for $Y \in \mathbf{g}_j$ that there exist $Z, W \in \mathbf{g}_j$ such that $Y = [Z, W]$, and for $X \in \mathbf{g}_i$ we conclude that

$$D(X,Y) = D(X,[Z,W]) = D([X,Z],W) = 0.$$

Therefore $\mathbf{g}_i \perp \mathbf{g}_j$ for all $i \neq j$ with respect to D. From this it follows that

$$D = B\Big|_{\mathbf{g}_1} \oplus \cdots \oplus B\Big|_{\mathbf{g}_l}.$$

Now we use the classification of $Ad(G)$-invariant bilinear forms on a simple Lie algebra given in theorems 1 and 2. From this the result follows. □

From [3, Ch.11]) we obtain a relation between the geometry of G and its Lie algebra $\mathbf{g}$. Also we can deduce that the geometry of all bi-invariant metrics on G share most of the pseudo-Riemannian invariant.

We are going to give a classification of the bi-invariant pseudoRiemannian metrics on a semisimple Lie group based on the classification of the bilinear $Ad(G)$-invariant forms on a semisimple Lie algebra.

Theorem 4. *Let G be a semisimple Lie group such that $Lie(G) = \mathbf{g} = \mathbf{g}_1 \oplus \mathbf{g}_2 \oplus \cdots \oplus \mathbf{g}_l$, where each $\mathbf{g}_i$ is a simple ideal of the Lie algebra $\mathbf{g}$. We shall suppose the following:*

- *The complexification of each $\mathbf{g}_i$, for $i = 1, \ldots, k$ is simple; and*
- *The complexification of each $\mathbf{g}_i$, for $i = k+1, \ldots, l$ is not simple and so there exists a complex structure J_i for each $\mathbf{g}_i$.*

*Then every bi-invariant pseudoRiemannian metric ϕ on $\mathbf{g}$ is given by $\phi = \lambda_1 B_{\mathbf{g}_1} \oplus \cdots \oplus \lambda_k B_{\mathbf{g}_k}$
$\oplus (\mu_1^{k+1} B_{\mathbf{g}_{k+1}} + \mu_2^{k+1} B_{\mathbf{g}_{k+1}}^{J_{k+1}}) \oplus \cdots \oplus (\mu_1^l B_{\mathbf{g}_l} + \mu_2^l B_{\mathbf{g}_l}^{J_l})$, where each $B_{\mathbf{g}_i}$ is the Killing–Cartan form on $\mathbf{g}_i$, for $i = 1, \ldots, l$, all λ_i and μ_i^j are real numbers, and $B_{\mathbf{g}_j}^{J_i}(X,Y) = B_{\mathbf{g}_j}(X, J_i Y)$.*

3. Group action

From now on $G = G_1 \cdots G_l$ will be a connected noncompact semisimple Lie group with Lie algebra $\mathbf{g} = \mathbf{g}_1 \oplus \mathbf{g}_2 \oplus \cdots \oplus \mathbf{g}_l$. We know that G admits bi-invariant pseudoRiemannian metrics and all of them can be described in terms of the Killing-Cartan form on each $\mathbf{g}_i$.

Let M be a connected compact smooth manifold. We always assume that the action of G on M is smooth, faithful, and preserve a finite measure on M.

Definition 2. *The dimension of maximal lightlike tangent subspaces for M will be denoted with $m_0 = \min\{m_1, m_2\}$, where (m_1, m_2) represent the signature of M, i.e., that m_1 correspond to the dimension of maximal timelike tangent subspaces and m_2 correspond to the dimension of the maximal spacelike tangent subspaces.*

Definition 3. *The dimension of maximal lightlike tangent subspaces for G_i, $i = 1, \ldots, l$, will be denoted with $n_0^i = \min\{n_1^i, n_2^i\}$, where (n_1^i, n_2^i) represent the signature of G_i.*

Gromov remarked in [2] that if (n_1, n_2) is the signature of the metric given by the Killing–Cartan form on $\mathbf{g}$, then any other bi-invariant pseudoRiemannian metric on G has signature given by either (n_1, n_2) or (n_2, n_1).

We are interested in comparing the numbers m_0 and $n_0^1 + \cdots n_0^l$. To better understand this result we are going to prove the following very useful lemma.

Lemma 2. *Let (V, g) be a scalar product space, i.e, V is a finite dimensional vector space and g a nondegenerate symmetric bilinear form. Suppose that $V = V_1 \oplus \cdots \oplus V_l$, where each V_i is a subspace of V and $g = g_1 \oplus \cdots \oplus g_l$, where each g_i is a scalar product in V_i, for $i = 1, \ldots, l$. Let n_0 be the dimension of the maximal subspace of null vectors with respect to g in V, and n_0^i, for $i = 1, \ldots, l$, is defined in a similar way for each V_i. Then the following inequality holds: $n_0 \geq n_0^1 + \cdots + n_0^l$*

Proof. The idea for this is to realize that for each i we have $n_0^i = \min\{n_-^i, n_+^i\}$, where n_-^i is the number of -1 and n_+^i the number of $+1$ when g is diagonalized. Without loss of generality we can suppose that for $i = 1, \ldots, k$ we have that $n_0^i = n_-^i$, and for $j = k+1, \ldots, l$ also $n_0^j = n_+^j$.

It follows that $n_- = n_-^1 + \cdots + n_-^l$
$\geq n_0^1 + \cdots + n_0^l$, and in a similar way we have $n_+ = n_+^1 + \cdots + n_+^l$
$\geq n_0^1 + \cdots + n_0^l$.

From this it follows that $n_0 = \min\{n_-, n_+\} \geq n_0^1 + \cdots + n_0^l$. $\Box$

We will denote with TO the tangent bundle to the orbits of the G-action on M. If $X \in \mathbf{g}$, we define the infinitesimal generator X^* as the vector field on M induced by X. This new vector field is given by

$$X_p^* = \frac{d}{dt}|_{t=0} \exp{(tX)} \cdot p \, .$$

It is clear that X^* is a Killing vector field, and $X_p^* \in T_p(G \cdot p)$, for $p \in M$.

We will use the following two maps: $\varphi : M \times \mathbf{g} \to TO$, given by $\varphi(p, X) = X_p^*$, and $\psi : M \to \mathbf{g}^* \otimes \mathbf{g}^*$, given by $\psi(p) = B_p$, where $B_p(X,Y) = h_p(X_p^*, Y_p^*)$, and h is the metric on M.

We can conclude from the next lemma that $\psi(p)$ is an $Ad(G)$-invariant bilinear form on $\mathbf{g}$, for every $p \in M$.

Lemma 3. *For every $p \in M$, $\psi(p)$ is $Ad(G)$-invariant.*

Proof. This is a consequence of lemma 1 if we prove the following: $\psi(p)(ad(W)X, X) = 0$, for all $X, W \in \mathbf{g}$.

$\psi(p)(ad(W)X, X) = h_p((ad(W)X)_p^*, X_p^*)$
$= h_p(-(ad(W^*)X^*)_p, X_p^*)$
$= -h_p(X_p^*, (ad(W)X)_p^*)$
$= -\psi(p)(ad(W)X, X)$. $\Box$

We obtain a foliation of M by orbits from the action of G on M. If we restrict the given metric on M to each orbit of M we obtain a nondegenerate metric, therefore we can apply the classification given in the past section.

Theorem 1. *For G and M as before suppose G acts topologically transitively on M, i.e., there is a dense G-orbit, preserving its pseudoRiemannian metric and satisfying $n_0^1 + \cdots + n_0^l = m_0$. Then G acts everywhere locally free with nondegenerate orbits.*

Proof. Since the action is topologically transitively on M, it follows from a result in [5] that the action is everywhere locally free. open subset $U \subset M$, so that the the G-orbit of every point in U is nondegenerate.

We are going to prove that there exist a G-invariant open subset U on M so that the G-orbit of every point in U is nondegenerate.

Every basis of $\mathbf{g}$ induces at every point $p \in M$ a family of vectors that also defines a base for the tangent space to the orbit at M. In particular, φ trivializes TO.

We consider the G-action on $M \times \mathbf{g}$ given by $g(p, X) = (gp, Ad(g)(X))$. The map φ is G-equivariant,

$$
\begin{aligned}
&\mathrm{g}\varphi(p, X) = gX_p^* \\
&= g\tfrac{d}{dt}\,|_{t=0}(\exp{(tX)}p) \\
&= \tfrac{d}{dt}\,|_{t=0}(g\exp{(tX)}p) \\
&= \tfrac{d}{dt}\,|_{t=0}(g\exp{(tX)}g^{-1}gp) \\
&= \tfrac{d}{dt}\,|_{t=0}(\exp{tAd(g)(X)gp}) \\
&= Ad(g)(X)^*_{gp} = \varphi(g(p, X)).
\end{aligned}
$$

The map ψ, defined above, is G-equivariant. For $g \in G$, $X, Y \in \mathbf{g}$ and $p \in M$ we have:

$$
\begin{aligned}
&\psi(gp)(X, Y) = h_{gp}(X^*_{gp}, Y^*_{gp}) \\
&= h_p(Ad(g^{-1})(X)^*_p, Ad(g^{-1})(Y)^*_p) \\
&= \psi(p)(Ad(g^{-1})(X), Ad(g^{-1})(Y)).
\end{aligned}
$$

Hence, since the G-action is tame on $\mathbf{g}^* \otimes \mathbf{g}^*$, such map is essentially constant on the support of almost every ergodic component of M, see [6, Ch.2]).

By the lemma 3, there is an $Ad(G)$-invariant bilinear form B_U on $\mathbf{g}$ so that the metric on $TO|_U \cong U \times \mathbf{g}$ induced by M is almost everywhere given by B_U on each fiber, where U is the support of one ergodic component of M. It is very easy to prove that the kernel of B_U is an ideal of $\mathbf{g}$. If such kernel is $\mathbf{g}$, then $TO|_U$ is lightlike which implies $\dim \mathbf{g} \leq m_0$. This contradicts the condition $n_0^1 + \cdots + n_0^l = m_0$ since $n_0 < \dim \mathbf{g} = \dim \mathbf{g}_1 + \cdots + \dim \mathbf{g}_l$. Hence, being $\mathbf{g}$ semisimple, it follows that B_U is nondegenerate, and so almost every G-orbit contained in S is nondegenerate. We can conclude that almost every G-orbit in M is nondegenerate. As a consequence, the set U as defined before is conull and so nonempty.

The previous argument shows that the image under ψ of a conull and hence dense, subset of M lies in the set of $Ad(G)$-invariant elements of $\mathbf{g}^* \otimes \mathbf{g}^*$. It follows that $\psi(M)$ lies on it, since such set is closed. In particular, on every G-orbit the metric induced from that of M is given by $Ad(G)$-invariant symmetric bilinear form on $\mathbf{g}$.

Using topological transitivity, we obtain an G-orbit O_α. This G-orbit is dense and so it must intersect U. It is clear that $O_\alpha \subset U$ since U is G-invariant.

The metric restricted to O_α, under the map φ, is given by the nondegenerate bilinear form B_α on $\mathbf{g}$. It follows that $\psi(O_\alpha) = B_\alpha$ and so the

the continuity of ψ together with the density of O_α imply that ψ is the constant map given by B_α.

□

We now prove the principal result in this work.

Theorem 2. *Let $G = G_1 \cdots G_l$ be a connected semisimple Lie group without compact factors acting by isometries on a finite volume pseudoRiemannian manifold M and no factor of G acts trivially. Then $n_0^1 + \cdots + n_0^l \leq m_0$.*

Proof. By results in [6] we have local freeness on an open subset $U \subset X$. Then the map $\psi: U \to \mathbf{g}^* \otimes \mathbf{g}^*$, defined above, is constant on the ergodic components in U for the G-action. On any such ergodic component, the metric along the G-orbits comes from an $Ad(G)$-invariant bilinear form B_U on $\mathbf{g}$.

Let η be the kernel of B_U. It is known that η is an ideal of $\mathbf{g}$. Since $\mathbf{g}$ is semisimple we have to consider the following cases.

(1) $\eta = \mathbf{g}$. In this case it follows easily that $B_U = 0$, then $\dim \mathbf{g} \leq m_0$. Since, $\dim \mathbf{g}_i \geq n_0^i$ and $\dim \mathbf{g} = \dim \mathbf{g}_1 + \cdots + \dim \mathbf{g}_l$, we conclude that

$$n_0^1 + \cdots + n_0^l < m_0.$$

(2) $\eta = \{0\}$. In this case B_U is nondegenerate and the G-orbits are nondegenerate submanifolds of X. Then we have that $n_0 \leq m_0$ and by Lemma 2 the claim follows.

(3) $\eta = \bigoplus_{j \in J} \mathbf{g}_j$ where $J \subset \{1, \ldots, l\}$. From this it follows that there is a subspace of null vectors in the tangent space to the G-orbits which has a dimension $\dim \bigoplus_{j \in J} \mathbf{g}_j + n_0^{j_1} + \cdots + n_0^{j_s}$. Therefore,

$$\dim \bigoplus_{j \in J} \mathbf{g}_j + n_0^{j_1} + \cdots + n_0^{j_s} \leq m_0,$$

where $j_1, \ldots, j_s \in \{1, \ldots, l\} \setminus J$. On the other hand,

$$\dim \bigoplus_{j \in J} \mathbf{g}_j + n_0^{j_1} + \cdots + n_0^{j_s} > n_0^1 + \cdots + n_0^l.$$

□

References

[1] S. Helgason, Differential Geometry, Lie Groups and Symmetric Spaces, Academic Press, New York, (1978).

[2] M. Gromov, Rigid transformations groups, Géométrie différentielle (Paris 1986), Travaux en Cours **33**, Hermann, Paris, pp. 65–139, (1988).

[3] B. O'neill, SEMI-RIEMANNIAN GEOMETRY, Academic Press, New York, (1983).

[4] J. Rosales-Ortega, The Gromov's Centralizer theorem for semisimple Lie group actions. Ph.D. Thesis, CINVESTAV-IPN, (2005).

[5] J. Szaro, Isotropy of semisimple group actions on manifolds with geometric structures, Amer. J. Math.**120**, pp. 129–158, (1998).

[6] R. J. Zimmer, Ergodic Theory and Semisimple Lie Groups, Birkhäuser, Boston, (1984).

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