Proyecciones Journal of Mathematics
Vol. 31, N^o 1, pp. 29-38, March 2012.
Universidad Católica del Norte
Antofagasta - Chile
DOI: 10.4067/S0716-09172012000100004

Improving some sequences convergent to Euler-Mascheroni constant

NECDET BATIR
Nevşehır University, Nevşehır, TURKEY
and
CHAO-PING CHEN
Henan Polytechnic University, CHINA
Received : November 2011. Accepted : December 2011

Abstract

We obtain the following very fast sequences convergent to Euler-Mascheroni constant:

$$\theta_n = H_n - \log\left(n + \frac{1}{2} + \frac{1}{24n} - \frac{1}{48n^2} + \frac{23}{5760n^3} + \frac{17}{3840n^4} - \frac{10099}{2903040n^5}\right)$$

and

$$\phi_n = H_n - \log\left(n + \frac{1}{2} + \frac{1}{24(n+\frac{1}{2})} - \frac{37}{5760(n+\frac{1}{2})^3} + \frac{10313}{2903040(n+\frac{1}{2})^5}\right),$$

where H_n *are the harmonic numbers defined by* $H_n = 1+\frac{1}{2}+\frac{1}{3}+...+\frac{1}{n}$.

subjclass [2000] : *Primary 11Y60; Secondary 40A05.*

Keywords : *Euler-Mascheroni constant, harmonic numbers, inequalities,asymptotic expansion.*

1. Introduction

Euler's constant (or Euler Mascheroni constant) γ was introduced by Leonhard Euler (1707-1783) in 1734 as the limit of the sequence

$$D_n = 1 + \frac{1}{2} + \frac{1}{3} + ... + \frac{1}{n} - \log n. \tag{1.1}$$

This constant is known to be the third most important mathematical constant, next to π and e. It appears in a lot of places in mathematics such as number theory, analysis, theory of probability, special functions, and differential equations. The convergence of D_n to γ is very slowly since

$$\frac{1}{2(n+1)} < D_n - \gamma < \frac{1}{2n} \quad \text{(R.M. Young [12])},$$

which shows that it converges to γ as n^{-1}. A faster convergent sequence to γ were introduced by DeTemle in [10, 11]. He proved that the sequence R_n defined by

$$R_n = 1 + \frac{1}{2} + \frac{1}{3} + ... + \frac{1}{n} - \log\left(n + \frac{1}{2}\right). \tag{1.2}$$

converges to γ with the speed like n^{-2}, since

$$\frac{1}{24(n+1)^2} < R_n - \gamma < \frac{1}{24n^2}. \tag{1.3}$$

Recently Chen [3] obtained sharp form of (1.3) as follows: For all $n \in \mathbf{N}$

$$\frac{1}{24(n+a)^2} < R_n - \gamma < \frac{1}{24(n+b)^2}$$

with the best possible constants

$$a = \frac{1}{\sqrt{24(1-\gamma-\log(3/2))}} = 0.55106..., \text{ and } b = \frac{1}{2}.$$

In 1997, Negoi [9] introduced the sequence

$$T_n = 1 + \frac{1}{2} + \frac{1}{3} + ... + \frac{1}{n} - \log\left(n + \frac{1}{2} + \frac{1}{24n}\right)$$

and showed that

$$\frac{1}{48(n+1)^3} < T_n - \gamma < \frac{1}{48n^3},$$

which shows that the approximation $T_n \approx \gamma$ has a significient superiority over the approximation $R_n \approx \gamma$. For other faster convergences of Euler-Mascheroni constant we refer to [1, 3, 5, 6, 7, 8]. To accelerate the sequence (T_n), Chen and Mortici [2] established the following approximation formula

$$\gamma \approx H_n - \log\left(n + \frac{1}{2} + \frac{1}{24n} - \frac{1}{48n^2} + \frac{23}{5760n^3} + ...\right). \tag{1.4}$$

Our first aim here is to improve (1.4) and obtain bounds for H_n in this form. Our second aim is to establish similar formulas to improve (R_n). For this purpose we shall consider the following sequences:

$$A_n = H_n - \log\left(n + \frac{1}{2} + \frac{1}{24n} + \frac{\alpha}{n^2} + \frac{\beta}{n^3} + \frac{\delta}{n^4} + \frac{\epsilon}{n^5}\right), \tag{1.5}$$

and

$$\mathrm{B}_n = H_n - \log\left(n + \frac{1}{2} + \frac{a}{n+\frac{1}{2}} + \frac{b}{(n+\frac{1}{2})^2} + \frac{c}{(n+\frac{1}{2})^3} + \frac{c}{(n+\frac{1}{2})^3} + \frac{d}{(n+\frac{1}{2})^4} + \frac{e}{(n+\frac{1}{2})^5}\right),$$

where $\alpha, \beta, \delta, \epsilon$ and a, b, c, d, e are real parameters. Precisely, we introduce the sequences (θ_n) and (ϕ_n) by

$$\theta_n = H_n - \log\left(n + \frac{1}{2} + \frac{1}{24n} - \frac{1}{48n^2} + \frac{23}{5760n^3} + \frac{17}{3840n^4} - \frac{10099}{2903040n^5}\right),$$

and

$$\phi_n = H_n - \log\left(n + \frac{1}{2} + \frac{1}{24(n+\frac{1}{2})} - \frac{37}{5760(n+\frac{1}{2})^3} + \frac{10313}{2903040(n+\frac{1}{2})^5}\right),$$

both of which converge to γ like n^{-7}, since

$$\lim_{n\to\infty} n^7(\theta_n - \gamma) = -\frac{2501}{1161216}$$

and

$$\lim_{n\to\infty} n^7\phi_n = -\frac{5509121}{1393459200}.$$

In order to prove our main results we need the following lemma, which is a strong tool to measure and improve the speed of convergence of some sequences having limit equal to zero.

Lemma 1.1. *If (s_n) is convergent to zero and there exists the limit*

$$\lim_{n\to\infty} n^k(s_n - s_{n+1}) = c \in \mathbf{R} \tag{1.6}$$

then there exists the limit

$$\lim_{n\to\infty} n^{k-1}s_n = \frac{c}{k-1},$$

see [4]. From this lemma it is clear that the speed of convergence of the sequence (s_n) is as higher as the value of k satisfying (1.6) is as greater as.

2. Main results

Let A_n be the sequence defined by (1.5). We are interested to find the values of α, β, δ and ϵ which provide the fastest sequence A_n. First, let us write

$$\mathrm{A}_n - A_{n+1} = -\tfrac{1}{n+1} - \log\left(n + \tfrac{1}{2} + \tfrac{1}{24n} + \tfrac{\alpha}{n^2} + \tfrac{\beta}{n^3}\tfrac{\delta}{n^4} + \tfrac{\epsilon}{n^5}\right)$$

$$+\log\left(n + 1 + \tfrac{1}{2} + \tfrac{1}{24(n+1)} + \tfrac{\alpha}{(n+1)^2} + \tfrac{\delta}{(n+1)^4} + \tfrac{\epsilon}{(n+1)^5}\right).$$

We are concentrated to compute a limit of the form (1.6). In order to do this we use a computer software to obtain the following power series representation in $\frac{1}{n}$:

$$\mathrm{A}_n - A_{n+1} = \left(-\tfrac{1}{16} - 3\alpha\right)\tfrac{1}{n^4} + \left(\tfrac{263}{1440} + 8\alpha - 4\beta\right)\tfrac{1}{n^5}$$

$$+\left(-\tfrac{139}{384} - \tfrac{385\alpha}{24} + \tfrac{25\beta}{2} - 5\delta\right)\tfrac{1}{n^6} + \left(\tfrac{3685}{6048} + \tfrac{229\alpha}{8} + 3\alpha^2 - \tfrac{115\beta}{4} + 18\delta - 6\epsilon\right)\tfrac{1}{n^7}$$

$$+\left(-\tfrac{8663}{9216} - \tfrac{27517\alpha}{576} - 14\alpha^2 + \tfrac{1379\beta}{24} + 7\alpha\beta - \tfrac{1127\delta}{24} + \tfrac{49\epsilon}{2}\right)\tfrac{1}{n^8} + O(n^{-9}).$$

By Lemma 1.1 faster convergences are obtained by imposing the conditions that the first four coefficients vanish. Now this results in

$$\begin{cases} -\frac{1}{16} - 3\alpha = 0, \\ \frac{263}{1440} + 8\alpha - 4\beta = 0, \\ \frac{139}{384} - \frac{385\alpha}{24} + \frac{25\beta}{2} - 5\delta = 0, \\ \frac{3685}{6048} + \frac{229\alpha}{8} + 3\alpha^2 - \frac{115\beta}{4} + 18\delta - 6\epsilon = 0, \\ -\frac{8663}{9216} - \frac{27517\alpha}{576} - 14\alpha^2 + \frac{1379\beta}{24} + 7\alpha\beta - \frac{1127\delta}{24} + \frac{49\epsilon}{2} = 0, \end{cases}$$

namely,

$$\alpha = -\frac{1}{48}, \quad \beta = \frac{23}{5760}, \quad \delta = \frac{17}{3840}, \quad \text{and} \quad \epsilon = -\frac{10099}{2903040}. \tag{2.1}$$

These solutions correspond to the following sequence

$$\theta_n = H_n - \log\left(n + \frac{1}{2} + \frac{1}{24n} - \frac{1}{48n^2} + \frac{23}{5760n^3} + \frac{17}{3840n^4} - \frac{10099}{2903040n^5}\right). \tag{2.2}$$

By replacing the solutions (2.1) above

$$\theta_n - \theta_{n+1} = -\frac{2501}{165888n^8} + O(n^{-9}).$$

Now we can state the following

Theorem 2.1. *Let (θ_n) be the sequence defined by (2.2). Then*

$$\lim_{n\to\infty} n^8(\theta_n - \theta_{n+1}) = -\frac{2501}{165888} \quad \text{and} \quad \lim_{n\to\infty} n^7(\theta_n - \gamma) = -\frac{2501}{1161206},$$

namely, the speed of convergences of the sequence (θ_n) is like n^{-7}

Let (B_n) be as defined by (1). Then we have

$$B_n - B_{n+1} = -\frac{1}{n+1} - \log\left(n + \frac{1}{2} + \frac{a}{(n+\frac{1}{2})} + \frac{b}{(n+\frac{1}{2})^2} + \frac{c}{(n+\frac{1}{2})^3} + \frac{d}{(n+\frac{1}{2})^4} + \frac{e}{(n+\frac{1}{2})^5}\right) + \log\left((n+1) + \frac{1}{2} + \frac{a}{(n+\frac{3}{2})} + \frac{b}{(n+\frac{3}{2})^2} + \frac{c}{(n+\frac{3}{2})^3} + \frac{d}{(n+\frac{3}{2})^4} + \frac{e}{(n+\frac{3}{2})^5}\right). \tag{2.3}$$

Using again a computer software we get

$$B_n - B_{n+1} = \left(\tfrac{1}{12} - 2a\right)\tfrac{1}{n^3} + \left(-\tfrac{1}{4} + 6a - 3b\right)\tfrac{1}{n^4}$$
$$+\left(\tfrac{41}{80} - 13a + 2a^2 + 12b - 4c\right)\tfrac{1}{n^5}$$
$$+\left(-\tfrac{43}{48} + 25a - 10a^2 - \tfrac{65b}{2} + 5ab + 20c - 5d\right)\tfrac{1}{n^6}$$
$$+\left(\tfrac{645}{448} - \tfrac{363a}{8} + \tfrac{65a^2}{2} - 2a^3 + 75b - 30ab + 3b^2 - 65c + 6ac + 30d - 6e\right)\tfrac{1}{n^7}$$
$$\Bigg(-\tfrac{141}{64} + \tfrac{637a}{8} - \tfrac{175a^2}{2} + 14a^3 - \tfrac{2541b}{16} + \tfrac{455ab}{4} - 7a^2b - 21b^2 + 175c$$
$$-42ac + 7bc - \tfrac{455d}{4} + 7ad + 42e\Bigg)\tfrac{1}{n^8} + O(n^{-9}).$$

(2.4)

According to Lemma 1.1 we can see that the fastest sequence ϕ_n is obtained in the case when as many of the first coefficients of (2.3) is cancelled. As we have five paremeters a, b, c, d, e, they produce the best result if and only if

$$\tfrac{1}{12} - 2a = 0,$$

$$-\tfrac{1}{4} + 6a - 3b = 0,$$

$$\tfrac{41}{80} - 13a + 2a^2 + 12b - 4c = 0,$$

$$-\tfrac{43}{48} + 25a - 10a^2 - \tfrac{65b}{2} + 5ab + 20c - 5d = 0,$$

$$\tfrac{645}{448} - \tfrac{363a}{8} + \tfrac{65a^2}{2} - 2a^3 + 75b - 30ab + 3b^2 - 65c$$

$$+6ac + 30d - 6e = 0,$$

$$-\tfrac{141}{64} + \tfrac{637a}{8} - \tfrac{175a^2}{2} + 14a^3 - \tfrac{2541b}{16} + \tfrac{455ab}{4} - 7a^2b$$

$$-21b^2 + 175c - 42ac + 7bc - \tfrac{455d}{4} + 7ad + 42e = 0.$$

From these we obtain the following solutions:

$$a = \frac{1}{24}, \quad b = 0, \quad c = -\frac{37}{5760}, \quad d = 0, \quad e = \frac{10313}{2903040}, \tag{2.5}$$

and these solutions correspond to the following sequence

$$\phi_n = H_n - \log\left(n + \frac{1}{2} + \frac{1}{24(n+\frac{1}{2})} - \frac{37}{5760(n+\frac{1}{2})^3} + \frac{10313}{2903040(n+\frac{1}{2})^5}\right). \tag{2.6}$$

By replacing the solutions given in (2.5) in (2.4) we get

$$\phi_n - \phi_{n+1} = -\frac{5509121}{174182400n^8} + O(n^{-9}).$$

These can be summarized as follow.

Theorem 2.2. *Let (ϕ_n) be the sequence given by (2.6). Then it holds that*

$$\lim_{n\to\infty} n^8(\phi_n - \phi_{n+1}) = -\frac{5509121}{174182400} \quad \textit{and} \quad \lim_{n\to\infty} n^7\phi_n = -\frac{5509121}{1393459200},$$

that is, the speed of convergence of (ϕ_n) is like n^{-7}.

Theorem 2.3. *Let the sequences (θ_n) and (ϕ_n) be as defined (2.2) and (2.6). Then, both (θ_n) and (ϕ_n) are strictly decreasing for $n \geq 2$ and all natural numbers n, respectively.*

Proof. We set $\theta_n - \theta_{n+1} = f(n)$, where

$$\begin{aligned} f(x) &= -\tfrac{1}{x+1} - \log\left(x + \tfrac{1}{2} + \tfrac{1}{24x} - \tfrac{1}{48x^2} + \tfrac{23}{5760x^3} + \tfrac{17}{3840x^4} - \tfrac{10099}{2903040x^5}\right) \\ &\quad + \log\left(x + \tfrac{3}{2} + \tfrac{1}{24(x+1)} - \tfrac{1}{48(x+1)^2} + \tfrac{23}{5760(x+1)^3} + \tfrac{17}{3840(x+1)^4} - \tfrac{10099}{2903040(x+1)^5}\right). \end{aligned}$$

Differentiation gives

$$f'(x) = \frac{p(x)}{q(x)}, \tag{2.7}$$

where

$$\begin{aligned} p(x) &= -223661795575 - 1556403370554x - 4175585115408x^2 \\ &-4951284518880x^3 - 1613300443776x^4 + 1495234411776x^5 \\ &+1016470425600x^6 \end{aligned}$$

and

$$q(x) = -44732359115x - 285157435534x^2 - 655259062139x^3 -595777525560x^4 - 16441205760x^5 + 7510856117184x^6 +67212592098624x^7 + 276219358571520x^8 + 661238937354240x^9 +1014009112166400x^{10} + 1032643563356160x^{11} +698089616179200x^{12} + 301990477824000x^{13} +75848771174400x^{14} + 8427641241600x^{15},$$

By expanding $p(x)$ and $q(x)$ as a power series of $x-2$ we get

$$p(x) = 27439716165461 + 185481302397702(x-2) +290969152206768(x-2)^2 + 204586956497952(x-2)^3 +74327269209984(x-2)^4 + 13692879518976(x-2)^5 +1016470425600(x-2)^6,$$

and

$$q(x) = 10423677493515991770 + 62668051134141291321(x-2) +... + 328678008422400(x-2)^{14} + 8427641241600(x-2)^{15},$$

which is a polynomial with all positive coefficients. It follows $f'(x) > 0$ for $x \geq 2$, so that f is strictly increasing in $(2, \infty)$ with $\lim_{x\to\infty} f(x) = 0$. it results that $f(x) < 0$ for $x \geq 2$, namely θ_n is strictly increasing for $n \geq 2$. This completes the first part of Theorem 2.3. To prove the second part of the theorem we denote $\phi_n - \phi_{n+1} = g(n)$, where

$$g(x) = -\frac{1}{x+1} - \log\left(x + \frac{1}{2} + \frac{1}{24(x+\frac{1}{2})} - \frac{37}{5760(x+\frac{1}{2})^3} +\frac{10313}{2903040(x+\frac{1}{2})^5}\right) + \log\left(x + \frac{3}{2} + \frac{1}{24(x+\frac{3}{2})} - \frac{37}{5760(x+\frac{3}{2})^3} + \frac{10313}{2903040(x+\frac{3}{2})^5}\right).$$

By differentiation we get

$$g'(x) = \frac{t(x)}{s(x)}, \tag{2.8}$$

where

$$t(x) = 9678358492223 + 57880272188784x + 144357200961720x^2 +192184418280960x^3 + 144005296337280x^4 + 57575515060224x^5 +9595919176704x^6,$$

and

$$s(x) = 5912418259515 + 110278703811038x + 996749749920191x^2 +... + 539369039462400x^{15} + 33710564966400x^{16},$$

which is a polynomial with all positive coefficients. Since both $t(x)$ and $s(x)$ are positive for $x \geq 1$, g is strictly increasing with $\lim_{x\to\infty} g(x) = 0$, consequently, the sequence (ϕ_n) is strictly increasing for $n = 1, 2, 3, \ldots$.. This completes the proof of Theorem 2.3. $\square$

As a direct consequence of the fact that θ is strictly increasing for $n = 2, 3, 4, \ldots$ we have $\theta_2 \leq \theta_n < \lim_{n\to\infty} \theta_n = \gamma$ for all $n \geq 2$. As $\theta_2 = \frac{3}{2} - \log\left(\frac{58804553}{23224320}\right)$, we have

Corollary 2.4. *Let $n \geq 2$ be an integer. Then we have*

$$\alpha + \log\left(n + \tfrac{1}{2} + \tfrac{1}{24n} - \tfrac{1}{48n^2} + \tfrac{23}{5760n^3} + \tfrac{17}{3840n^4} - \tfrac{10099}{2903040n^5}\right)$$

$$\leq H_n < \beta + \log\left(n + \tfrac{1}{2} + \tfrac{1}{24n} - \tfrac{1}{48n^2}\right.$$

$$\left. + \tfrac{23}{5760n^3} + \tfrac{17}{3840n^4} - \tfrac{10099}{2903040n^5}\right),$$

where $\alpha = \frac{3}{2} - \log\left(\frac{58804553}{23224320}\right) = 0.5709807216\ldots$ and $\beta = \gamma = 0.5772156\ldots$ are the best possible.

Similarly from monotonic increase of the sequence (ϕ_n) with $\lim\limits_{n\to\infty} \phi_n = \gamma$ and $\phi_1 = 1 - \log\left(\frac{6729631}{4408992}\right) = 0.57712577887\ldots$, we get

$$\alpha^* + \log\left(n + \tfrac{1}{2} + \tfrac{1}{24(n+\frac{1}{2})} - \tfrac{37}{5760(n+\frac{1}{2})^3} + \tfrac{10313}{2903040(n+\frac{1}{2})^5}\right) \leq H_n$$

$$\beta^* + \log\left(n + \tfrac{1}{2} + \tfrac{1}{24(n+\frac{1}{2})} - \tfrac{37}{5760(n+\frac{1}{2})^3} + \tfrac{10313}{2903040(n+\frac{1}{2})^5}\right),$$

where $\alpha^* = 1 - \log\left(\frac{6729631}{4408992}\right) = 0.57712577887\ldots$ and $\beta^* = \gamma = 0.5772156$ are the best possible constants.

References

[1] N. Batir, Sharp bounds for the psi function and harmonic numbers, Math. Inequal. Appl., No. 4, pp. 917-925, (2011).

[2] C-P Chen, C. Mortici, New sequences converging towards the Euler-Mascheroni constant, Computer and Mathematics with Applications, doi:10.1016/j.camwa.2011.03.099, (2011).

[3] C-P. Chen, Inequalities for the Euler-Mascheroni constant, Appl. Math. Lett., 23, pp. 161-164, (2010).

[4] C. Mortici, New approximation of the gamma function in terms of the digamma function, Appl. Math. Lett., 23, No. 1, pp. 97-100, (2010).

[5] C. Mortici, Fast convergences toward Euler-Mascheroni constant, Comput. Appl. Math., 29, No. 3, pp. 479-491, (2010).

[6] C. Mortici, On new sequences converging towards the Euler-Mascheroni constant, Computer Math. Appl., 59, No. 8, pp. 2610-2614, (2010).

[7] C. Mortici, Optimizing the rate of convergence of some new classes of sequences convergent to Euler constant, Analysis Appl., 8, No. 1, pp. 99-107, (2010).

[8] C. Mortici, A quicker convergence toward the constant with the logarithm term involving the constant e, Carpathian J. Math., 26, No. 1, pp. 86-91, (2010).

[9] T. Negoi, A faster convergence to the constant of Euler, Gazeta Matematica, Seria A, 15, No. 94, pp. 113, (1997).

[10] D. W. Temple, A geometric look at sequences that converge to Euler's constant, College Math. J., 37, pp. 128-131, (2006).

[11] D. W. Temple, A quicker convergences to Euler's constant, Amer. Math. Monthly, 100 (5), pp. 468-470,(1993).

[12] R. M. Young, Euler's constant, Math. Gaz., 75, pp. 187-190, (1991).

Necdet Batir
Department of Mathematics,
Faculty of Arts and Sciences,
Nevşehır University, Nevşehır,
Turkey
e-mail : nbatir@hotmail.com

and

Chao-Ping Chen
School of Mathematics and Informatics,
Henan Polytechnic University,
Jiaozuo City 454003,
Henan Province,
People's Republic of China
e-mail : chenchaoping@sohu.com