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Generalized difference entire sequence spaces

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Abstract

In this paper we introduce difference entire sequence spaces and difference analytic sequence spaces defined by a sequence of modulus function $F = (f_k)$ *and study some topological properties and some inclusion relations between these spaces. We also make an effort to study some properties and inclusion relation between the spaces* $\Gamma_F(\Delta_s^m, u, p, q, ||., \cdots, .||)$ *and* $\Lambda_F(\Delta_s^m, u, p, q, ||., \cdots, .||)$.

Subjclass [2000] : *40A05, 40C05, 40D05.*

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1. Introduction and Preliminaries

The notion of difference sequence spaces was introduced by Kızmaz [11], who studied the difference sequence spaces $l_\infty(\Delta)$, $c(\Delta)$ and $c_o(\Delta)$. The notion was further generalized by Et and Çolak [5] by introducing the spaces $l_\infty(\Delta^n)$, $c(\Delta^n)$ and $c_o(\Delta^n)$. Let w be the space of all complex or real sequences $x = (x_k)$ and let m, s be non-negative integers, then for $Z = l_\infty$, c, c_0 we have sequence spaces

$$Z(\Delta_s^m) = \{x = (x_k) \in w : (\Delta_s^m x_k) \in Z\},$$

where $\Delta_s^m x = (\Delta_s^m x_k) = (\Delta_s^{m-1} x_k - \Delta_s^{m-1} x_{k+1})$ and $\Delta_s^0 x_k = x_k$ for all $k \in \mathbf{N}$, which is equivalent to the following binomial representation

$$\Delta_s^m x_k = \sum_{v=0}^{m} (-1)^v \begin{pmatrix} m \\ v \end{pmatrix} x_{k+sv}.$$

Taking $s = 1$, we get the spaces which were studied by Et and Çolak [5]. Taking $m = s = 1$, we get the spaces which were introduced and studied by Kızmaz [11].

A complex sequence, whose k^{th} term is x_k, is denoted by (x_k). Let φ be the set of all finite sequences. A sequence $x = (x_k)$ is said to be analytic if $\sup_k |x_k|^{\frac{1}{k}} < \infty$. The vector space of all analytic sequences will be denoted by Λ. A sequence $x = (x_k)$ is called entire sequence if $\lim_{k\to\infty} |x_k|^{\frac{1}{k}} = 0$. The vector space of all entire sequences will be denoted by Γ.
A modulus function is a function $f : [0, \infty) \to [0, \infty)$ such that

1. $f(x) = 0$ if and only if $x = 0$,
2. $f(x + y) \leq f(x) + f(y)$ for all $x \geq 0$, $y \geq 0$,
3. f is increasing
4. f is continuous from right at 0.

It follows that f must be continuous everywhere on $[0, \infty)$. The modulus function may be bounded or unbounded. For example, if we take $f(x) = \frac{x}{x+1}$, then $f(x)$ is bounded. If $f(x) = x^p$, $0 < p < 1$, then the modulus $f(x)$ is unbounded. Subsequentially, modulus function has been discussed in ([1], [2], [3], [4], [12], [13], [17], [18]) and references therein. Let $F = (f_k)$ be a sequence of modulus function.

The space consisting of all those sequences x in w such that $f_k\left(\frac{|x_k|^{1/k}}{\rho}\right) \to 0$ as $k \to \infty$ for some arbitrary fixed $\rho > 0$ is denoted by Γ_F and is known as a space of entire sequences defined by a sequence of modulus function. The space Γ_F is a metric space with the metric $d(x, y) = \sup_k f_k\left(\frac{|x_k - y_k|^{1/k}}{\rho}\right)$ for all $x = (x_k)$ and $y = (y_k)$ in Γ_F. The space consisting of all those sequences x in w such that $\left(\sup_k \left(f_k\left(\frac{|x_k|^{1/k}}{\rho}\right)\right)\right) < \infty$ for some arbitrarily fixed $\rho > 0$ is denoted by Λ_F and is known as a space of analytic sequences defined by a sequence of modulus function.

A sequence space E is said to be solid or normal if $(\alpha_k x_k) \in E$ whenever $(x_k) \in E$ and for all sequences of scalars (α_k) with $|\alpha_k| \leq 1$ (see [10]).

Let X be a linear metric space. A function $p : X \to \mathbf{R}$ is called paranorm, if

1. $p(x) \geq 0$, for all $x \in X$,
2. $p(-x) = p(x)$, for all $x \in X$,
3. $p(x + y) \leq p(x) + p(y)$, for all $x, y \in X$,
4. if (λ_n) is a sequence of scalars with $\lambda_n \to \lambda$ *as* $n \to \infty$ and (x_n) is a sequence of vectors with $p(x_n - x) \to 0$ *as* $n \to \infty$, then $p(\lambda_n x_n - \lambda x) \to 0$ *as* $n \to \infty$.

A paranorm p for which $p(x) = 0$ implies $x = 0$ is called total paranorm and the pair (X, p) is called a total paranormed space. It is well known that the metric of any linear metric space is given by some total paranorm (see [19], Theorem 10.4.2, P-183).
The following inequality will be used throughout the paper. Let $p = (p_k)$ be a sequence of positive real numbers with $0 \leq p_k \leq \sup p_k = G$, $K = \max(1, 2^{G-1})$ then

$$|a_k + b_k|^{p_k} \leq K\{|a_k|^{p_k} + |b_k|^{p_k}\}$$

for all k and $a_k, b_k \in \mathbf{C}$. Also $|a|^{p_k} \leq \max(1, |a|^G)$ for all $a \in \mathbf{C}$.

Let $F = (f_k)$ be a sequence of modulus functions and X be locally convex Hausdorff topological linear space whose topology is determined by a set of continuous seminorms q. The symbol $\Lambda(X)$ and $\Gamma(X)$ denotes the space of all analytic and entire sequences respectively defined over X. If $p = (p_k)$ be bounded sequences of strictly positive real numbers and $u = (u_k)$ be sequences of positive real numbers, then we define the following sequence spaces:

$$\Lambda_F(\Delta_s^m, u, p, q) = \Big\{x \in \Lambda(X) : \sup_n \frac{1}{n} \sum_{k=1}^{n} \Big[f_k\Big(\Big(\frac{|(u_k \Delta_s^m x_k)^{1/k}|}{\rho}\Big)\Big)\Big]^{p_k} < \infty,$$
$$\text{for some } \ \rho > 0\Big\}$$

and

$$\Gamma_F(\Delta_s^m, u, p, q) = \Big\{x \in \Gamma(X) : \frac{1}{n} \sum_{k=1}^{n} \Big[f_k\Big(q\Big(\frac{|(u_k \Delta_s^m x_k)^{1/k}|}{\rho}\Big)\Big)\Big]^{p_k} \to 0 \ \text{ as}$$
$$n \to \infty, \text{for some } \ \rho > 0\Big\}.$$

If we take If we take $p = (p_k) = 1$, we get

$$\Lambda_F(\Delta_s^m, u, q) = \Big\{x \in \Lambda(X) : \sup_n \frac{1}{n} \sum_{k=1}^{n} \Big[f_k\Big(\Big(\frac{|(u_k \Delta_s^m x_k)^{1/k}|}{\rho}\Big)\Big)\Big] < \infty,$$
$$\text{for some } \ \rho > 0\Big\}$$

and

$$\Gamma_F(\Delta_s^m, u, q) = \Big\{x \in \Gamma(X) : \frac{1}{n} \sum_{k=1}^{n} \Big[f_k\Big(q\Big(\frac{|(u_k \Delta_s^m x_k)^{1/k}|}{\rho}\Big)\Big)\Big] \to 0 \ \text{ as}$$
$$n \to \infty, \text{for some } \ \rho > 0\Big\}.$$

The purpose of this paper is to introduce and study a concept of difference entire sequence spaces and difference analytic sequence spaces using sequence of modulus functions. We examine some topological properties and inclusion relation between the spaces $\Lambda_F(\Delta_s^m, u, p, q)$ and $\Gamma_F(\Delta_s^m, u, p, q)$ in the second section and third section devoted to the study of some properties of n-normed spaces $\Lambda_F(\Delta_s^m, u, p, q, ||., \cdots, .||)$ and $\Gamma_F(\Delta_s^m, u, p, q, ||., \cdots, .||)$.

2. Some Topological properties of the spaces $\Lambda_F(\Delta_s^m, u, p, q)$ and $\Gamma_F(\Delta_s^m, u, p, q)$

In this section of the paper we study very interesting properties like linearity, paranorm and some attractive inclusion relations between the spaces $\Lambda_F(\Delta_s^m, u, p, q)$ and $\Gamma_F(\Delta_s^m, u, p, q)$.

Theorem 2.1 Let $F = (f_k)$ be a sequence of modulus functions and $p = (p_k)$ be bounded sequence of strictly positive real numbers, then $\Gamma_F(\Delta_s^m, u, p, q)$ and $\Lambda_F(\Delta_s^m, u, p, q)$ are linear spaces over the set of complex numbers **C**.

Proof. Let $x = (x_k), y = (y_k) \in \Gamma_F(\Delta_s^m, u, p, q)$ and $\alpha, \beta \in \mathbf{C}$. In order to prove the result, we need to find some $\rho_3 > 0$ such that

$$\frac{1}{n}\sum_{k=1}^{n}\left[f_k\left(q\left(\frac{(|u_k\Delta_s^m(\alpha x_k+\beta y_k)|)^{\frac{1}{k}}}{\rho_3}\right)\right)\right]^{p_k} \to 0 \text{ as } n\to\infty.$$

Since $x = (x_k), y = (y_k) \in \Gamma_F(\Delta_s^m, u, p, q)$, there exist some positive ρ_1 and ρ_2 such that

$$\frac{1}{n}\sum_{k=1}^{n}\left[f_k\left(q\left(\frac{(|u_k\Delta_s^m x_k|)^{\frac{1}{k}}}{\rho_1}\right)\right)\right]^{p_k} \to 0 \text{ as } n\to\infty$$

and

$$\frac{1}{n}\sum_{k=1}^{n}\left[f_k\left(q\left(\frac{(|u_k\Delta_s^m y_k|)^{\frac{1}{k}}}{\rho_2}\right)\right)\right]^{p_k} \to 0 \text{ as } n\to\infty.$$

Since $F=(f_k)$ is a non-decreasing function, q is a seminorm and Δ_s^m is linear, then

$$\frac{1}{n}\sum_{k=1}^{n}\Big[f_k\Big(q\Big(\frac{(|u_k\Delta_s^m(\alpha x_k+\beta y_k)|)^{\frac{1}{k}}}{\rho_3}\Big)\Big)\Big]^{p_k}$$
$$\leq \frac{1}{n}\sum_{k=1}^{n}\Big[f_k\Big(q\Big(\frac{|\alpha|^{\frac{1}{k}}(|u_k\Delta_s^m x_k|)^{\frac{1}{k}}}{\rho_3}+\frac{|\beta|^{\frac{1}{k}}(|u_k\Delta_s^m y_k|)^{\frac{1}{k}}}{\rho_3}\Big)\Big)\Big]^{p_k}$$

so that

$$\sum_{k=1}^{n}\Big[f_k\Big(q\Big(\frac{(|u_k\Delta_s^m(\alpha x_k+\beta y_k)|)^{\frac{1}{k}}}{\rho_3}\Big)\Big)\Big]^{p_k}$$
$$\leq \frac{1}{n}\sum_{k=1}^{n}\Big[f_k\Big(q\Big(\frac{|\alpha|(|u_k\Delta_s^m x_k|)^{\frac{1}{k}}}{\rho_3}+\frac{|\beta|(|u_k\Delta_s^m y_k|)^{\frac{1}{k}}}{\rho_3}\Big)\Big)\Big]^{p_k}.$$

Take $\rho_3>0$ such that $\frac{1}{\rho_3}=\min\Big\{\frac{1}{|\alpha|\,\rho_1},\frac{1}{|\beta|\,\rho_2}\Big\}$

$$\frac{1}{n}\sum_{k=1}^{n}\Big[f_k\Big(q\Big(\frac{(|u_k\Delta_s^m(\alpha x_k+\beta y_k)|)^{\frac{1}{k}}}{\rho_3}\Big)\Big)\Big]^{p_k}$$
$$\leq \frac{1}{n}\sum_{k=1}^{n}\Big[f_k\Big(q\Big(\frac{(|u_k\Delta_s^m x_k|)^{\frac{1}{k}}}{\rho_1}+\frac{(|u_k\Delta_s^m y_k|)^{\frac{1}{k}}}{\rho_2}\Big)\Big)\Big]^{p_k}$$
$$\frac{1}{n}\sum_{k=1}^{n}\Big[\Big[f_k\Big(q\Big(\frac{(|u_k\Delta_s^m x_k|)^{\frac{1}{k}}}{\rho_1}\Big)\Big)\Big]^{p_k}+\Big[f_k\Big(q\Big(\frac{(|u_k\Delta_s^m y_k|)^{\frac{1}{k}}}{\rho_2}\Big)\Big)\Big]^{p_k}\Big]$$
$$\leq K\frac{1}{n}\sum_{k=1}^{n}\Big[f_k\Big(q\Big(\frac{(|u_k\Delta_s^m x_k|)^{\frac{1}{k}}}{\rho_1}\Big)\Big)\Big]^{p_k}$$
$$+K\frac{1}{n}\sum_{k=1}^{n}\Big[f_k\Big(q\Big(\frac{(|u_k\Delta_s^m y_k|)^{\frac{1}{k}}|}{\rho_2}\Big)\Big)\Big]^{p_k}$$
$$\longrightarrow 0 \text{ as } n\to\infty.$$

Hence

$$\sum_{k=1}^{n}\Big[f_k\Big(q\Big(\frac{(|\alpha u_k\Delta_s^m x_k+\beta u_k\Delta_s^m y_k|)^{\frac{1}{k}}}{\rho_3}\Big)\Big)\Big]^{p_k}\to 0 \text{ as } n\to\infty.$$

This proves that $\Gamma_F(\Delta_s^m,u,p,q)$ is a linear space. Similarly, we can prove that $\Lambda_F(\Delta_s^m,u,p,q)$ is a linear space

Theorem 2.2 Let $F = (f_k)$ be a sequence of modulus functions and $p = (p_k)$ be bounded sequence of strictly positive real numbers. Then $\Gamma_F(\Delta_s^m, u, p, q)$ is a paranormed space with paranorm defined by

$$g(x) = \inf\left\{\rho^{\frac{p_m}{H}} : \sup_{k\geq 1}\left[f_k\left(q\left(\frac{|(u_k\Delta_s^m x_k|)^{\frac{1}{k}}}{\rho}\right)\right)\right]^{p_k} \leq 1,\ \ \rho > 0,\ \ m \in \mathbf{N}\right\},$$

where $H = \max(1, \sup_k p_k)$.

Proof. Clearly $g(x) \geq 0,\ g(x) = g(-x)$ and $g(\theta) = 0$, where θ is the zero sequence of X.

Let $(x_k), (y_k) \in \Gamma_F(\Delta_s^m, u, p, q)$. Let $\rho_1,\ \rho_2 > 0$ be such that

$$\sup_{k\geq 1}\left[f_k\left(q\left(\frac{(|u_k\Delta_s^m x_k|)^{\frac{1}{k}}}{\rho_1}\right)\right)\right]^{p_k} \leq 1$$

and

$$\sup_{k\geq 1}\left[f_k\left(q\left(\frac{(|u_k\Delta_s^m y_k|)^{\frac{1}{k}}}{\rho_2}\right)\right)\right]^{p_k} \leq 1.$$

Let $\rho = \rho_1 + \rho_2$.

Then by using Minkowski's inequality, we have

$$\sup_{k\geq 1}\left[f_k\left(q\left(\frac{(|u_k\Delta_s^m (x_k + y_k)|)^{\frac{1}{k}}}{\rho}\right)\right)\right]^{p_k}$$
$$\leq \left(\tfrac{\rho_1}{\rho_1+\rho_2}\right)\sup_{k\geq 1}\left[f_k\left(q\left(\frac{(|u_k\Delta_s^m x_k|)^{\frac{1}{k}}}{\rho_1}\right)\right)\right]^{p_k}$$
$$+\left(\tfrac{\rho_2}{\rho_1+\rho_2}\right)\sup_{k\geq 1}\left[f_k\left(q\left(\frac{(|u_k\Delta_s^m y_k|)^{\frac{1}{k}}}{\rho_2}\right)\right)\right]^{p_k}$$
$$\leq 1.$$

Hence

$g(x+y)$

$$\leq \inf\left\{(\rho_1+\rho_2)^{\frac{p_m}{H}} : \sup_{k\geq 1}\left[f_k\left(q\left(\frac{(|u_k\Delta_s^m x_k|)^{\frac{1}{k}}}{\rho_1+\rho_2}\right)\right)\right]^{p_k} \leq 1, \rho_1,\ \rho_2 > 0,\ \ m \in N\right\}$$

$$\leq \inf\Big\{(\rho_1)^{\frac{p_m}{H}} : \sup_{k\geq 1}\Big[f_k\Big(q\Big(\frac{(|u_k\Delta_s^m x_k|)^{\frac{1}{k}}}{\rho_1}\Big)\Big)\Big]^{p_k} \leq 1, \rho_1 > 0, \ \ m \in \mathbf{N}\Big\}$$
$$+ \inf\Big\{(\rho_2)^{\frac{p_m}{H}} : \sup_{k\geq 1}\Big[f_k\Big(q\Big(\frac{(|u_k\Delta_s^m y_k|)^{\frac{1}{k}}}{\rho_2}\Big)\Big)\Big]^{p_k} \leq 1, \rho_2 > 0, \ \ m \in \mathbf{N}\Big\}.$$

Thus we have

$g(x+y) \leq g(x) + g(y)$. Hence g satisfies the triangle inequality.

$g(\lambda x) =$

$$\inf\Big\{(\rho)^{\frac{p_m}{H}} : \sup_{k\geq 1}\Big[f_k\Big(q\Big(\frac{(|\lambda u_k\Delta_s^m x_k|)^{\frac{1}{k}}}{\rho}\Big)\Big)\Big]^{p_k} \leq 1, \rho > 0, \ \ m \in \mathbf{N}\Big\}$$

$$= \inf\Big\{(r|\lambda|)^{\frac{p_m}{H}} : \sup_{k\geq 1}\Big[f_k\Big(q\Big(\frac{(|u_k\Delta_s^m x_k|)^{\frac{1}{k}}}{r}\Big)\Big)\Big]^{p_k} \leq 1, r > 0, \ \ m \in \mathbf{N}\Big\},$$
where $r = \frac{\rho}{|\lambda|}$.

Hence $\Gamma_F(\Delta_s^m, u, p, q)$ is a paranormed space.

Theorem 2.3 Let $F' = (f'_k)$ and $F'' = (f''_k)$ be two sequences of modulus functions. Then

$$\Gamma_{F'}(\Delta_s^m, u, p, q) \cap \Gamma_{F''}(\Delta_s^m, u, p, q) \subseteq \Gamma_{F'+F''}(\Delta_s^m, u, p, q).$$

Proof. Let $x = (x_k) \in \Gamma_{F'}(\Delta_s^m, u, p, q) \cap \Gamma_{F''}(\Delta_s^m, u, p, q)$.
Then there exist ρ_1 and ρ_2 such that

$$\frac{1}{n}\sum_{k=1}^{n}\Big[f'_k\Big(q\Big(\frac{(|u_k\Delta_s^m x_k|)^{\frac{1}{k}}}{\rho_1}\Big)\Big)\Big]^{p_k} \to 0 \text{ as } n \to \infty.$$

and

$$\frac{1}{n}\sum_{k=1}^{n}\Big[f''_k\Big(q\Big(\frac{(|u_k\Delta_s^m x_k|)^{\frac{1}{k}}}{\rho_2}\Big)\Big)\Big]^{p_k} \to 0 \text{ as } n \to \infty.$$

Since $\rho > 0$ such that $\frac{1}{\rho} = \min\Big(\frac{1}{\rho_1}, \frac{1}{\rho_2}\Big)$. Then we have $\frac{1}{n}\sum_{k=1}^{n}\Big[(f'_k + f''_k)\Big(q\Big(\frac{(|u_k\Delta_s^m x_k|)^{\frac{1}{k}}}{\rho}\Big)\Big)\Big]^{p_k}$

$$\leq K\Big[\frac{1}{n}\sum_{k=1}^{n}\Big[f'_k\Big(q\Big(\frac{(|u_k\Delta_s^m x_k|)^{\frac{1}{k}}}{\rho_1}\Big)\Big)\Big]^{p_k}\Big]$$

$$+K\Big[\frac{1}{n}\sum_{k=1}^{n}\Big[f''_k\Big(q\Big(\frac{(|u_k\Delta_s^m x_k|)^{\frac{1}{k}}}{\rho_2}\Big)\Big)\Big]^{p_k}\Big]$$

$\to 0$ as $n \to \infty$

Then

$$\frac{1}{n}\sum_{k=1}^{n}\Big[(f'_k + f''_k)\Big(q\Big(\frac{(|u_k\Delta_s^m x_k|)^{\frac{1}{k}}}{\rho}\Big)\Big)\Big]^{p_k} \to 0 \text{ as } n \to \infty.$$

Therefore $x = (x_k) \in \Gamma_{F'+F''}(\Delta_s^m, u, p, q)$.

Theorem 2.4 Let $m \geq 1$. Then we have the following inclusions:
(i) $\Gamma_F(\Delta_s^{m-1}, u, p, q) \subseteq \Gamma_F(\Delta_s^m, u, p, q)$,
(ii) $\Lambda_F(\Delta_s^{m-1}, u, p, q) \subseteq \Lambda_F(\Delta_s^m, u, p, q)$.

Proof. Let $x = (x_k) \in \Gamma_F(\Delta_s^{m-1}, u, p, q)$. Then we have

$$\frac{1}{n}\sum_{k=1}^{n}\Big[f_k\Big(q\Big(\frac{(|u_k\Delta_s^{m-1} x_k|)^{\frac{1}{k}}}{\rho}\Big)\Big)\Big]^{p_k} \to 0 \text{ as } n \to \infty, \text{ for some } \rho > 0.$$

Since $F = (f_k)$ is non-decreasing and q is a seminorm, we have

$$\frac{1}{n}\sum_{k=1}^{n}\Big[f_k\Big(q\Big(\frac{(|u_k\Delta_s^m x_k|)^{\frac{1}{k}}}{\rho}\Big)\Big)\Big]^{p_k}$$

$$\leq \frac{1}{n}\sum_{k=1}^{n}\Big[f_k\Big(q\Big(\frac{(|u_k\Delta_s^{m-1} x_k - u_k\Delta_s^{m-1} x_{k+1}|)^{\frac{1}{k}}}{\rho}\Big)\Big)\Big]^{p_k}$$

$$\leq K\Big\{\frac{1}{n}\sum_{k=1}^{n}\Big[f_k\Big(q\Big(\frac{(|u_k\Delta_s^{m-1} x_k|)^{\frac{1}{k}}}{\rho}\Big)\Big)\Big]^{p_k}$$

$$+\frac{1}{n}\sum_{k=1}^{n}\Big[f_k\Big(q\Big(\frac{(|u_k\Delta_s^{m-1}x_{k+1}|)^{\frac{1}{k}}}{\rho}\Big)\Big)\Big]^{p_k}\Big\}$$
$\longrightarrow 0$ as $n \to \infty$.

Therefore $\frac{1}{n}\sum_{k=1}^{n}\Big[f_k\Big(q\Big(\frac{(|u_k\Delta_s^{m}x_{k}|)^{\frac{1}{k}}}{\rho}\Big)\Big)\Big]^{p_k} \to 0$ as $n \to \infty$.

Hence $x \in \Gamma_F(\Delta_s^m, u, p, q)$. This completes the proof of (i). Similarly, we can prove (ii).

Theorem 2.5 Let $0 \leq p_k \leq r_k$ and let $\{\frac{r_k}{p_k}\}$ be bounded. Then $\Gamma_F(\Delta_s^m, u, r, q) \subset \Gamma_F(\Delta_s^m, u, p, q)$.

Proof. Let $x = (x_k) \in \Gamma_F(\Delta_s^m, u, r, q)$. Then

$$\frac{1}{n}\sum_{k=1}^{n}\Big[f_k\Big(q\Big(\frac{(|u_k\Delta_s^{m}x_{k}|)^{\frac{1}{k}}}{\rho}\Big)\Big)\Big]^{r_k} \quad \to 0 \text{ as } n \to \infty.$$

Let $t_k = \frac{1}{n}\sum_{k=1}^{n}\Big[f_k\Big(q\Big(\frac{(|u_k\Delta_s^{m}x_{k}|)^{\frac{1}{k}}}{\rho}\Big)\Big)\Big]^{q_k}$
and $\lambda_k = \frac{p_k}{r_k}$.

Since $p_k \leq r_k$, we have $0 \leq \lambda_k \leq 1$. Take $0 < \lambda < \lambda_k$. Define

$$u_k = \begin{cases} t_k & \text{if } t_k \geq 1 \\ 0 & \text{if } t_k < 1 \end{cases}$$

and

$$v_k = \begin{cases} 0 & \text{if } t_k \geq 1 \\ t_k & \text{if } t_k < 1 \end{cases}$$

$t_k = u_k + v_k$, $\quad t_k^{\lambda_k} = u_k^{\lambda_k} + v_k^{\lambda_k}$. It follows that $u_k^{\lambda_k} \leq u_k \leq t_k$, $v_k^{\lambda_k} \leq v_k^{\lambda}$. Since $t_k^{\lambda_k} = u_k^{\lambda_k} + v_k^{\lambda_k}$, then $t_k^{\lambda_k} \leq t_k + v_k^{\lambda}$. Thus

$$\frac{1}{n}\sum_{k=1}^{n}\Big[f_k\Big(q\Big(\frac{(|u_k\Delta_s^{m}x_{k}|)^{\frac{1}{k}}}{\rho}\Big)\Big)^{r_k}\Big]^{\lambda_k}$$
$$\leq \frac{1}{n}\sum_{k=1}^{n}\Big[f_k\Big(q\Big(\frac{(|u_k\Delta_s^{m}x_{k}|)^{\frac{1}{k}}}{\rho}\Big)\Big)\Big]^{r_k}$$

$$\Longrightarrow \tfrac{1}{n}\sum_{k=1}^{n}\left[f_k\Big(q\Big(\frac{(|u_k\Delta_s^m x_k|)^{\frac{1}{k}}}{\rho}\Big)\Big)^{r_k}\right]^{p_k/r_k}$$
$$\leq \tfrac{1}{n}\sum_{k=1}^{n}\left[f_k\Big(q\Big(\frac{(|u_k\Delta_s^m x_k|)^{\frac{1}{k}}}{\rho}\Big)\Big)\right]^{r_k}$$
$$\Longrightarrow \tfrac{1}{n}\sum_{k=1}^{n}\left[f_k\Big(q\Big(\frac{(|u_k\Delta_s^m x_k|)^{\frac{1}{k}}}{\rho}\Big)\Big)\right]^{p_k}$$
$$\leq \tfrac{1}{n}\sum_{k=1}^{n}\left[f_k\Big(q\Big(\frac{(|u_k\Delta_s^m x_k|)^{\frac{1}{k}}}{\rho}\Big)\Big)\right]^{r_k}.$$

But

$$\frac{1}{n}\sum_{k=1}^{n}\left[f_k\Big(q\Big(\frac{(|u_k\Delta_s^m x_k|)^{\frac{1}{k}}}{\rho}\Big)\Big)\right]^{r_k} \to 0 \text{ as } n\to\infty.$$

Therefore

$$\frac{1}{n}\sum_{k=1}^{n}\left[f_k\Big(q\Big(\frac{(|u_k\Delta_s^m x_k|)^{\frac{1}{k}}}{\rho}\Big)\Big)\right]^{p_k} \to 0 \text{ as } n\to\infty.$$

Hence $x=(x_k)\in\Gamma_F(\Delta_s^m,u,p,q)$. Thus, we have

$\Gamma_F(\Delta_s^m,u,r,q)\subset\Gamma_F(\Delta_s^m,u,p,q)$.

Theorem 2.6

(i) Let $0<\inf p_k\leq p_k\leq 1$*. Then*
$\Gamma_F(\Delta_s^m,u,p,q)\subset\Gamma_F(\Delta_s^m,u,q)$,

(ii) Let $1\leq p_k\leq\sup p_k<\infty$*. Then* $\Gamma_F(\Delta_s^m,u,q)\subset\Gamma_F(\Delta_s^m,u,p,q)$.

Proof. (i) Let $x=(x_k)\in\Gamma_F(\Delta_s^m,u,p,q)$. Then

$$\frac{1}{n}\sum_{k=1}^{n}\left[f_k\Big(q\Big(\frac{(|u_k\Delta_s^m x_k|)^{\frac{1}{k}}}{\rho}\Big)\Big)\right]^{p_k} \to 0 \text{ as } n\to\infty.$$

Since $0<\inf p_k\leq p_k\leq 1$,

$$\frac{1}{n}\sum_{k=1}^{n}\left[f_k\Big(q\Big(\frac{(|u_k\Delta_s^m x_k|)^{\frac{1}{k}}}{\rho}\Big)\Big)\right] \leq \frac{1}{n}\sum_{k=1}^{n}\left[f_k\Big(q\Big(\frac{(|u_k\Delta_s^m x_k|)^{\frac{1}{k}}}{\rho}\Big)\Big)\right]^{p_k} \to 0$$

as $n \to \infty$.

Thus, it follows that, $x = (x_k) \in \Gamma_F(\Delta_s^m, u, q)$. Thus $\Gamma_F(\Delta_s^m, u, p, q) \subset \Gamma_F(\Delta_s^m, u, q)$.

(ii) Let $p_k \geq 1$ for each k and $\sup p_k < \infty$ and let $x = (x_k) \in \Gamma_F(\Delta_s^m, u, q)$. Then

$$\frac{1}{n}\sum_{k=1}^{n}\left[f_k\left(q\left(\frac{(|u_k\Delta_s^m x_k|)^{\frac{1}{k}}}{\rho}\right)\right)\right] \to 0 \text{ as } n \to \infty$$

Since $1 \leq p_k \leq \sup p_k < \infty$, we have

$$\frac{1}{n}\sum_{k=1}^{n}\left[f_k\left(q\left(\frac{(|u_k\Delta_s^m x_k|)^{\frac{1}{k}}}{\rho}\right)\right)\right]^{p_k} \leq \frac{1}{n}\sum_{k=1}^{n}\left[f_k\left(q\left(\frac{(|u_k\Delta_s^m x_k|)^{\frac{1}{k}}}{\rho}\right)\right)\right]$$

$$\frac{1}{n}\sum_{k=1}^{n}\left[f_k\left(q\left(\frac{(|u_k\Delta_s^m x_k|)^{\frac{1}{k}}}{\rho}\right)\right)\right]^{p_k} \to 0 \text{ as } n \to \infty.$$

This implies that $x = (x_k) \in \Gamma_F(\Delta_s^m, u, p, q)$. Therefore

$\Gamma_F(\Delta_s^m, u, q) \subset \Gamma_F(\Delta_s^m, u, p, q)$.

Theorem 2.7 Suppose $\frac{1}{n}\sum_{k=1}^{n}\left[f_k\left(q\left(\frac{(|u_k\Delta_s^m x_k|)^{\frac{1}{k}}}{\rho}\right)\right)\right]^{p_k} \leq |x_k|^{1/k}$, then $\Gamma \subset \Gamma_F(\Delta_s^m, u, p, q)$.

Proof. Let $x = (x_k) \in \Gamma$. Then we have,

$$|x_k|^{1/k} \to 0 \text{ as } k \to \infty.$$

But $\frac{1}{n}\sum_{k=1}^{n}\left[f_k\left(q\left(\frac{(|u_k\Delta_s^m x_k|)^{\frac{1}{k}}}{\rho}\right)\right)\right]^{p_k} \leq |x_k|^{1/k}$, by our assumption, implies that

$$\frac{1}{n}\sum_{k=1}^{n}\left[f_k\left(q\left(\frac{(|u_k\Delta_s^m x_k|)^{\frac{1}{k}}}{\rho}\right)\right)\right]^{p_k} \to 0 \text{ as } n \to \infty$$

Then $x = (x_k) \in \Gamma_F(\Delta_s^m, u, p, q)$ and $\Gamma \subset \Gamma_F(\Delta_s^m, u, p, q)$.

Theorem 2.8 $\Gamma_F(\Delta_s^m, u, p, q)$ is solid.

Proof. Let $|x_k| \leq |y_k|$ and let $y = (y_k) \in \Gamma_F(\Delta_s^m, u, p, q)$, because $F = (f_k)$ is non-decreasing

$$\frac{1}{n}\sum_{k=1}^{n}\left[f_k\Big(q\Big(\frac{(|u_k\Delta_s^m x_k|)^{\frac{1}{k}}}{\rho}\Big)\Big)\right]^{p_k} \leq \frac{1}{n}\sum_{k=1}^{n}\left[f_k\Big(q\Big(\frac{(|u_k\Delta_s^m y_k|)^{\frac{1}{k}}}{\rho}\Big)\Big)\right]^{p_k}$$

Since $y = (y_k) \in \Gamma_F(\Delta_s^m, u, p, q)$. Therefore,

$$\frac{1}{n}\sum_{k=1}^{n}\left[f_k\Big(q\Big(\frac{(|u_k\Delta_s^m y_k|)^{\frac{1}{k}}}{\rho}\Big)\Big)\right]^{p_k} \to 0 \text{ as } n \to \infty$$

and so that

$$\frac{1}{n}\sum_{k=1}^{n}\left[f_k\Big(q\Big(\frac{(|u_k\Delta_s^m x_k|)^{\frac{1}{k}}}{\rho}\Big)\Big)\right]^{p_k} \to 0 \text{ as } n \to \infty.$$

Therefore $x = (x_k) \in \Gamma_F(\Delta_s^m, u, p, q)$.

Theorem 2.9 $\Gamma_F(\Delta_s^m, u, p, q)$ is monotone.

Proof. It is trivial so we omit it.

3. Difference Entire sequence spaces over n- normed spaces

The concept of 2-normed spaces was initially developed by Gähler[6] in the mid of 1960's, while that of n-normed spaces one can see in Misiak[14]. Since then, many others have studied this concept and obtained various results, see Gunawan ([7],[8]) and Gunawan and Mashadi [9]. For more details about the sequence spaces over n-normed spaces see ([15],[16]).

Let $n \in \mathbf{N}$ and X be a linear space over the field $\mathbf{K}$, where $\mathbf{K}$ is field of real or complex numbers of dimension d, where $d \geq n \geq 2$. A real valued function $||\cdot, \cdots, \cdot||$ on X^n satisfying the following four conditions:

1. $||x_1, x_2, \cdots, x_n|| = 0$ if and only if $x_1, x_2, \cdots, x_n$ are linearly dependent in X;
2. $||x_1, x_2, \cdots, x_n||$ is invariant under permutation;

3. $||\alpha x_1, x_2, \cdots, x_n|| = |\alpha| \ ||x_1, x_2, \cdots, x_n||$ for any $\alpha \in \mathbf{K}$, and

4. $||x + x', x_2, \cdots, x_n|| \leq ||x, x_2, \cdots, x_n|| + ||x', x_2, \cdots, x_n||$

is called an n-norm on X, and the pair $(X, ||\cdot, \cdots, \cdot||)$ is called a n-normed space over the field $\mathbf{K}$. For example, we may take $X = \mathbf{R}^n$ being equipped with the n-norm $||x_1, x_2, \cdots, x_n||_E =$ the volume of the n-dimensional parallelopiped spanned by the vectors $x_1, x_2, \cdots, x_n$ which may be given explicitly by the formula

$$||x_1, x_2, \cdots, x_n||_E = |\det(x_{ij})|,$$

where $x_i = (x_{i1}, x_{i2}, \cdots, x_{in}) \in \mathbf{R}^n$ for each $i = 1, 2, \cdots, n$.

Let $(X, ||\cdot, \cdots, \cdot||)$ be an n-normed space of dimension $d \geq n \geq 2$ and $\{a_1, a_2, \cdots, a_n\}$ be linearly independent set in X. Then the following function $||\cdot, \cdots, \cdot||_\infty$ on X^{n-1} defined by

$$||x_1, x_2, \cdots, x_{n-1}||_\infty = \max\{||x_1, x_2, \cdots, x_{n-1}, a_i|| : i = 1, 2, \cdots, n\}$$

defines an $(n-1)$-norm on X with respect to $\{a_1, a_2, \cdots, a_n\}$.

A sequence (x_k) in a n-normed space $(X, ||\cdot, \cdots, \cdot||)$ is said to converge to some $L \in X$ if

$$\lim_{k\to\infty} ||x_k - L, z_1, \cdots, z_{n-1}|| = 0 \text{ for every } z_1, \cdots, z_{n-1} \in X.$$

A sequence (x_k) in a n-normed space $(X, ||\cdot, \cdots, \cdot||)$ is said to be Cauchy if

$$\lim_{k,p\to\infty} ||x_k - x_p, z_1, \cdots, z_{n-1}|| = 0 \text{ for every } z_1, \cdots, z_{n-1} \in X.$$

If every cauchy sequence in X converges to some $L \in X$, then X is said to be complete with respect to the n-norm. Any complete n-normed space is said to be n-Banach space.

Let $F = (f_k)$ be a sequence of modulus functions and let X be locally convex Hausdorff topological linear space whose topology is determined by a set of continuous seminorms q. The symbol $\Lambda(X)$, $\Gamma(X)$ denotes the space of all analytic and entire sequences respectively defined over X. In this section we define the following sequences spaces:

$$\Lambda_F(\Delta_s^m, u, p, q, ||., \cdots, .||) =$$

$$\Big\{x \in \Lambda(X) : \sup_n \frac{1}{n} \sum_{k=1}^{n} \Big[f_k\Big(q\Big(||\frac{(u_k \Delta_s^m x_k)^{1/k}}{\rho}, z_1, \cdots, z_{n-1}||\Big)\Big)\Big]^{p_k}$$
$$< \infty, \ \text{ for some } \ \rho > 0\Big\},$$

$$\Gamma_F(\Delta_s^m, u, p, q, ||., \cdots, .||) =$$

$$\Big\{x \in \Gamma(X) : \tfrac{1}{n} \sum_{k=1}^{n} \Big[f_k\Big(q\Big(||\frac{(u_k \Delta_s^m x_k)^{1/k}}{\rho}, z_1, \cdots, z_{n-1}||\Big)\Big)\Big]^{p_k}$$
$$\to 0 \ \text{ as } n \to \infty, \text{for some } \ \rho > 0\Big\}.$$

If we take $p = (p_k) = 1$, we get

$$\Lambda_F(\Delta_s^m, u, q, ||., \cdots, .||) =$$
$$\Big\{x \in \Lambda(X) : \sup_n \frac{1}{n} \sum_{k=1}^{n} \Big[f_k\Big(q\Big(||\frac{(u_k \Delta_s^m x_k)^{1/k}}{\rho}, z_1, \cdots, z_{n-1}||\Big)\Big)\Big]$$
$$< \infty, \ \text{ for some } \ \rho > 0\Big\},$$

$$\Gamma_F(\Delta_s^m, u, q, ||., \cdots, .||) =$$

$$\Big\{x \in \Gamma(X) : \tfrac{1}{n} \sum_{k=1}^{n} \Big[f_k\Big(q\Big(||\frac{(u_k \Delta_s^m x_k)^{1/k}}{\rho}, z_1, \cdots, z_{n-1}||\Big)\Big)\Big]$$
$$\to 0 \ \text{ as } n \to \infty, \text{for some } \ \rho > 0\Big\}.$$

In this section of the paper we study some topological properties of the spaces $\Lambda_F(\Delta_s^m, u, p, q, ||., \cdots, .||)$ and $\Gamma_F(\Delta_s^m, u, p, q, ||., \cdots, .||)$. We also examine some inclusion relation between these spaces.

Theorem 3.1 Let $F = (f_k)$ be a sequence of modulus functions and $p = (p_k)$ be bounded sequence of strictly positive real numbers, then $\Gamma_F(\Delta_s^m, u, p, q, ||., \cdots, .||)$ and $\Lambda_F(\Delta_s^m, u, p, q, ||., \cdots, .||)$ are linear spaces over the set of complex numbers **C**.

Proof. $x = (x_k),\ y = (y_k) \in \Gamma_F(\Delta_s^m, u, p, q, ||., \cdots, .||)$ and $\alpha, \beta \in \mathbf{C}$. In order to prove the result, we need to find some $\rho_3 > 0$ such that

$$\frac{1}{n}\sum_{k=1}^{n}\Big[f_k\Big(q\Big(||\frac{(u_k\Delta_s^m(\alpha x_k+\beta y_k))^{\frac{1}{k}}}{\rho_3}, z_1, \cdots, z_{n-1}||\Big)\Big)\Big]^{p_k} \to 0 \text{ as } n \to \infty.$$

Since $x = (x_k), y = (y_k) \in \Gamma_F(\Delta_s^m, u, p, q, ||., \cdots, .||)$, there exist some positive ρ_1 and ρ_2 such that

$$\tfrac{1}{n}\sum_{k=1}^{n}\Big[f_k\Big(q\Big(||\frac{(u_k\Delta_s^m x_k)^{\frac{1}{k}}}{\rho_1}, z_1, \cdots, z_{n-1}||\Big)\Big)\Big]^{p_k} \to 0 \text{ as } n \to \infty$$

and

$$\frac{1}{n}\sum_{k=1}^{n}\Big[f_k\Big(q\Big(||\frac{(u_k\Delta_s^m y_k)^{\frac{1}{k}}}{\rho_2}, z_1, \cdots, z_{n-1}||\Big)\Big)\Big]^{p_k} \to 0 \text{ as } n \to \infty.$$

Since $F = (f_k)$ is a non-decreasing function, q is a seminorm and Δ_s^m is linear, then

$$\tfrac{1}{n}\sum_{k=1}^{n}\Big[f_k\Big(q\Big(||\frac{(u_k\Delta_s^m(\alpha x_k+\beta y_k))^{\frac{1}{k}}}{\rho_3}, z_1, \cdots, z_{n-1}||\Big)\Big)\Big]^{p_k}$$
$$\le \tfrac{1}{n}\sum_{k=1}^{n}\Big[f_k\Big(q\Big(||\frac{\alpha^{\frac{1}{k}}(u_k\Delta_s^m x_k)^{\frac{1}{k}}}{\rho_3}, z_1, \cdots, z_{n-1}||+$$
$$||\tfrac{\beta^{\frac{1}{k}}(u_k\Delta_s^m y_k)^{\frac{1}{k}}}{\rho_3}, z_1, \cdots, z_{n-1}||\Big)\Big)\Big]^{p_k}$$

so that

$$\frac{1}{n}\sum_{k=1}^{n}\Big[f_k\Big(q\Big(||\frac{(u_k\Delta_s^m(\alpha x_k+\beta y_k))^{\frac{1}{k}}}{\rho_3}, z_1, \cdots, z_{n-1}||\Big)\Big)\Big]^{p_k}$$
$$\le \tfrac{1}{n}\sum_{k=1}^{n}\Big[f_k\Big(q\Big(||\frac{\alpha(u_k\Delta_s^m x_k)^{\frac{1}{k}}}{\rho_3}, z_1, \cdots, z_{n-1}||$$
$$+||\tfrac{\beta(u_k\Delta_s^m y_k)^{\frac{1}{k}}}{\rho_3}, z_1, \cdots, z_{n-1}||\Big)\Big)\Big]^{p_k}.$$

Since $\rho_3 > 0$ such that $\frac{1}{\rho_3} = \min\left\{\frac{1}{|\alpha|\,\rho_1}, \frac{1}{|\beta|\,\rho_2}\right\}$

$$\frac{1}{n}\sum_{k=1}^{n}\left[f_k\left(q\left(||\frac{(u_k\Delta_s^m(\alpha x_k+\beta y_k))^{\frac{1}{k}}}{\rho_3}, z_1, \cdots, z_{n-1}||\right)\right)\right]^{p_k}$$

$$\leq \quad \frac{1}{n}\sum_{k=1}^{n}\left[f_k\left(q\left(||\left(\frac{(u_k\Delta_s^m x_k)^{\frac{1}{k}}}{\rho_1} + \frac{(|u_k\Delta_s^m y_k|)^{\frac{1}{k}}}{\rho_2}\right), z_1, \cdots, z_{n-1}||\right)\right)\right]^{p_k}$$

$$\leq \quad \frac{1}{n}\sum_{k=1}^{n}\left[\left[f_k\left(q\left(||\frac{(u_k\Delta_s^m x_k)^{\frac{1}{k}}}{\rho_1}, z_1, \cdots, z_{n-1}||\right)\right)\right]^{p_k}\right.$$

$$\left.+\left[f_k\left(q\left(||\frac{(u_k\Delta_s^m y_k)^{\frac{1}{k}}}{\rho_2}, z_1, \cdots, z_{n-1}||\right)\right)\right]^{p_k}\right]$$

$$\leq K\frac{1}{n}\sum_{k=1}^{n}\left[f_k\left(q\left(||\frac{(u_k\Delta_s^m x_k)^{\frac{1}{k}}}{\rho_1}, z_1, \cdots, z_{n-1}||\right)\right)\right]^{p_k}$$

$$+K\frac{1}{n}\sum_{k=1}^{n}\left[f_k\left(q\left(||\frac{(u_k\Delta_s^m y_k)^{\frac{1}{k}}}{\rho_1}, z_1, \cdots, z_{n-1}||\right)\right)\right]^{p_k}$$

$$\longrightarrow 0 \text{ as } n \to \infty.$$

Hence

$$\sum_{k=1}^{n}\left[f_k\left(q\left(||\frac{(u_k\alpha\Delta_s^m x_k + \beta u_k\Delta_s^m y_k)^{\frac{1}{k}}}{\rho_3}, z_1, \cdots, z_{n-1}||\right)\right)\right]^{p_k} \to 0 \text{ as } n \to \infty.$$

This proves that $\Gamma_F(\Delta_s^m, u, p, q, ||., \cdots, .||)$ is a linear space. Similarly, we can prove that $\Lambda_F(\Delta_s^m, u, p, q, ||., \cdots, .||)$ is a linear space.

Theorem 3.2 Let $F = (f_k)$ be a sequence of modulus functions and $p = (p_k)$ be bounded sequence of strictly positive real numbers, $\Gamma_F(\Delta_s^m, u, p, q, ||., \cdots, .||)$ is paranormed space with paranorm defined by

$$g(x) = \inf\left\{\rho^{\frac{p_m}{H}} : \sup_{k\geq 1}\left[f_k\left(q\left(||\frac{(u_k\Delta_s^m x_k)^{\frac{1}{k}}}{\rho}, z_1, \cdots, z_{n-1}||\right)\right)\right]^{p_k} \leq 1,\right.$$
$$\left.\rho > 0,\ m \in \mathbf{N}\right\},$$

where $H = \max(1, \sup_k p_k)$.

Proof. Clearly $g(x) \geq 0,\ g(x) = g(-x)$ and $g(\theta) = 0$, where θ is the zero sequence of X.

Let $(x_k), (y_k) \in \Gamma_F(\Delta_s^m, u, p, q, ||., \cdots, .||)$. Let $\rho_1,\ \rho_2 > 0$ be such that

$$\sup_{k\geq 1}\Big[f_k\Big(q\Big(||\frac{(u_k\Delta_s^m x_k)^{\frac{1}{k}}}{\rho_1}, z_1, \cdots, z_{n-1}||\Big)\Big)\Big]^{p_k} \leq 1$$

and

$$\sup_{k\geq 1}\Big[f_k\Big(q\Big(||\frac{(u_k\Delta_s^m y_k)^{\frac{1}{k}}}{\rho_2}, z_1, \cdots, z_{n-1}||\Big)\Big)\Big]^{p_k} \leq 1.$$

Let $\rho = \rho_1 + \rho_2$. Then by using Minkowski's inequality, we have

$$\begin{aligned}
&\sup_{k\geq 1}\Big[f_k\Big(q\Big(||\frac{(u_k\Delta_s^m (x_k + y_k))^{\frac{1}{k}}}{\rho}, z_1, \cdots, z_{n-1}||\Big)\Big)\Big]^{p_k}\\
\leq\ & \Big(\frac{\rho_1}{\rho_1+\rho_2}\Big)\sup_{k\geq 1}\Big[f_k\Big(q\Big(||\frac{(u_k\Delta_s^m x_k)^{\frac{1}{k}}}{\rho_1}, z_1, \cdots, z_{n-1}||\Big)\Big)\Big]^{p_k}\\
+\ & \Big(\frac{\rho_2}{\rho_1+\rho_2}\Big)\sup_{k\geq 1}\Big[f_k\Big(q\Big(||\frac{(u_k\Delta_s^m y_k)^{\frac{1}{k}}}{\rho_2}, z_1, \cdots, z_{n-1}||\Big)\Big)\Big]^{p_k}\\
\leq\ & 1.
\end{aligned}$$

Hence

$$\begin{aligned}
&g(x+y)\\
\leq\ & \inf\Big\{(\rho_1+\rho_2)^{\frac{p_m}{H}} : \sup_{k\geq 1}\Big[f_k\Big(q\Big(||\frac{(u_k\Delta_s^m x_k)^{\frac{1}{k}}}{\rho_1+\rho_2}, z_1, \cdots, z_{n-1}||\Big)\Big)\Big]^{p_k} \leq 1,\\
&\rho_1,\ \rho_2 > 0,\ \ m \in N\Big\}\\
&\leq \inf\Big\{(\rho_1)^{\frac{p_m}{H}} : \sup_{k\geq 1}\Big[f_k\Big(q\Big(||\frac{(u_k\Delta_s^m x_k)^{\frac{1}{k}}}{\rho_1}, z_1, \cdots, z_{n-1}||\Big)\Big)\Big]^{p_k} \leq 1,\\
&\rho_1 > 0,\ \ m \in N\Big\}\\
&+ \inf\Big\{(\rho_2)^{\frac{p_m}{H}} : \sup_{k\geq 1}\Big[f_k\Big(q\Big(||\frac{(u_k\Delta_s^m y_k)^{\frac{1}{k}}}{\rho_2}, z_1, \cdots, z_{n-1}||\Big)\Big)\Big]^{p_k} \leq 1,\\
&\rho_2 > 0,\ \ m \in N\Big\}.
\end{aligned}$$

Thus we have $g(x+y) \leq g(x)+g(y)$. Hence g satisfies the triangle inequality.

$$g(\lambda x) = \inf\Big\{(\rho)^{\frac{p_m}{H}} : \sup_{k\geq 1}\Big[f_k\Big(q\Big(||\frac{(\lambda u_k \Delta_s^m x_k)^{\frac{1}{k}}}{\rho}, z_1, \cdots, z_{n-1}||\Big)\Big)\Big]^{p_k} \leq 1,$$

$$\rho > 0, \;\; m \in N\Big\}$$

$$= \inf\Big\{(r|\lambda|)^{\frac{p_m}{H}} : \sup_{k\geq 1}\Big[f_k\Big(q\Big(||\frac{(u_k \Delta_s^m x_k)^{\frac{1}{k}}}{r}, z_1, \cdots, z_{n-1}||\Big)\Big)\Big]^{p_k} \leq 1,$$

$$r > 0, \;\; m \in N\Big\},$$

where $r = \frac{\rho}{|\lambda|}$.

Hence $\Gamma_F(\Delta_s^m, u, p, q, ||., \cdots, .||)$ is a paranormed space.

Theorem 3.3 Let $F' = (f'_k)$ and $F'' = (f''_k)$ be two sequences of modulus functions.

Then $\Gamma_{F'}(\Delta_s^m, u, p, q, ||., \cdots, .||) \cap \Gamma_{F''}(\Delta_s^m, u, p, q, ||., \cdots, .||)$

$\subseteq \Gamma_{F'+F''}(\Delta_s^m, u, p, q, ||., \cdots, .||)$.

Proof. Let $x = (x_k) \in \Gamma_{F'}(\Delta_s^m, u, p, q, ||., \cdots, .||) \cap \Gamma_{F''}(\Delta_s^m, u, p, q, ||., \cdots, .||)$.

Then there exist ρ_1 and ρ_2 such that

$$\frac{1}{n}\sum_{k=1}^{n}\Big[f'_k\Big(q\Big(||\frac{(u_k \Delta_s^m x_k)^{\frac{1}{k}}}{\rho_1}, z_1, \cdots, z_{n-1}||\Big)\Big)\Big]^{p_k} \to 0 \text{ as } n \to \infty.$$

and

$$\frac{1}{n}\sum_{k=1}^{n}\Big[f''_k\Big(q\Big(||\frac{(u_k \Delta_s^m x_k)^{\frac{1}{k}}}{\rho_2}, z_1, \cdots, z_{n-1}||\Big)\Big)\Big]^{p_k} \to 0 \text{ as } n \to \infty.$$

Let $\frac{1}{\rho} = \min\Big(\frac{1}{\rho_1}, \frac{1}{\rho_2}\Big)$. Then we have

$$\frac{1}{n}\sum_{k=1}^{n}\Big[(f'_k + f''_k)\Big(q\Big(||\frac{(u_k \Delta_s^m x_k)^{\frac{1}{k}}}{\rho}, z_1, \cdots, z_{n-1}||\Big)\Big)\Big]^{p_k}$$

$$\leq \quad K\Big[\frac{1}{n}\sum_{k=1}^{n}\Big[f_k'\Big(q\Big(||\frac{(u_k\Delta_s^m x_k)^{\frac{1}{k}}}{\rho_1}, z_1, \cdots, z_{n-1}||\Big)\Big)\Big]^{p_k}\Big]$$
$$+ \quad K\Big[\frac{1}{n}\sum_{k=1}^{n}\Big[f_k''\Big(q\Big(||\frac{(u_k\Delta_s^m x_k)^{\frac{1}{k}}}{\rho_2}, z_1, \cdots, z_{n-1}||\Big)\Big)\Big]^{p_k}\Big]$$
$$\to 0 \text{ as } n \to \infty$$

Then

$$\frac{1}{n}\sum_{k=1}^{n}\Big[(f_k'+f_k'')\Big(q\Big(||\frac{(u_k\Delta_s^m x_k)^{\frac{1}{k}}}{\rho}, z_1, \cdots, z_{n-1}||\Big)\Big)\Big]^{p_k} \to 0 \text{ as } n \to \infty.$$

Therefore $x = (x_k) \in \Gamma_{F'+F''}(\Delta_s^m, u, p, q, ||., \cdots, .||)$.

Theorem 3.4 *Let $m \geq 1$. Then we have the following inclusions:*
(i) $\Gamma_F(\Delta_s^{m-1}, u, p, q, ||., \cdots, .||) \subseteq \Gamma_F(\Delta_s^m, u, p, q, ||., \cdots, .||)$,
(ii) $\Lambda_F(\Delta_s^{m-1}, u, p, q, ||., \cdots, .||) \subseteq \Lambda_F(\Delta_s^m, u, p, q, ||., \cdots, .||)$.

Proof. Let $x = (x_k) \in \Gamma_F(\Delta_s^{m-1}, u, p, q, ||., \cdots, .||)$. Then we have
$\frac{1}{n}\sum_{k=1}^{n}\Big[f_k\Big(q\Big(||\frac{(u_k\Delta_s^{m-1}x_k)^{\frac{1}{k}}}{\rho}, z_1, \cdots, z_{n-1}||\Big)\Big)\Big]^{p_k} \to 0 \text{ as } n \to \infty,$ for some $\rho > 0$.

Since $F = (f_k)$ is non-decreasing and q is a seminorm, we have

$$\frac{1}{n}\sum_{k=1}^{n}\Big[f_k\Big(q\Big(||\frac{(u_k\Delta_s^m x_k)^{\frac{1}{k}}}{\rho}, z_1, \cdots, z_{n-1}||\Big)\Big)\Big]^{p_k}$$
$$\leq \frac{1}{n}\sum_{k=1}^{n}\Big[f_k\Big(q\Big(||\frac{(u_k\Delta_s^{m-1}x_k - u_k\Delta_s^{m-1}x_{k+1})^{\frac{1}{k}}}{\rho}, z_1, \cdots, z_{n-1}||\Big)\Big)\Big]^{p_k}$$
$$\leq K\Big\{\frac{1}{n}\sum_{k=1}^{n}\Big[f_k\Big(q\Big(||\frac{(u_k\Delta_s^{m-1}x_k)^{\frac{1}{k}}}{\rho}, z_1, \cdots, z_{n-1}||\Big)\Big)\Big]^{p_k}$$
$$+\frac{1}{n}\sum_{k=1}^{n}\Big[f_k\Big(q\Big(||\frac{(u_k\Delta_s^{m-1}x_{k+1})^{\frac{1}{k}}}{\rho}, z_1, \cdots, z_{n-1}||\Big)\Big)\Big]^{p_k}\Big\}$$
$$\longrightarrow 0 \text{ as } n \to \infty.$$

Therefore $\frac{1}{n}\sum_{k=1}^{n}\Big[f_k\Big(q\Big(||\frac{(u_k\Delta_s^m x_k)^{\frac{1}{k}}}{\rho}, z_1, \cdots, z_{n-1}||\Big)\Big)\Big]^{p_k} \to 0$ as $n \to \infty$.

Hence $x = (x_k) \in \Gamma_F(\Delta_s^m, u, p, q, ||., \cdots, .||)$. This completes the proof of (i). Similarly, we can prove (ii).

Theorem 3.5 Let $0 \leq p_k \leq r_k$ and let $\{\frac{r_k}{p_k}\}$ be bounded. Then

$$\Gamma_F(\Delta_s^m, u, r, q, ||., \cdots, .||) \subset \Gamma_F(\Delta_s^m, u, p, q, ||., \cdots, .||).$$

Proof. Let $x \in \Gamma_F(\Delta_s^m, u, r, q, ||., \cdots, .||)$. Then

$$\frac{1}{n}\sum_{k=1}^{n}\left[f_k\Big(q\Big(||\frac{(u_k\Delta_s^m x_k)^{\frac{1}{k}}}{\rho}, z_1, \cdots, z_{n-1}||\Big)\Big)\right]^{r_k} \to 0 \text{ as } n \to \infty.$$

Let $t_k = \frac{1}{n}\sum_{k=1}^{n}\left[f_k\Big(q\Big(||\frac{(u_k\Delta_s^m x_k)^{\frac{1}{k}}}{\rho}, z_1, \cdots, z_{n-1}||\Big)\Big)\right]^{q_k}$ and $\lambda_k = \frac{p_k}{r_k}$.
Since $p_k \leq r_k$, we have $0 \leq \lambda_k \leq 1$. Take $0 < \lambda < \lambda_k$. Define

$$u_k = \begin{cases} t_k & \text{if } t_k \geq 1 \\ 0 & \text{if } t_k < 1 \end{cases}$$

and

$$v_k = \begin{cases} 0 & \text{if } t_k \geq 1 \\ t_k & \text{if } t_k < 1 \end{cases}$$

$t_k = u_k + v_k$, $\quad t_k^{\lambda_k} = u_k^{\lambda_k} + v_k^{\lambda_k}$. It follows that $u_k^{\lambda_k} \leq u_k \leq t_k$, $v_k^{\lambda_k} \leq v_k^{\lambda}$. Since $t_k^{\lambda_k} = u_k^{\lambda_k} + v_k^{\lambda_k}$, then $t_k^{\lambda_k} \leq t_k + v_k^{\lambda}$. So that

$$\frac{1}{n}\sum_{k=1}^{n}\left[f_k\Big(q\Big(||\frac{(u_k\Delta_s^m x_k)^{\frac{1}{k}}}{\rho}, z_1, \cdots, z_{n-1}||\Big)\Big)^{r_k}\right]^{\lambda_k}$$
$$\leq \frac{1}{n}\sum_{k=1}^{n}\left[f_k\Big(q\Big(||\frac{(u_k\Delta_s^m x_k)^{\frac{1}{k}}}{\rho}, z_1, \cdots, z_{n-1}||\Big)\Big)\right]^{r_k}$$

This implies that

$$\frac{1}{n}\sum_{k=1}^{n}\left[f_k\Big(q\Big(||\frac{(u_k\Delta_s^m x_k)^{\frac{1}{k}}}{\rho}, z_1, \cdots, z_{n-1}||\Big)\Big)^{r_k}\right]^{p_k/r_k}$$
$$\leq \frac{1}{n}\sum_{k=1}^{n}\left[f_k\Big(q\Big(||\frac{(u_k\Delta_s^m x_k)^{\frac{1}{k}}}{\rho}, z_1, \cdots, z_{n-1}||\Big)\Big)\right]^{r_k}$$

$$\Longrightarrow \tfrac{1}{n}\sum_{k=1}^{n}\Big[f_k\Big(q\Big(||\frac{(u_k\Delta_s^m x_k)^{\frac{1}{k}}}{\rho}, z_1,\cdots,z_{n-1}||\Big)\Big)\Big]^{p_k}$$
$$\leq \tfrac{1}{n}\sum_{k=1}^{n}\Big[f_k\Big(q\Big(||\frac{(u_k\Delta_s^m x_k)^{\frac{1}{k}}}{\rho}, z_1,\cdots,z_{n-1}||\Big)\Big)\Big]^{r_k}.$$

But

$$\frac{1}{n}\sum_{k=1}^{n}\Big[f_k\Big(q\Big(||\frac{(u_k\Delta_s^m x_k)^{\frac{1}{k}}}{\rho}, z_1,\cdots,z_{n-1}||\Big)\Big)\Big]^{r_k} \to 0 \text{ as } n\to\infty.$$

Therefore

$$\frac{1}{n}\sum_{k=1}^{n}\Big[f_k\Big(q\Big(||\frac{(u_k\Delta_s^m x_k)^{\frac{1}{k}}}{\rho}, z_1,\cdots,z_{n-1}||\Big)\Big)\Big]^{p_k} \to 0 \text{ as } n\to\infty.$$

Hence $x=(x_k)\in\Gamma_F(\Delta_s^m,u,p,q,||.,\cdots,.||)$. Thus, we get

$$\Gamma_F(\Delta_s^m,u,r,q,||.,\cdots,.||)\subset\Gamma_F(\Delta_s^m,u,p,q,||.,\cdots,.||).$$

Theorem 3.6 *(i) Let* $0<\inf p_k\leq p_k\leq 1$. *Then*

$\Gamma_F(\Delta_s^m,u,p,q,||.,\cdots,.||)\subset\Gamma_F(\Delta_s^m,u,q,||.,\cdots,.||)$,
(ii) Let $1\leq p_k\leq\sup p_k<\infty$. *Then*

$$\Gamma_F(\Delta_s^m,u,q,||.,\cdots,.||)\subset\Gamma_F(\Delta_s^m,u,p,q,||.,\cdots,.||).$$

Proof. (i) Let $x=(x_k)\in\Gamma_F(\Delta_s^m,u,p,q,||.,\cdots,.||)$. Then

$$\frac{1}{n}\sum_{k=1}^{n}\Big[f_k\Big(q\Big(||\frac{(u_k\Delta_s^m x_k)^{\frac{1}{k}}}{\rho}, z_1,\cdots,z_{n-1}||\Big)\Big)\Big]^{p_k} \to 0 \text{ as } n\to\infty.$$

Since $0<\inf p_k\leq p_k\leq 1$, $\tfrac{1}{n}\sum_{k=1}^{n}\Big[f_k\Big(q\Big(||\frac{(u_k\Delta_s^m x_k)^{\frac{1}{k}}}{\rho}, z_1,\cdots,z_{n-1}||\Big)\Big)\Big]$
$$\leq \tfrac{1}{n}\sum_{k=1}^{n}\Big[f_k\Big(q\Big(||\frac{(u_k\Delta_s^m x_k)^{\frac{1}{k}}}{\rho}, z_1,\cdots,z_{n-1}||\Big)\Big)\Big]^{p_k}$$
$\to 0$ as $n\to\infty$.

Thus, it follows that, $x = (x_k) \in \Gamma_F(\Delta_s^m, u, q, ||., \cdots, .||)$.
Thus $\Gamma_F(\Delta_s^m, u, p, q, ||., \cdots, .||) \subset \Gamma_F(\Delta_s^m, u, q, ||., \cdots, .||)$.
(ii) Let $p_k \geq 1$ for each k and $\sup p_k < \infty$ and let

$x = (x_k) \in \Gamma_F(\Delta_s^m, u, q, ||., \cdots, .||)$. Then

$$\tfrac{1}{n}\sum_{k=1}^{n}\left[f_k\Big(q\Big(||\frac{(u_k\Delta_s^m x_k)^{\frac{1}{k}}}{\rho}, z_1, \cdots, z_{n-1}||\Big)\Big)\right] \to 0 \ \text{ as } \ n \to \infty$$

Since $1 \leq p_k \leq \sup p_k < \infty$, we have

$$\tfrac{1}{n}\sum_{k=1}^{n}\left[f_k\Big(q\Big(||\frac{(u_k\Delta_s^m x_k)^{\frac{1}{k}}}{\rho}, z_1, \cdots, z_{n-1}||\Big)\Big)\right]^{p_k}$$
$$\leq \tfrac{1}{n}\sum_{k=1}^{n}\left[f_k\Big(q\Big(||\frac{(u_k\Delta_s^m x_k)^{\frac{1}{k}}}{\rho}, z_1, \cdots, z_{n-1}||\Big)\Big)\right].$$

Hence

$$\tfrac{1}{n}\sum_{k=1}^{n}\left[f_k\Big(q\Big(||\frac{(u_k\Delta_s^m x_k)^{\frac{1}{k}}}{\rho}, z_1, \cdots, z_{n-1}||\Big)\Big)\right]^{p_k} \to 0 \ \text{ as } \ n \to \infty.$$

This implies that $x = (x_k) \in \Gamma_F(\Delta_s^m, u, p, q, ||., \cdots, .||)$. Therefore $\Gamma_F(\Delta_s^m, u, q, ||., \cdots, .||) \subset \Gamma_F(\Delta_s^m, u, p, q, ||., \cdots, .||)$.

Theorem 3.7 Suppose

$$\tfrac{1}{n}\sum_{k=1}^{n}\left[f_k\Big(q\Big(||\frac{(u_k\Delta_s^m x_k)^{\frac{1}{k}}}{\rho}, z_1, \cdots, z_{n-1}||\Big)\Big)\right]^{p_k} \leq |x_k|^{1/k},$$

then $\Gamma \subset \Gamma_F(\Delta_s^m, u, p, q, ||., \cdots, .||)$.

Proof. Let $x = (x_k) \in \Gamma$. Then we have,

$$|x_k|^{1/k} \to 0 \ \text{ as } \ k \to \infty.$$

But $\tfrac{1}{n}\sum_{k=1}^{n}\left[f_k\Big(q\Big(||\frac{(u_k\Delta_s^m x_k)^{\frac{1}{k}}}{\rho}, z_1, \cdots, z_{n-1}||\Big)\Big)\right]^{p_k} \leq |x_k|^{1/k}$, by our assumption, implies that

$$\frac{1}{n}\sum_{k=1}^{n}\left[f_k\Big(q\Big(||\frac{(u_k\Delta_s^m x_k)^{\frac{1}{k}}}{\rho}, z_1, \cdots, z_{n-1}||\Big)\Big)\right]^{p_k} \to 0 \ \text{ as } \ n \to \infty \quad \text{by}(10)$$

Then $x = (x_k) \in \Gamma_F(\Delta_s^m, u, p, q, ||., \cdots, .||)$ and

$\Gamma \subset \Gamma_F(\Delta_s^m, u, p, q, ||., \cdots, .||)$.

Theorem 3.8 $\Gamma_F(\Delta_s^m, u, p, q, ||., \cdots, .||)$ is solid.

Proof. Let $|x_k| \leq |y_k|$ and let $y = (y_k) \in \Gamma_F(\Delta_s^m, u, p, q, ||., \cdots, .||)$, because $F = (f_k)$ is non-decreasing, so that

$$\frac{1}{n}\sum_{k=1}^{n}\Big[f_k\Big(q\Big(||\frac{(u_k\Delta_s^m x_k)^{\frac{1}{k}}}{\rho}, z_1, \cdots, z_{n-1}||\Big)\Big)\Big]^{p_k}$$
$$\leq \frac{1}{n}\sum_{k=1}^{n}\Big[f_k\Big(q\Big(||\frac{(u_k\Delta_s^m y_k)^{\frac{1}{k}}}{\rho}, z_1, \cdots, z_{n-1}||\Big)\Big)\Big]^{p_k}$$

Since $y \in \Gamma_F(\Delta_s^m, u, p, q, ||., \cdots, .||)$. Therefore,

$$\frac{1}{n}\sum_{k=1}^{n}\Big[f_k\Big(q\Big(||\frac{(u_k\Delta_s^m y_k)^{\frac{1}{k}}}{\rho}, z_1, \cdots, z_{n-1}||\Big)\Big)\Big]^{p_k} \to 0 \text{ as } n \to \infty$$

and

$$\frac{1}{n}\sum_{k=1}^{n}\Big[f_k\Big(q\Big(||\frac{(u_k\Delta_s^m x_k)^{\frac{1}{k}}}{\rho}, z_1, \cdots, z_{n-1}||\Big)\Big)\Big]^{p_k} \to 0 \text{ as } n \to \infty.$$

Therefore $x = (x_k) \in \Gamma_F(\Delta_s^m, u, p, q)$.

Theorem 3.9 $\Gamma_F(\Delta_s^m, u, p, q, ||., \cdots, .||)$ is monotone.

Proof. It is trivial so we omit it.

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