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Hochschild-Serre Statement for the total cohomology

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Abstract

Let M be a complex manifold and $\mathcal{F}$ a O_M-module with a $\mathbf{g}$-holomorphic action where $\mathbf{g}$ is a complex Lie algebra (cf. [3]). We denote by $\mathbf{H}(\mathbf{g}, \mathcal{F})$ the "total cohomology" as defined in [1] [2]. Then we prove that, for any ideal $\mathbf{a} \subset \mathbf{g}$, the module $\mathbf{H}^\bullet(\mathbf{a}, \mathcal{F})$ viewed as a $\mathbf{g}/\mathbf{a}$-module, we have a spectral sequence which converges to $\mathbf{H}(\mathbf{g}, \mathcal{F})$ and whose E_2-term is $E_2^{p,q} = H^p\big(\mathbf{g}/\mathbf{a}; \mathbf{H}^\mathbf{q}(\mathbf{a}, \mathcal{F})\big)$.

Let $\mathbf{g}$ be a finite dimensional complex Lie algebra and M a complex analytic manifold of finite dimension. Suppose that a holomorphic field $\mathbf{u_M}$ of tangents $(1,0)$-vectors on M is associated to each $\mathbf{u} \in \mathbf{g}$. If this transformation satisfies the condition $[\mathbf{u_M}, \mathbf{v_M}] = [\mathbf{u}, \mathbf{v}]_\mathbf{M}$, we shall say that it defines a holomorphic $\mathbf{g}$-action on M. To be more precise, the real parts of these fields $\mathbf{u_M}$ are the opposite of the Killing fields of a local holomorphic action of some complex Lie group. Let $\mathcal{F}$ be an O_M-module and, for all $\mathbf{u} \in \mathbf{g}$, let $\gamma_*(\mathbf{u}) : \mathcal{F} \to \mathcal{F}$ be a morphism of C-sheaf.

Definition 0.1. *If, for any local section σ of $\mathcal{F}$ and any local holomorphic function f on M, we have:*
(i) $\gamma_*([\mathbf{u}, \mathbf{v}]) = [\gamma_*(\mathbf{u}), \gamma_*(\mathbf{v})]$
(ii) $\gamma_*(\mathbf{u})(\mathbf{f}\sigma) = \mathbf{L}_{\mathbf{u_M}}\mathbf{f}\sigma + \mathbf{f}\gamma_*(\mathbf{u})\sigma$,
we say that $\mathcal{F}$ is an O_M-module with a holomorphic $\mathbf{g}$*-action.*

Now, denote by $U(\mathbf{g}, \mathbf{C})$ be the envelopping algebra of the complex Lie algebra $\mathbf{g}$.

In [3], we have introduced the sheaf of crossed algebras $U(\mathbf{g}, \mathbf{O_M}) \stackrel{\mathbf{def}}{=} \mathbf{O_M} \otimes_\mathbf{C} \mathbf{U}(\mathbf{g}, \mathbf{C})$ with the use of the commutation formula: $(1 \otimes \mathbf{u})(\varphi \otimes \mathbf{1}) \stackrel{\mathbf{def}}{=} \mathbf{L}_{\mathbf{u_M}}\varphi \otimes \mathbf{1} + \varphi \otimes \mathbf{u}$. Then, we see immediately that the O_M-modules with a holomorphic $\mathbf{g}$-action, are exactly the $U(\mathbf{g}, \mathbf{O_M})$-modules, objects which make some Abelian category denoted $Mod\Big(U(\mathbf{g}, \mathbf{O_M})\Big)$. On the other hand, in [1] and [2], we have defined, for any holomorphically G-equivariant vector bundle $E \to M$ (G is a complex Lie group with Lie algebra $\mathbf{g}$), the total cohomology denoted $\mathbf{H}^*(\mathbf{g}, \mathbf{E})$. In [3], we have generalized this total cohomology to any $U(\mathbf{g}, \mathbf{O_M})$-module $\mathcal{F}$ and we have showed indeed that the total cohomology is a derived functor; more precisely, we have proved that:

$$\mathbf{H}^*(\mathbf{g}, \mathbf{E}) \approx \mathbf{Ext}^*_{\mathbf{U}(\mathbf{g}, \mathbf{O_M})}(\mathbf{O_M}, \mathbf{E})$$

Proposition 0.2. *Let M, $\mathbf{g}$, and so on... be like above. Let $\mathcal{F}$ be a left $U(\mathbf{g}, \mathbf{O_M})$-module and $\mathbf{a}$ an ideal of the complex Lie algebra $\mathbf{g}$. Then:*
(i) The total cohomology $\mathbf{H}(\mathbf{a}, \mathcal{F})$ is naturally a left $(\mathbf{g}/\mathbf{a})$-module.
(ii) There is a Hochschild-Serre spectral sequence E_r whose E_2-term is given by $H^p\Big(\mathbf{g}/\mathbf{a}, \mathbf{H}^\mathbf{q}(\mathbf{a}, \mathcal{F})\Big)$ and which converges to $\mathbf{H}^{\mathbf{p}+\mathbf{q}}(\mathbf{g}, \mathcal{F})$

Proof. (i) It is well known, by the Poincaré-Birkhoff-Witt formula, that $U(\mathbf{g}, \mathbf{O_M})$ is a free left $U(\mathbf{a}, O_M)$-module, and then also, by the anti-isomorphism T (see [3]), a free right $U(\mathbf{a}, O_M)$-module. From this we deduce the exactness of the change of rings functor:

$$U(\mathbf{g}, \mathbf{O_M}) \otimes_{\mathbf{U}(\mathbf{a},\mathbf{O_M})} - : \mathbf{Mod}\Big(\mathbf{U}(\mathbf{a}, \mathbf{O_M})\Big) \to \mathbf{Mod}\Big(\mathbf{U}(\mathbf{g}, \mathbf{O_M})\Big)$$

By functor adjunction (see [3]), this exactness allows us to show that the 'forget functor': $Mod\Big(U(\mathbf{g}, \mathbf{O_M})\Big) \to \mathbf{Mod}\Big(\mathbf{U}(\mathbf{a}, \mathbf{O_M})\Big)$ preserves injective objects. Also, taking the cohomology of the complex of global $\mathbf{a}$ - invariant sections of an injective resolution for an $U(\mathbf{g}, \mathbf{O_M})$-module $\mathcal{F}$, we obtain the total cohomology $\mathbf{H}^\bullet(\mathbf{a}, \mathcal{F})$ which is then a $(\mathbf{g}/\mathbf{a})$-module and does not depend of the auxiliary choice of the resolution.

(ii) The Grothendieck composition theorem of functors shows that it is sufficient to prove that, if $\mathcal{I}$ is an injective $U(\mathbf{g}, \mathbf{O_M})$-module, then the Chevalley-Eilenberg cohomology $H^p\Big(\mathbf{g}/\mathbf{a}, \mathbf{H^0}(\mathbf{a}, \mathcal{I})\Big)$ of the $(\mathbf{g}/\mathbf{a})$-module $\mathbf{H^0}(\mathbf{a}, \mathcal{I})$ is zero for $p \geq 1$. For this, we know that it will be enough - and we shall make it - to show that the $\mathbf{H^0}(\mathbf{a}, \mathcal{I})$ is an injective $(\mathbf{g}/\mathbf{a})$-module.

Indeed, let $0 \to \mathbf{M'} \xrightarrow{\mathbf{j}} \mathbf{M}$ be a monomorphism of $U(\mathbf{g}/\mathbf{a}, \mathbf{C})$-module. We must factorize each $(\mathbf{g}/\mathbf{a})-$morphism $\mathbf{M'} \xrightarrow{\mathbf{u}} \mathbf{H^0}(\mathbf{a}, \mathcal{I})$ through the monomorphism $\mathbf{j}$. Let us consider $\mathbf{M'}$ and $\mathbf{M}$ as $\mathbf{g}$-modules with an in-effectiveness $\mathbf{a}$; we introduce, as in [3], the $U(\mathbf{g}, \mathbf{O_M})$-modules $O_M \otimes_C \mathbf{M'}$ and $O_M \otimes_C \mathbf{M}$, defined by the formula:

$$\gamma_*(\mathbf{u})(\mathbf{f} \otimes \mathbf{m}) = \mathbf{L_{u_M}f} \otimes \mathbf{m} + \mathbf{f} \otimes \gamma_*(\mathbf{u})\mathbf{m}.$$

But, $\mathbf{j}$ enlarges it naturally in an arrow of $U(\mathbf{g}, \mathbf{O_M})$-modules $j : O_M \otimes_C \mathbf{M'} \to \mathbf{O_M} \otimes_\mathbf{C} \mathbf{M}$. In more, u allows to define naturally some arrow $O_M \otimes_C \mathbf{M'} \to \mathcal{I}$ which, by the injectivity of $\mathcal{I}$, factorizes itself by $\mathbf{j}$ with the use of one arrow: $O_M \otimes_C \mathbf{M} \to \mathcal{I}$.

Last arrow that defines one other: $\mathbf{H^0}(\mathbf{a}, \mathbf{O_M} \otimes_\mathbf{C} \mathbf{M}) \to \mathbf{H^0}(\mathbf{a}, \mathcal{I})$. But, then , by restriction of this last arrow to $\mathbf{M} \subset \mathbf{H^0}(\mathbf{a}, \mathbf{O_M} \otimes_\mathbf{C} \mathbf{M})$, we see easily that this answers the question.

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