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Double lacunary sequence spaces of double sequence of interval numbers

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Abstract

In this paper we introduce the concepts of double lacunary strongly convergence and double lacunary statistical convergence of double interval numbers. We prove some inclusion relations and study some of their properties.

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1. Introduction

Interval arithmetic was first suggested by Dwyer [12] in 1951. Development of interval arithmetic as a formal system and evidence of its value as a computational device was provided by Moore [15] in 1959 and Moore and Yang [16] 1962. Furthermore, Moore and others [12] , [13] , [14] , [17] and [18] have developed applications to differential equations.

Chiao in [9] introduced sequence of interval numbers and defined usual convergence of sequences of interval number. Recently Esi $[1-3]$ and Şengönül and Eryilmaz in [11] introduced and studied some sequence spaces of interval numbers.

The idea of statistical convergence for ordinary sequences was introduced by Fast [7] in 1951. Schoenberg [8] studied statistical convergence as a summability method and listed some of elamantary properties of statistical convergence. Both of these authors noted that if bounded sequence is statistically convergent, then it is Cesaro summable. Existing work on statistical convergence appears to have been restricted to real or complex sequence, but several authors extended the idea to apply to sequences of fuzzy numbers and also introduced and discussed the concept of statistically sequences of fuzzy numbers.

2. Preliminaries

A double sequence of real numbers is a function $x : \mathbf{N} \times \mathbf{N} \to \mathbf{R}$. We shall use the notation $x = (x_{k,l})$.

A double sequence $x = (x_{k,l})$ has a *Pringsheim limit* L (denoted by $P - limx = L$) provided that given an $\varepsilon > 0$ there exists an $N \in \mathbf{N}$ such that $|x_{k,l} - L| < \varepsilon$ whenever $k, l > N$. We shall describe such an $x = (x_{k,l})$ more briefly as "$P - convergent$" [4].The double sequence $x = (x_{k,l})$ is *bounded* if there exists a positive number M such that $|x_{k,l}| < M$ for all k and l,

$$\|x\| = \sup_{k,l} |x_{k,l}| < \infty.$$

Let $p = (p_{k,l})$ be a double sequence of positive real numbers. If $0 < h = \inf_{k,l} p_{k,l} \le p_{k,l} \le H = \sup_{k,l} p_{k,l} < \infty$ and $D = max\left(1, 2^{H-1}\right)$, then for all $a_{k,l}, b_{k,l} \in \mathbf{C}$ for all $k, l \in \mathbf{N}$, we have

$$|a_{k,l} + b_{k,l}|^{p_{k,l}} \le D\left(|a_{k,l}|^{p_{k,l}} + |b_{k,l}|^{p_{k,l}}\right).$$

We should note that in contrast to the case for single sequences, a convergent double sequence need not be bounded.The concept of statistical convergence was introduced by Fast [7] in 1951. A sequence $x = (x_k)$ is said to be statistically convergent to the number L if for every $\varepsilon > 0$

$$\lim_n \frac{1}{n} |\{ \ k \leq n : \ \ |x_k - L| \geq \varepsilon\}| = 0,$$

where the vertical bars indicate the number of elements in the enclosed set. Later, Mursaleen and Edely [10] defined the statistical analogue for double sequence $x = (x_{k,l})$ as follows: A real double sequence $x = (x_{k,l})$ is said to be $P-$ *statistical convergence to* L provided that for each $\varepsilon > 0$

$$P - \lim_{m,n} \frac{1}{mn} |\{ \ (k,l) : \ k < m, l < n; \ \ |x_{k,l} - L| \geq \varepsilon\}| = 0.$$

In this case, we write $St_2 - \lim_{k,l} x_{k,l} = L$ and we denote the set of all $P-$ *statistical convergent double sequences* by St_2.

By a lacunary $\theta = (k_r)$; $r = 0, 1, 2, ... where\ k_o = 0$, we shall mean an increasing sequence of non-negative integers with $h_r = k_r - k_{r-1} \to \infty$ as $r \to \infty$. The intervals determined by θ will be denoted by $I_r = (k_{r-1}, k_r]$.The ratio $\frac{k_r}{k_{r-1}}$ will be denoted by q_r. The space of lacunary strongly convergent sequence space N_θ was defined by Freedman et.al. [5] as follows:

$$N_\theta = \left\{ x = (x_k) : \ \lim_r \frac{1}{h_r} \sum_{k \in I_r} |x_k - L| = 0, \text{ for some } L \right\}.$$

The double sequence $\theta_{r,s} = \{(k_r, l_s)\}$ is called double lacunary sequence if there exist two increasing sequences of integers such that

$$k_o = 0, \ h_r = k_r - k_{r-1} \to \infty \ \text{ as } r \to \infty$$

and

$$l_o = 0, \ \overline{h_s} = l_s - l_{s-1} \to \infty \ \text{ as } s \to \infty.$$

Notations: $k_{r,s} = k_r l_s$, $h_{r,s} = h_r \overline{h_s}$ and $\theta_{r,s}$ is determined by

$$I_{r,s} = \{(k,l) : \ k_{r-1} < k \leq k_r \text{ and } l_{s-1} < l \leq l_s\},$$

$$q_r = \frac{k_r}{k_{r-1}}, \overline{q_s} = \frac{l_s}{l_{s-1}} \ and\ q_{r,s} = q_r \overline{q_s}. \ [6]$$

The set of all double lacunary sequences denoted by $N_{\theta_{r,s}}$ and defined by Savaş and Patterson [6] as follows:

$$N_{\theta_{r,s}}=\left\{x=(x_{k,l}):\ P-\lim_{r,s}\frac{1}{h_{r,s}}\sum_{(k,l)\in I_{r,s}}|x_{k,l}-L|=0,\ \text{for some L}\right\}.$$

We denote the set of all real valued closed intervals by I**R**. Any elements of I**R** is called interval number and denoted by $\overline{x}=[x_l,x_r]$. Let x_l and x_r be first and last points of $\overline{x}$ interval number, respectively. For $\overline{x}_1,\overline{x}_2\in$I**R**, we have
$\overline{x}_1=\overline{x}_2\Leftrightarrow x_{1_l}{=}x_{2_l}, x_{1_r}{=}x_{2_r}$. $\overline{x}_1{+}\overline{x}_2=\{x\in\mathbf{R}:x_{1_l}+x_{2_l}\leq x\leq x_{1_r}+x_{2_r}\}$, and if $\alpha\geq 0$, then $\alpha\overline{x}=\{x\in\mathbf{R}:\ \alpha x_{1_l}\leq x\leq\alpha x_{1_r}\}$ and if $\alpha<0$, then $\alpha\overline{x}=\{x\in\mathbf{R}:\ \alpha x_{1_r}\leq x\leq\alpha x_{1_l}\}$,

$$\overline{x}_1.\overline{x}_2$$

$$=\{x\in\mathbf{R}:\min\{x_{1_l}.x_{2_l},x_{1_l}.x_{2_r},x_{1_r}.x_{2_l},x_{1_r}.x_{2_r}\}\leq x$$

$$\leq\max\{x_{1_l}.x_{2_l},x_{1_l}.x_{2_r},x_{1_r}.x_{2_l},x_{1_r}.x_{2_r}\}.$$

The set of all interval numbers I**R** is a complete metric space defined by

$$d(\overline{x}_1,\overline{x}_2)=\max\{|x_{1_l}-x_{2_l}|,|x_{1_r}-x_{2_r}|\}\ \ [15].$$

In the special case $\overline{x}_1=[a,a]$ and $\overline{x}_2=[b,b]$, we obtain usual metric of **R**.

Now we give the definition of convergence of interval numbers:

Definition 1.1. [9] A sequence $\overline{x}=(\overline{x}_k)$ of interval numbers is said to be convergent to the interval number $\overline{x}_o$ if for each $\varepsilon>0$ there exists a positive integer k_o such that $d(\overline{x}_k,\overline{x}_o)<\varepsilon$ for all $k\geq k_o$ and we denote it by $\lim_k\overline{x}_k=\overline{x}_o$.

Thus, $\lim_k\overline{x}_k=\overline{x}_o\Leftrightarrow\lim_k x_{k_l}=x_{o_l}$ and $\lim_k x_{k_r}=x_{o_r}$.

Let's define transformation $\overline{x}$ from **N**×**N** to I**R** by $k,l\to\overline{x}(k,l)=\overline{x}_{k,l}$. We shall use the notation $\overline{x}=(\overline{x}_{k,l})$. Then $\overline{x}=(\overline{x}_{k,l})$ is called double sequence of interval numbers. The $\overline{x}_{k,l}$ is called $(k,l)^{th}$ term of sequence $\overline{x}=(\overline{x}_{k,l})$.

In this paper, we introduce and study the concepts of double lacunary strongly convergence and double lacunary statistically convergence for interval numbers.

3. Main Results

In this section we give some definition and prove the results of this paper.

Definition 3.1. Let $\theta_{r,s} = \{(k_r, l_s)\}$ be a double lacunary sequence and $p = (p_{k,l})$ be any double sequence of strictly positive real numbers. A double sequence $\overline{x} = (\overline{x}_{k,l})$ of interval numbers is said to be double lacunary strongly convergent if there is a double interval number $\overline{x}_o$ such that

$$P - \lim_{r,s} \frac{1}{h_{r,s}} \sum_{k \in I_{r,s}} [d(\overline{x}_{k,l}, \overline{x}_o)]^{p_{k,l}} = 0.$$

In this case we write $\overline{x}_{k,l} \to \overline{x}_o \left({}_2\overline{N}^p_{\theta_{r,s}} \right)$ or ${}_2\overline{N}^p_\theta - \lim \overline{x}_{k,l} = \overline{x}_o$. We denote with ${}_2\overline{N}^p_{\theta_{r,s}}$ the set of all lacunary strongly convergent double sequences of interval numbers. In the special case $\theta_{r,s} = \{(2^r, 2^s)\}$, we shall write ${}_2\overline{N}^p$ instead of ${}_2\overline{N}^p_{\theta_{r,s}}$.

Definition 3.2. Let $\theta_{r,s} = \{(k_r, l_s)\}$ be a double lacunary sequence. A double sequence $\overline{x} = (\overline{x}_{k,l})$ of interval numbers is said to be double lacunary statistically convergent to interval number $\overline{x}_o$ if for every $\varepsilon > 0$

$$P - \lim_{r,s} \frac{1}{h_{r,s}} \left| \{(k,l) \in I_{r,s} : \ d(\overline{x}_{k,l}, \overline{x}_o) \geq \varepsilon\} \right| = 0.$$

In this case we write $\overline{x}_{k,l} \to \overline{x}_o \left(\overline{s}_{\theta_{r,s}} \right)$ or $\overline{s}_{\theta_{r,s}} - \lim \overline{x}_{k,l} = \overline{x}_o$. The set of all double lacunary statistically convergent sequences of interval number is denoted by $\overline{s}_{\theta_{r,s}}$. In the special case $\theta_{r,s} = \{(2^r, 2^s)\}$, we shall write $\overline{s}$ instead of $\overline{s}_{\theta_{r,s}}$.

Theorem 3.1. Let $\overline{x} = (\overline{x}_{k,l})$ and $\overline{y} = \left(\overline{y}_{k,l}\right)$ be double sequences of interval numbers.

(i) If $\overline{s}_{\theta_{r,s}} - \lim \overline{x}_{k,l} = \overline{x}_o$ and $\alpha \in \mathbf{R}$, then $\overline{s}_{\theta_{r,s}} - \lim \alpha \overline{x}_{k,l} = \alpha \overline{x}_o$.

(ii) If $\overline{s}_{\theta_{r,s}} - \lim \overline{x}_{k,l} = \overline{x}_o$ and $\overline{s}_{\theta_{r,s}} - \lim \overline{y}_{k,l} = \overline{y}_o$, then $\overline{s}_{\theta_{r,s}} - \lim \left(\overline{x}_{k,l} + \overline{y}_{k,l} \right) = \overline{x}_o + \overline{y}_o$.

Proof. **(i)** Let $\alpha \in = |\alpha| \, d(\overline{x}_{k,l}, \overline{x}_o)$. For a given $\varepsilon > 0$

$$\frac{1}{h_{r,s}} \left| \{(k,l) \in I_{r,s} : \ d(\alpha \overline{x}_{k,l}, \alpha \overline{x}_o) \geq \varepsilon\} \right|$$

$$= \frac{1}{h_{r,s}} \left| \left\{ (k,l) \in I_{r,s} : \ d\left(\overline{x}_{k,l}, \overline{x}_o\right) \geq \frac{\varepsilon}{|\alpha|} \right\} \right|.$$

Hence $\overline{s}_{\theta_{r,s}} - \lim \alpha \overline{x}_{k,l} = \alpha \overline{x}_o$.

(ii) Suppose that $\overline{s}_{\theta_{r,s}} - \lim \overline{x}_{k,l} = \overline{x}_o$ and $\overline{s}_{\theta_{r,s}} - \lim \overline{y}_{k,l} = \overline{y}_o$. We have

$$d\left(\overline{x}_{k,l} + \overline{y}_{k,l}, \overline{x}_o + \overline{y}_o\right)$$

$$\leq d\left(\overline{x}_{k,l}, \overline{x}_o\right) + d\left(\overline{y}_{k,l}, \overline{y}_o\right).$$

Therefore given $\varepsilon > 0$, we have

$$\frac{1}{h_{r,s}} \left| \left\{ (k,l) \in I_{r,s} : d\left(\overline{x}_{k,l} + \overline{y}_{k,l}, \overline{x}_o + \overline{y}_o\right) \geq \varepsilon \right\} \right|$$

$$\leq \frac{1}{h_{r,s}} \left| \left\{ (k,l) \in I_{r,s} : d\left(\overline{x}_{k,l}, \overline{x}_o\right) + d\left(\overline{y}_{k,l}, \overline{y}_o\right) \geq \varepsilon \right\} \right|$$

$$\leq \frac{1}{h_{r,s}} \left| \left\{ (k,l) \in I_{r,s} : \ d\left(\overline{x}_{k,l}, \overline{x}_o\right) \geq \frac{\varepsilon}{2} \right\} \right| + \frac{1}{h_{r,s}} \left| \left\{ (k,l) \in I_{r,s} : \ d\left(\overline{y}_{k,l}, \overline{y}_o\right) \geq \frac{\varepsilon}{2} \right\} \right|.$$

Thus, $\overline{s}_{\theta_{r,s}} - \lim \left(\overline{x}_{k,l} + \overline{y}_{k,l}\right) = \overline{x}_o + \overline{y}_o$.

Theorem 3.2. Let $\theta_{r,s} = \{(k_r, l_s)\}$ be a double lacunary sequence and $\overline{x} = (\overline{x}_{k,l})$ be a double sequence of interval numbers. Then

(i) $\overline{x}_{k,l} \to \overline{x}_o \left(\overline{N}^p_{\theta_{r,s}}\right)$ implies $\overline{x}_{k,l} \to \overline{x}_o \left(\overline{s}_{\theta_{r,s}}\right)$,

(ii) $\overline{x} = (\overline{x}_{k,l}) \in \overline{m}$ and $\overline{x}_{k,l} \to \overline{x}_o \left(\overline{s}_{\theta_{r,s}}\right)$ imply $\overline{x}_{k,l} \to \overline{x}_o \left(\overline{N}^p_{\theta_{r,s}}\right)$,

(iii) If $\overline{x} = (\overline{x}_{k,l}) \in \overline{m}$, then $\overline{x}_{k,l} \to \overline{x}_o \left(\overline{N}^p_{\theta_{r,s}}\right)$ if and only if $\overline{x}_{k,l} \to \overline{x}_o \left(\overline{s}_{\theta_{r,s}}\right)$, where $\overline{m} = \left\{ \overline{x} = (\overline{x}_{k,l}) : \ \sup_{k,l} d\left(\overline{x}_{k,l}, \overline{x}_o\right) < \infty \right\}$.

Proof. **(i)** Let $\varepsilon > 0$ and $\overline{x}_{k,l} \to \overline{x}_o \left(\overline{N}^p_{\theta_{r,s}}\right)$. Then we write

$$\left| \left\{ (k,l) \in I_{r,s} : d\left(\overline{x}_{k,l}, \overline{x}_o\right) \geq \varepsilon \right\} \right| \leq \sum_{(k,l) \in I_{r,s} d\left(\overline{x}_{k,l}, \overline{x}_o\right) \geq \varepsilon} d\left(\overline{x}_{k,l}, \overline{x}_o\right)$$

and

$$P - \lim_{r,s} \frac{1}{h_{r,s}} \sum_{k \in I_{r,s}} \left[d\left(\overline{x}_{k,l}, \overline{x}_o\right) \right]^{p_{k,l}} = 0.$$

This implies that

$$P - \lim_{r,s} \frac{1}{h_{r,s}} \left| \left\{ (k,l) \in I_{r,s} : d\left(\overline{x}_{k,l}, \overline{x}_o\right) \geq \varepsilon \right\} \right| = 0.$$

This completes the proof (i).

(ii) Suppose that $\overline{x} = (\overline{x}_{k,l}) \in \overline{m}$ and $\overline{x}_{k,l} \to \overline{x}_o \left(\overline{s}_{\theta_{r,s}}\right)$. Since $\overline{x} = (\overline{x}_{k,l}) \in \overline{m}$, there is a constant $C > 0$ such that $d(\overline{x}_{k,l}, \overline{x}_o) \leq C$. Given $\varepsilon > 0$, we have

$$\frac{1}{h_{r,s}} \sum_{(k,l)\in I_{r,s}} [d(\overline{x}_{k,l}, \overline{x}_o)]^{p_k}$$

$$= \frac{1}{h_{r,s}} \sum_{d\left(\overline{x}_{k,l},\overline{x}_o\right)\geq\varepsilon (k,l)\in I_{r,s}} [d(\overline{x}_k, \overline{x}_o)]^{p_k} + \frac{1}{h_{r,s}} \sum_{d\left(\overline{x}_{k,l},\overline{x}_o\right)<\varepsilon (k,l)\in I_{r,s}} [d(\overline{x}_k, \overline{x}_o)]^{p_k}$$

$$\leq \frac{1}{h_{r,s}} \sum_{d\left(\overline{x}_{k,l},\overline{x}_o\right)\geq\varepsilon (k,l)\in I_{r,s}} \max\left(C^h, C^H\right) + \frac{1}{h_{r,s}} \sum_{d\left(\overline{x}_{k,l},\overline{x}_o\right)<\varepsilon (k,l)\in I_{r,s}} \varepsilon^{p_k}$$

$$\leq \max\left(C^h, C^H\right) \frac{1}{h_{r,s}} \left|\{(k,l) \in I_{r,s} :\ d(\overline{x}_{k,l}, \overline{x}_o) \geq \varepsilon\}\right| + \max\left(\varepsilon^h, \varepsilon^H\right).$$

Thus we obtain $\overline{x}_{k,l} \to \overline{x}_o \left(\overline{N}^p_{\theta_{r,s}}\right)$.

(iii) It follows from (i) and (ii).

Theorem 3.3. Let $\theta_{r,s} = \{(k_r, l_s)\}$ be a double lacunary sequence and $\overline{x} = (\overline{x}_{k,l})$ be a double sequence of interval numbers. Then

(i) For $\liminf_r q_r > 1$ and $\liminf_s \overline{q}_s > 1$ then $\overline{x}_{k,l} \to \overline{x}_o(\overline{s})$ implies $\overline{x}_{k,l} \to \overline{x}_o \left(\overline{s}_{\theta_{r,s}}\right)$,

(ii) For $\limsup_r q_r < \infty$ and $\limsup_s \overline{q}_s < \infty$ then $\overline{x}_{k,l} \to \overline{x}_o \left(\overline{s}_{\theta_{r,s}}\right)$ implies $\overline{x}_{k,l} \to \overline{x}_o(\overline{s})$,

(iii) If $1 < \liminf_{r,s} q_{r,s} \leq \limsup_{r,s} \overline{q}_{r,s} < \infty$, then $\overline{x}_{k,l} \to \overline{x}_o(\overline{s})$ if and only if $\overline{x}_{k,l} \to \overline{x}_o \left(\overline{s}_{\theta_{r,s}}\right)$.

Proof. **(i)** Suppose that $\liminf_r q_r > 1$ and $\liminf_s \overline{q}_s > 1$ then there exists a $\delta > 0$ such that $q_r \geq 1 + \delta$, $\overline{q}_s \geq 1 + \delta$ for sufficiently large r and s which implies

$$\frac{h_r}{k_r} \geq \frac{\delta}{1+\delta} \text{ and } \frac{\overline{h}_s}{l_s} \geq \frac{\delta}{1+\delta}.$$

Since $h_{r,s} = k_r l_s - k_r l_{s-1} - k_{r-1} l_s - k_{r-1} l_{s-1}$ we granted the following

$$\frac{k_r l_s}{h_{r,s}} \leq \frac{(1+\delta)^2}{\delta^2} \text{ and } \frac{k_{r-1} l_{s-1}}{h_{r,s}} \leq \frac{1}{\delta}$$

Now, let $\overline{x}_{k,l} \to \overline{x}_o(\overline{s})$. We are going to prove $\overline{x}_{k,l} \to \overline{x}_o\left(\overline{s}_{\theta_{r,s}}\right)$. Then for sufficiently large r and s, we have

$$\frac{1}{k_r l_s}\left|\left\{(k,l) \in I_{r,s}; k \leq k_r \text{ and } l \leq l_s : d\left(\overline{x}_{k,l}, \overline{x}_o\right) \geq \varepsilon\right\}\right|$$

$$\geq \frac{1}{k_r l_s}\left|\left\{(k,l) \in I_{r,s} : d\left(\overline{x}_{k,l}, \overline{x}_o\right) \geq \varepsilon\right\}\right|$$

$$\geq \frac{(1+\delta)^2}{\delta^2}.\frac{1}{h_{r,s}}\left|\left\{(k,l) \in I_{r,s} : d\left(\overline{x}_{k,l}, \overline{x}_o\right) \geq \varepsilon\right\}\right|.$$

Hence $\overline{x}_{k,l} \to \overline{x}_o\left(\overline{s}_{\theta_{r,s}}\right)$.

(ii) If $\limsup_r q_r < \infty$ and $\limsup_s \overline{q}_s < \infty$ then there exists $C > 0$ such that $q_r < C$ and $\overline{q}_s < C$ for all $r, s \geq 1$. Let $\overline{x}_{k,l} \to \overline{x}_o\left(\overline{s}_{\theta_{r,s}}\right)$ and $\varepsilon > 0$. Then there exist $r_o < 0$ and $s_o > 0$ such that for every $i \geq r_o$ and $j \geq s_o$

$$B_{i,j} = \frac{1}{h_{i,j}}\left|\left\{(k,l) \in I_{i,j} : d\left(\overline{x}_{k,l}, \overline{x}_o\right) \geq \varepsilon\right\}\right| < \varepsilon.$$

Let $M = \max\left\{B_{i,j} :\ 1 \leq i \leq r_o \text{ and } 1 \leq j \leq s_o\right\}$ and m and n be such that $k_{r-1} < m \leq k_r$ and $l_{s-1} < n \leq l_s$. Thus we obtain the following

$$\frac{1}{mn}\left|\left\{(k,l) \in I_{i,j}; k \leq m \text{ and } l \leq n : d\left(\overline{x}_{k,l}, \overline{x}_o\right) \geq \varepsilon\right\}\right|$$

$$\leq \frac{1}{k_{r-1} l_{s-1}}\left|\left\{(k,l) \in I_{i,j}; k \leq k_r \text{ and } l \leq l_s : d\left(\overline{x}_{k,l}, \overline{x}_o\right) \geq \varepsilon\right\}\right|$$

$$\leq \frac{1}{k_{r-1} l_{s-1}} \sum_{t,u=1,1}^{r_o,s_o} h_{t,u} B_{t,u} + \frac{1}{k_{r-1} l_{s-1}} \sum_{(r_o<t\leq r)\cup(s_o<u\leq s)} h_{t,u} B_{t,u}$$

$$\leq \frac{M}{k_{r-1} l_{s-1}} \sum_{t,u=1,1}^{r_o,s_o} h_{t,u} + \frac{1}{k_{r-1} l_{s-1}} \sum_{(r_o<t\leq r)\cup(s_o<u\leq s)} h_{t,u} B_{t,u}$$

$$\leq \frac{M k_{r_o} l_{s_o} r_o s_o}{k_{r-1} l_{s-1}} + \frac{1}{k_{r-1} l_{s-1}} \sum_{(r_o<t\leq r)\cup(s_o<u\leq s)} h_{t,u} B_{t,u}$$

$$\leq \frac{M k_{r_o} l_{s_o} r_o s_o}{k_{r-1} l_{s-1}} + \left(\sup_{t\geq r_o \cup u\geq s_o} B_{t,u}\right) \frac{1}{k_{r-1} l_{s-1}} \sum_{(r_o<t\leq r)\cup(s_o<u\leq s)} h_{t,u}$$

$$\leq \frac{Mk_{r_o}l_{s_o}r_os_o}{k_{r-1}l_{s-1}} + \frac{\varepsilon}{k_{r-1}l_{s-1}} \sum_{(r_o<t\leq r)\cup(s_o<u\leq s)} h_{t,u}$$
$$\leq \frac{Mk_{r_o}l_{s_o}r_os_o}{k_{r-1}l_{s-1}} + \varepsilon C^2.$$

Since k_r and l_s both approach infinity as both m and n approach infinity it follows that

$$\frac{1}{mn}\left|\{(k,l) \in I_{i,j}; k \leq m \text{ and } l \leq n : d\left(\overline{x}_{k,l}, \overline{x}_o\right) \geq \varepsilon\}\right| \to 0.$$

This completes the proof.

(iii) It follows from (i) and (ii).

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