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# New numerical radius inequalities for certain operaor matrices

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**Abstract**

*In this paper we prove some upper and lower bounds for the numerical radius of the off-diagonal part of* $3 \times 3$ *operator matrices and some bounds for the numerical radius inequalities of the general* $3 \times 3$ *operator matrix.*

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## 1. Introduction

Let $H$ be a complex Hilbert space with inner product $\langle .,. \rangle$ and let $B(H)$ be the space of all bounded linear operators on $H$. For $A \in B(H)$, let $\omega(A)$ and $\|A\|$ denote the numerical radius and the usual operator norm, respectively. Recall that

$$\omega(A) = \sup\{|\lambda| : \lambda \in W(A)\},$$

where $W(A)$ is the numerical range of $A$ which is a subset of the complex numbers, and

$$\|A\| = \sup\{\|Ax\| : \|x\| = 1\}.$$

It is well-known that $\omega(.)$ defines a norm on $B(H)$, which is equivalent to the usual operator norm $\|A\|$. In fact, for $A \in B(H)$,we have

$$\frac{1}{2}\|A\| \leq \omega(A) \leq \|A\|.$$

(1.1)

These inequalities are sharp. The first inequality becomes an equality if $A^2 = 0$, and the second inequality becomes an equality if $A$ is normal.

One of the important properties of $\omega(.)$ is that it is weakly unitarily invariant, that is, for $A \in B(H)$, we have

$$\omega(UAU^*) = \omega(A),$$

(1.2)

for every unitary $U \in B(H)$.

This improvement of the seconed inequality in (1.1) has been given in [6]. It says that for $A \in B(H)$, we have

$$\omega(A) \leq \frac{1}{2}\left(\|A\| + \left\|A^2\right\|^{\frac{1}{2}}\right),$$

(1.3)

consequently, if $A^2 = 0$, then

$$\omega(A) = \frac{1}{2}\|A\|. \tag{1.4}$$

The equality (1.4) follows from the inequality (1.3) and the first inequality in (1.1).

A fundamental inequality for the numerical radius is the power inequality, which says that for $A \in B(H)$, we have

$$\omega(A^n) \le (\omega(A))^n, \tag{1.5}$$

for $n = 1, 2, 3, ...$ (see, e.g., [4, p. 118]).

Recent numerical radius equalities and inequalities for operator matrices can be found in [1, 2], and [5].

In this paper, we give some new numerical radius inequalities for certain $3 \times 3$ operator matrices. In section 2, we establish upper and lower bounds for the numerical radii of the off-diagonal parts of $3 \times 3$ operator matrices. In section 3, we establish upper and lower bounds for the numerical radii of general $3 \times 3$ operator matrices.

## 2. Numerical radius inequalities for the operator matrix $\begin{bmatrix} 0 & 0 & A \\ 0 & B & 0 \\ C & 0 & 0 \end{bmatrix}$.

Our goal in this section is to give bounds for the numerical radius of the off-diagonal part $\begin{bmatrix} 0 & 0 & A_{13} \\ 0 & A_{22} & 0 \\ A_{31} & 0 & 0 \end{bmatrix}$ of a $3 \times 3$ operator matrix $\begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}$ defined on $H \oplus H \oplus H$. To achieve our goal, we need two basic lemmas. Part (a) of the first lemma is well-known, and it can be found in [3]. Part (b) is also known (see, e.g., [1]) and it follows by applying the identity (1.2) to the operator matrix $\begin{bmatrix} A & B & C \\ C & A & B \\ B & C & A \end{bmatrix}$ and the unitary operator $\frac{1}{\sqrt{3}}\begin{bmatrix} I & I & I \\ I & \alpha I & \alpha^2 I \\ I & \alpha^2 I & \alpha I \end{bmatrix}$ which is defined on $H \oplus H \oplus H$, where $1, \alpha, \alpha^2$ are the cubic roots of unity.

**Lemma 1.** *Let $A, B, C \in B(H)$. Then*

*(a)* $\omega\left(\begin{bmatrix} A & 0 & 0 \\ 0 & B & 0 \\ 0 & 0 & C \end{bmatrix}\right) = \max\left(\omega(A), \omega(B), \omega(C)\right).$

*(b)* $\omega\left(\begin{bmatrix} A & B & C \\ C & A & B \\ B & C & A \end{bmatrix}\right) =$

$\max\left(\omega(A+B+C), \omega\left(A+\alpha B+\alpha^2 C\right), \omega\left(A+\alpha^2 B+\alpha C\right)\right).$

**Lemma 2.** *Let $A, B, C \in B(H)$. Then*

$$(a) \quad \omega\left(\begin{bmatrix} 0 & 0 & A \\ 0 & B & 0 \\ C & 0 & 0 \end{bmatrix}\right)$$

$$= \omega\left(\begin{bmatrix} 0 & 0 & C \\ 0 & B & 0 \\ A & 0 & 0 \end{bmatrix}\right) = \omega\left(\begin{bmatrix} B & 0 & 0 \\ 0 & 0 & C \\ 0 & A & 0 \end{bmatrix}\right) = \omega\left(\begin{bmatrix} B & 0 & 0 \\ 0 & 0 & A \\ 0 & C & 0 \end{bmatrix}\right)$$

$$= \omega\left(\begin{bmatrix} 0 & A & 0 \\ C & 0 & 0 \\ 0 & 0 & B \end{bmatrix}\right) = \omega\left(\begin{bmatrix} 0 & C & 0 \\ A & 0 & 0 \\ 0 & 0 & B \end{bmatrix}\right)$$

$$= \left(\begin{bmatrix} 0 & 0 & \alpha^2 C \\ 0 & B & 0 \\ \alpha A & 0 & 0 \end{bmatrix}\right) = \omega\left(\begin{bmatrix} 0 & 0 & \alpha C \\ 0 & B & 0 \\ \alpha^2 A & 0 & 0 \end{bmatrix}\right)$$

$$(b) \quad \omega\left(\begin{bmatrix} 0 & 0 & A \\ 0 & A & 0 \\ A & 0 & 0 \end{bmatrix}\right) = \omega(A).$$

**Proof.** To prove part (a), let

$$U_1 = \begin{bmatrix} 0 & 0 & I \\ 0 & I & 0 \\ I & 0 & 0 \end{bmatrix}, U_2 = \begin{bmatrix} 0 & I & 0 \\ 0 & 0 & I \\ I & 0 & 0 \end{bmatrix}, U_3 = \begin{bmatrix} 0 & I & 0 \\ I & 0 & 0 \\ 0 & 0 & I \end{bmatrix}, U_4 = \begin{bmatrix} I & 0 & 0 \\ 0 & 0 & I \\ 0 & I & 0 \end{bmatrix},$$

$$U_5 = \begin{bmatrix} 0 & I & 0 \\ \alpha I & 0 & 0 \\ 0 & 0 & \alpha^2 I \end{bmatrix}, U_6 = \begin{bmatrix} 0 & 0 & I \\ 0 & \alpha I & 0 \\ \alpha^2 I & 0 & 0 \end{bmatrix}, \text{and } U_7 = \begin{bmatrix} 0 & 0 & \alpha I \\ 0 & I & 0 \\ \alpha^2 I & 0 & 0 \end{bmatrix}.$$

Then $U_1, U_2, U_3, U_4, U_5, U_6$, and $U_7$ are unitary operator matrices, where $I$ is the identity operator in $B(H)$.

Consider $X = \begin{bmatrix} 0 & 0 & A \\ 0 & B & 0 \\ C & 0 & 0 \end{bmatrix}$.

Now, it is easy to prove the following identities

$$U_1 X U_1^* = \begin{bmatrix} 0 & 0 & A \\ 0 & B & 0 \\ C & 0 & 0 \end{bmatrix}, U_2 X U_2^* = \begin{bmatrix} B & 0 & 0 \\ 0 & 0 & C \\ 0 & A & 0 \end{bmatrix},$$

$$\mathrm{U}_3 X U_3^* = \begin{bmatrix} B & 0 & 0 \\ 0 & 0 & A \\ 0 & C & 0 \end{bmatrix}, U_4 X U_4^* = \begin{bmatrix} 0 & A & 0 \\ C & 0 & 0 \\ 0 & 0 & B \end{bmatrix},$$

$$\mathrm{U}_2^* X U_2 = \begin{bmatrix} 0 & C & 0 \\ A & 0 & 0 \\ 0 & 0 & B \end{bmatrix}, U_7 X U_7^* = \begin{bmatrix} 0 & 0 & \alpha^2 C \\ 0 & B & 0 \\ \alpha A & 0 & 0 \end{bmatrix},$$

$$\mathrm{U}_6 X U_6^* = \begin{bmatrix} 0 & 0 & \alpha C \\ 0 & B & 0 \\ \alpha^2 A & 0 & 0 \end{bmatrix}, U_5 X U_5^* = \begin{bmatrix} B & 0 & 0 \\ 0 & 0 & \alpha^2 A \\ 0 & \alpha C & 0 \end{bmatrix}.$$

Hence, from the property (1.2), we obtain the required results.

Now, to prove part (b), take $U = \frac{1}{2} \begin{bmatrix} I & \sqrt{2}I & I \\ \sqrt{2}I & 0 & -\sqrt{2}I \\ I & -\sqrt{2}I & I \end{bmatrix}$. Then $U$ is unitary matrix. Thus,

$$U \begin{bmatrix} 0 & 0 & A \\ 0 & A & 0 \\ A & 0 & 0 \end{bmatrix} U^* = \begin{bmatrix} A & 0 & 0 \\ 0 & -A & 0 \\ 0 & 0 & A \end{bmatrix}.$$

Consequently,

$$\left(\begin{bmatrix} 0 & 0 & A \\ 0 & A & 0 \\ A & 0 & 0 \end{bmatrix}\right)$$

$$=$$

$$\omega\left(U\begin{bmatrix} 0 & 0 & A \\ 0 & A & 0 \\ A & 0 & 0 \end{bmatrix}U^{*}\right)$$

□

$$=$$

$$\omega\left(\begin{bmatrix} A & 0 & 0 \\ 0 & -A & 0 \\ 0 & 0 & A \end{bmatrix}\right)$$

$$=$$

$\omega(A)$ (by Lemma 1 (a)).
Our first result in this section can be stated as follows.

**Theorem 3.** Let $A, B, C \in B(H)$. Then

$$\sqrt[2n]{\max\left(\omega((AC)^{n}), \omega\left(B^{2n}\right), \omega((CA)^{n})\right)} \leq \omega\left(\begin{bmatrix} 0 & 0 & A \\ 0 & B & 0 \\ C & 0 & 0 \end{bmatrix}\right)$$

$\leq \frac{1}{2}\left(\|A\| + \|C\|\right) + \omega(B), for$n=1,2,3,.... (2.1)

**Proof.** To prove the first inequality in (2.1), let

$$X = \begin{bmatrix} 0 & 0 & A \\ 0 & B & 0 \\ C & 0 & 0 \end{bmatrix}. \text{ Then}$$

$$X^{2n} = \begin{bmatrix} (AC)^{n} & 0 & A \\ 0 & B^{2n} & 0 \\ C & 0 & (CA)^{n} \end{bmatrix},$$

for $n = 1, 2, 3, ...$, and so

$$\max\left(\omega((AC)^n), \omega\left(B^{2n}\right), \omega((CA)^n)\right)$$

$$= \omega\left(X^{2n}\right) \quad \text{(by Lemma 1 (a))}$$

$$\leq \omega^{2n}(X) \quad \text{(by the inequality (1.5))} \quad \square$$

$$= \omega^{2n}\left(\begin{bmatrix} 0 & 0 & A \\ 0 & B & 0 \\ C & 0 & 0 \end{bmatrix}\right).$$

Thus,

$$\sqrt[2n]{\max\left(\omega((AC)^n), \omega\left(B^{2n}\right), \omega((CA)^n)\right)} \leq \omega\left(\begin{bmatrix} 0 & 0 & A \\ 0 & B & 0 \\ C & 0 & 0 \end{bmatrix}\right),$$

for $n = 1, 2, 3, ....$ This completes the proof of the first inequality in (2.1).

Now, since $\begin{bmatrix} 0 & 0 & A \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}^2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ C & 0 & 0 \end{bmatrix}^2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, it follows by the identity (1.4) that

$$\begin{aligned}
\omega\left(\begin{bmatrix} 0 & 0 & A \\ 0 & B & 0 \\ C & 0 & 0 \end{bmatrix}\right) &\leq \omega\left(\begin{bmatrix} 0 & 0 & A \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\right) + \omega\left(\begin{bmatrix} 0 & 0 & 0 \\ 0 & B & 0 \\ 0 & 0 & 0 \end{bmatrix}\right) \\
&\quad + \omega\left(\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ C & 0 & 0 \end{bmatrix}\right) \\
&= \frac{1}{2}\left\|\begin{bmatrix} 0 & 0 & A \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\right\| + \omega(B) + \frac{1}{2}\left\|\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ C & 0 & 0 \end{bmatrix}\right\| \\
&= \frac{1}{2}\left(\|A\| + \|C\|\right) + \omega(B).
\end{aligned}$$

This proves the second inequality in (2.1).

Now, we give some inequalities that involve $\omega\left(\begin{bmatrix} 0 & 0 & A \\ 0 & B & 0 \\ C & 0 & 0 \end{bmatrix}\right)$.

**Theorem 4.** *Let $A, B, C \in B(H)$. Then*

$$\omega\left(\begin{bmatrix} 0 & 0 & A \\ 0 & B & 0 \\ C & 0 & 0 \end{bmatrix}\right) \geq \frac{1}{3} max\left(\omega(A+B+C), \omega\left(\alpha A + B + \alpha^2 C\right),\right.$$
$$\left.\omega\left(\alpha^2 A + B + \alpha C\right)\right) \tag{2.2}$$

*and*

$$\omega\left(\begin{bmatrix} 0 & 0 & A \\ 0 & B & 0 \\ C & 0 & 0 \end{bmatrix}\right) \leq \frac{1}{3}\left(\omega(A+B+C) + \omega\left(\alpha A + B + \alpha^2 C\right) + \right.$$
$$\left.\omega\left(\alpha^2 A + B + \alpha C\right)\right) \tag{2.3}$$

**Proof.** First, we prove the inequality (2.2). We have

$$\begin{aligned} & \omega\left(\begin{bmatrix} A+B+C & 0 & 0 \\ 0 & \alpha A + B + \alpha^2 C & 0 \\ 0 & 0 & \alpha^2 A + B + \alpha C \end{bmatrix}\right) \\ = \; & \omega\left(\begin{bmatrix} B & A & C \\ C & B & A \\ A & C & B \end{bmatrix}\right) \quad \text{(by Lemma 1 (b))} \\ = \; & \omega\left(\begin{bmatrix} B & 0 & 0 \\ 0 & 0 & A \\ 0 & C & 0 \end{bmatrix} + \begin{bmatrix} 0 & A & 0 \\ C & 0 & 0 \\ 0 & 0 & B \end{bmatrix} + \begin{bmatrix} 0 & 0 & C \\ 0 & B & 0 \\ A & 0 & 0 \end{bmatrix}\right) \\ \leq \; & \omega\left(\begin{bmatrix} B & 0 & 0 \\ 0 & 0 & A \\ 0 & C & 0 \end{bmatrix}\right) + \omega\left(\begin{bmatrix} 0 & A & 0 \\ C & 0 & 0 \\ 0 & 0 & B \end{bmatrix}\right) + \omega\left(\begin{bmatrix} 0 & 0 & C \\ 0 & B & 0 \\ A & 0 & 0 \end{bmatrix}\right) \\ = \; & \omega\left(\begin{bmatrix} 0 & 0 & A \\ 0 & B & 0 \\ C & 0 & 0 \end{bmatrix}\right) + \omega\left(\begin{bmatrix} 0 & 0 & A \\ 0 & B & 0 \\ C & 0 & 0 \end{bmatrix}\right) + \omega\left(\begin{bmatrix} 0 & 0 & A \\ 0 & B & 0 \\ C & 0 & 0 \end{bmatrix}\right) \end{aligned}$$

(by Lemma 2 (a))

$$=3\omega\left(\begin{bmatrix} 0 & 0 & A \\ 0 & B & 0 \\ C & 0 & 0 \end{bmatrix}\right),$$

and so

$$\tfrac{1}{3}\max(\omega\left(A+B+C\right),\omega\left(\alpha A+B+\alpha^2 C\right),\omega\left(\alpha^2 A+B+\alpha C\right))$$

$$\leq\omega\left(\begin{bmatrix} 0 & 0 & A \\ 0 & B & 0 \\ C & 0 & 0 \end{bmatrix}\right).$$

This completes the proof of the inequality (2.2).

Now, to prove the inequality (2.3), let $U=\frac{1}{\sqrt{3}}\begin{bmatrix} I & I & I \\ I & \alpha I & \alpha^2 I \\ I & \alpha^2 I & \alpha I \end{bmatrix}$ and $X=\begin{bmatrix} 0 & 0 & A \\ 0 & B & 0 \\ C & 0 & 0 \end{bmatrix}$. Then $U$ is unitary.

Consequently,

$$\begin{aligned}
\omega\left(X\right) &= \omega\left(UXU^{*}\right) \quad \text{(by the identity (1.2))} \\
&= \frac{1}{3}\omega\left(\begin{bmatrix} A+B+C & \alpha A+\alpha^2 B+C & \alpha^2 A+\alpha B+C \\ A+\alpha B+\alpha^2 C & \alpha A+B+\alpha^2 C & \alpha^2 A+\alpha^2 B+\alpha^2 C \\ A+\alpha^2 B+\alpha C & \alpha A+\alpha B+\alpha C & \alpha^2 A+B+\alpha C \end{bmatrix}\right) \\
&= \frac{1}{3}\omega\left(\begin{array}{l} \begin{bmatrix} 0 & 0 & \alpha^2 A+\alpha B+C \\ 0 & \alpha A+B+\alpha^2 C & 0 \\ A+\alpha^2 B+\alpha C & 0 & 0 \end{bmatrix} \\ +\begin{bmatrix} 0 & \alpha A+\alpha^2 B+C & 0 \\ A+\alpha B+\alpha^2 C & 0 & 0 \\ 0 & 0 & \alpha^2 A+B+\alpha C \end{bmatrix} \\ +\begin{bmatrix} A+B+C & 0 & 0 \\ 0 & 0 & \alpha^2 A+\alpha^2 B+\alpha^2 C \\ 0 & \alpha A+\alpha B+\alpha C & 0 \end{bmatrix} \end{array}\right)
\end{aligned}$$

$$\leq \frac{1}{3}\left(\begin{array}{c} \omega\left(\begin{bmatrix} 0 & 0 & \alpha^2 A+\alpha B+C \\ 0 & \alpha A+B+\alpha^2 C & 0 \\ A+\alpha^2 B+\alpha C & 0 & 0 \end{bmatrix}\right) \\ +\omega\left(\begin{bmatrix} 0 & \alpha A+\alpha^2 B+C & 0 \\ A+\alpha B+\alpha^2 C & 0 & 0 \\ 0 & 0 & \alpha^2 A+B+\alpha C \end{bmatrix}\right) \\ +\omega\left(\begin{bmatrix} A+B+C & 0 & 0 \\ 0 & 0 & \alpha^2 A+\alpha^2 B+\alpha^2 C \\ 0 & \alpha A+\alpha B+\alpha C & 0 \end{bmatrix}\right) \end{array}\right)$$

$$= \frac{1}{3}\left(\begin{array}{c} \omega\left(\begin{bmatrix} 0 & 0 & \alpha\left(\alpha A+B+\alpha^2 C\right) \\ 0 & \alpha A+B+\alpha^2 C & 0 \\ \alpha^2\left(\alpha A+B+\alpha^2 C\right) & 0 & 0 \end{bmatrix}\right) \\ +\omega\left(\begin{bmatrix} 0 & 0 & \alpha^2\left(\alpha^2 A+B+\alpha C\right) \\ 0 & \alpha^2 A+B+\alpha C & 0 \\ \alpha\left(\alpha^2 A+B+\alpha C\right) & 0 & 0 \end{bmatrix}\right) \\ +\omega\left(\begin{bmatrix} 0 & 0 & \alpha^2\left(A+B+C\right) \\ 0 & A+B+C & 0 \\ \alpha\left(A+B+C\right) & 0 & 0 \end{bmatrix}\right) \end{array}\right)$$

$$= \frac{1}{3}\left(\begin{array}{c} \omega\left(\begin{bmatrix} 0 & 0 & \alpha A+B+\alpha^2 C \\ 0 & \alpha A+B+\alpha^2 C & 0 \\ \alpha A+B+\alpha^2 C & 0 & 0 \end{bmatrix}\right) \\ +\omega\left(\begin{bmatrix} 0 & 0 & \alpha^2 A+B+\alpha C \\ 0 & \alpha^2 A+B+\alpha C & 0 \\ \alpha^2 A+B+\alpha C & 0 & 0 \end{bmatrix}\right) \\ +\omega\left(\begin{bmatrix} 0 & 0 & A+B+C \\ 0 & A+B+C & 0 \\ A+B+C & 0 & 0 \end{bmatrix}\right) \end{array}\right)$$

(by Lemma 2 (a))

$$= \frac{1}{3}\left(\omega\left(\alpha A+B+\alpha^2 C\right)+\omega\left(\alpha^2 A+B+\alpha C\right)+\omega\left(A+B+C\right)\right)$$

(by Lemma 2 (b)).

□

**Remark 5.** *If $A = B = C$, then the inequalities in (2.2) and (2.3) becomes equalities.*

In the following two results we give further upper and lower bounds for the numerical radius of $\begin{bmatrix} 0 & 0 & A \\ 0 & B & 0 \\ C & 0 & 0 \end{bmatrix}$. In these results, we use the observation that for $X \in B(H)$, we have $\begin{bmatrix} X & X & X \\ \alpha^2 X & \alpha^2 X & \alpha^2 X \\ \alpha X & \alpha X & \alpha X \end{bmatrix}^2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$.

So by the identity (1.4) we have

$$\omega\left(\begin{bmatrix} X & X \\ X & \\ \alpha^2 X & \alpha^2 X \\ \alpha^2 X & \\ \alpha X & \alpha X \\ \alpha X & \end{bmatrix}\right)$$

$$= \frac{1}{2}\left\|\begin{bmatrix} X & X \\ X & \\ \alpha^2 X & \alpha^2 X \\ \alpha^2 X & \\ \alpha X & \alpha X \\ \alpha X & \end{bmatrix}\right\|$$

$$= \frac{1}{2}\left\|\frac{1}{3}\begin{bmatrix} I & I \\ I & \\ I & \alpha^2 I \\ \alpha I & \\ I & \alpha I \\ \alpha^2 I & \end{bmatrix}\begin{bmatrix} X & X \\ X & \\ \alpha^2 X & \alpha^2 X \\ \alpha^2 X & \\ \alpha X & \alpha X \\ \alpha X & \end{bmatrix}\begin{bmatrix} I & I \\ I & \\ I & \alpha I \\ \alpha^2 I & \\ I & \alpha^2 I \\ \alpha I & \end{bmatrix}\right\|$$

$$= \frac{1}{6}\left\| \begin{bmatrix} 0 & 0 \\ 0 & \\ 0 & 0 \\ 0 & \\ 9X & 0 \\ 0 & \end{bmatrix} \right\| = \frac{3}{2}\|X\|$$

**Theorem 6.** *Let $A, B, C \in B(H)$. Then*

$$\omega(X) \leq$$

$$\left( \begin{array}{c} \left(\frac{1}{2}\right) min\left(\|A+B+C\|, \|\alpha^2 A + B + \alpha C\|, \|\alpha A + B + \alpha^2 C\|\right) \\ + \left(\frac{1}{3}\right) min \left( \begin{array}{c} \omega\left(\left(1+2\alpha^2\right)A + \left(2+\alpha^2\right)B\right) + \omega\left(\left(2+\alpha^2\right)A + \left(1+2\alpha^2\right)B\right), \\ \omega\left(\left(1+2\alpha^2\right)C + \left(2+\alpha^2\right)B\right) + \omega\left(\left(2+\alpha^2\right)C + \left(1+2\alpha^2\right)B\right), \\ \omega\left(\left(2+\alpha\right)C + \left(2+\alpha^2\right)B\right) + \omega\left(\left(1+2\alpha\right)C + \left(1+2\alpha^2\right)B\right), \\ \omega\left(\left(2\alpha+\alpha^2\right)C + \left(2+\alpha^2\right)B\right) + \omega\left(\left(\alpha+2\alpha^2\right)C + \left(1+2\alpha^2\right)B\right) \end{array} \right) \end{array} \right),$$

*where* $X = \begin{bmatrix} 0 & 0 & A \\ 0 & B & 0 \\ C & 0 & 0 \end{bmatrix}$.

**Proof.** Let $U = \frac{1}{\sqrt{3}} \begin{bmatrix} I & I & I \\ I & \alpha I & \alpha^2 I \\ I & \alpha^2 I & \alpha I \end{bmatrix}$ and $X = \begin{bmatrix} 0 & 0 & A \\ 0 & B & 0 \\ C & 0 & 0 \end{bmatrix}$.

Then $U$ is unitary. It follows that

$$\omega(X) = \omega(UXU^*) \quad \text{(by the identity (1.2))}$$

$$= \frac{1}{3}\omega\left( \begin{bmatrix} A+B+C & \alpha A + \alpha^2 B + C & \alpha^2 A + \alpha B + C \\ A + \alpha B + \alpha^2 C & \alpha A + B + \alpha^2 C & \alpha^2 A + \alpha^2 B + \alpha^2 C \\ A + \alpha^2 B + \alpha C & \alpha A + \alpha B + \alpha C & \alpha^2 A + B + \alpha C \end{bmatrix} \right)$$

0

$$= \frac{1}{3}\omega \left( \begin{array}{c} \begin{bmatrix} A+B+C & A+B+C & A+B+C \\ \alpha^2 (A+B+C) & \alpha^2 (A+B+C) & \alpha^2 (A+B+C) \\ \alpha (A+B+C) & \alpha (A+B+C) & \alpha (A+B+C) \end{bmatrix} + \\ \begin{bmatrix} 0 & (\alpha-1) A + (\alpha^2-1) B & 0 \\ (1-\alpha^2) A + (\alpha-\alpha^2) B & 0 & 0 \\ 0 & 0 & (\alpha^2-\alpha) A + (1-\alpha) B \end{bmatrix} \\ + \begin{bmatrix} 0 & 0 & (\alpha^2-1) A + (\alpha-1) B \\ 0 & (\alpha-\alpha^2) A + (1-\alpha^2) B & 0 \\ (1-\alpha) A + (\alpha^2-\alpha) B & 0 & 0 \end{bmatrix} \end{array} \right)$$

$$\leq \frac{1}{3} \left( \begin{array}{c} \omega\left( \begin{bmatrix} A+B+C & A+B+C & A+B+C \\ \alpha^2 (A+B+C) & \alpha^2 (A+B+C) & \alpha^2 (A+B+C) \\ \alpha (A+B+C) & \alpha (A+B+C) & \alpha (A+B+C) \end{bmatrix} \right) + \\ \omega\left( \begin{bmatrix} 0 & (\alpha-1) A + (\alpha^2-1) B & 0 \\ (1-\alpha^2) A + (\alpha-\alpha^2) B & 0 & 0 \\ 0 & 0 & (\alpha^2-\alpha) A + (1-\alpha) B \end{bmatrix} \right) \\ +\omega\left( \begin{bmatrix} 0 & 0 & (\alpha^2-1) A + (\alpha-1) B \\ 0 & (\alpha-\alpha^2) A + (1-\alpha^2) B & 0 \\ (1-\alpha) A + (\alpha^2-\alpha) B & 0 & 0 \end{bmatrix} \right) \end{array} \right)$$

$$= \frac{1}{3} \left( \begin{array}{c} \frac{3}{2} \|A+B+C\| \\ +\omega\left( (1+2\alpha^2) A + (2+\alpha^2) B \right) + \omega\left( (2+\alpha^2) A + (1+2\alpha^2) B \right) \end{array} \right)$$

(2.6)

(by the identity (2.4) and Lemma 2 (a) and (b)).

In a similar way, we can prove the following

$$\omega(X) = \omega\left(\begin{bmatrix} 0 & 0 & C \\ 0 & B & 0 \\ A & 0 & 0 \end{bmatrix}\right)$$
$$\leq \frac{1}{3}\begin{pmatrix} \frac{3}{2}\|A+B+C\| + \omega\left(\left(1+2\alpha^2\right)C + \left(2+\alpha^2\right)B\right) \\ +\omega\left(\left(2+\alpha^2\right)C + \left(1+2\alpha^2\right)B\right) \end{pmatrix}$$

(2.7)

$$\omega(X) = \omega\left(\begin{bmatrix} 0 & 0 & \alpha C \\ 0 & B & 0 \\ \alpha^2 A & 0 & 0 \end{bmatrix}\right)$$
$$\leq \frac{1}{3}\begin{pmatrix} \frac{3}{2}\|\alpha^2 A+B+\alpha C\| + \omega\left((2+\alpha)C + \left(2+\alpha^2\right)B\right) \\ +\omega\left((1+2\alpha)C + \left(1+2\alpha^2\right)B\right) \end{pmatrix}$$

(2.8)

$$\omega(X) = \omega\left(\begin{bmatrix} 0 & 0 & \alpha^2 C \\ 0 & B & 0 \\ \alpha A & 0 & 0 \end{bmatrix}\right)$$
$$\leq \frac{1}{3}\begin{pmatrix} \frac{3}{2}\|\alpha A+B+\alpha^2 C\| + \omega\left(\left(2\alpha+\alpha^2\right)C + \left(2+\alpha^2\right)B\right) \\ +\omega\left(\left(\alpha+2\alpha^2\right)C + \left(1+2\alpha^2\right)B\right) \end{pmatrix}$$

(2.9)

Now, the result follows from the inequalities (2.6), (2.7), (2,8), and (2.9). Thus,

$$\omega(X)$$
$$\leq \begin{pmatrix} \left(\frac{1}{2}\right)\min\left(\|A+B+C\|, \|\alpha^2 A+B+\alpha C\|, \|\alpha A+B+\alpha^2 C\|\right) \\ +\left(\frac{1}{3}\right)\min\begin{pmatrix} \omega\left(\left(1+2\alpha^2\right)A + \left(2+\alpha^2\right)B\right) + \omega\left(\left(2+\alpha^2\right)A + \left(1+2\alpha^2\right)B\right), \\ \omega\left(\left(1+2\alpha^2\right)C + \left(2+\alpha^2\right)B\right) + \omega\left(\left(2+\alpha^2\right)C + \left(1+2\alpha^2\right)B\right), \\ \omega\left((2+\alpha)C + \left(2+\alpha^2\right)B\right) + \omega\left((1+2\alpha)C + \left(1+2\alpha^2\right)B\right), \\ \omega\left(\left(2\alpha+\alpha^2\right)C + \left(2+\alpha^2\right)B\right) + \omega\left(\left(\alpha+2\alpha^2\right)C + \left(1+2\alpha^2\right)B\right) \end{pmatrix} \end{pmatrix} \square$$

**Remark 7.** *If $A = B = C$, then the inequalities in Theorem 6 becomes equalities.*

## 3. Upper and lower bounds for the numerical radius of the general $3 \times 3$ operator matrix.

We start our results by the following lemma which satisfies certain pinching inequalities (see, e.g., [3]).

**Lemma 1.** *Let $A_{ij} \in B(H)$, for all $i, j = 1, 2, 3$. Then*

$$(a)\ \omega\left(\begin{bmatrix} A_{11} & 0 & 0 \\ 0 & A_{22} & 0 \\ 0 & 0 & A_{33} \end{bmatrix}\right) \leq \omega\left(\begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}\right),$$

$$(b)\ \omega\left(\begin{bmatrix} 0 & 0 & A_{13} \\ 0 & 0 & 0 \\ A_{31} & 0 & 0 \end{bmatrix}\right) \leq \omega\left(\begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}\right),$$

$$(c)\ \omega\left(\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & A_{23} \\ 0 & A_{32} & 0 \end{bmatrix}\right) \leq \omega\left(\begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}\right),$$

$$(d)\ \omega\left(\begin{bmatrix} 0 & A_{12} & 0 \\ A_{21} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\right) \leq \omega\left(\begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}\right).$$

**Proof.** Let

$$U_1 = \begin{bmatrix} I & 0 & 0 \\ 0 & -I & 0 \\ 0 & 0 & I \end{bmatrix}, U_2 = \begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & -I \end{bmatrix}, U_3 = \begin{bmatrix} -I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix},$$

$$U_4 = \begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix}, \text{ and } X = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}.$$

Then, to prove part (c) for example, it is easy to prove that

$$U_1 X U_1^* + U_2 X U_2^* - U_3 X U_3^* - U_4 X U_4^* = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -4A_{23} \\ 0 & -4A_{32} & 0 \end{bmatrix},$$

and from the fact that the numerical radius is a norm, which is weakly unitarily invariant, we have

$$\omega\left(\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & A_{23} \\ 0 & A_{32} & 0 \end{bmatrix}\right) \leq \omega\left(\begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}\right).$$

□

Based on the Lemmas 1 and 8, we have our first result in this section.

**Theorem 2.** *Let $A, B, C \in B(H)$. Then*

$$\max\left(\omega(A), \omega(B), \omega(C)\right) \leq \omega\left(\begin{bmatrix} A & \alpha B & \alpha^2 C \\ B & \alpha C & \alpha^2 A \\ C & \alpha A & \alpha^2 B \end{bmatrix}\right) \leq \omega(A)+\omega(B)+\omega(C).$$

**Proof.** For the second inequality, we have

$$\begin{aligned} &\omega\left(\begin{bmatrix} A & \alpha B & \alpha^2 C \\ B & \alpha C & \alpha^2 A \\ C & \alpha A & \alpha^2 B \end{bmatrix}\right) \\ &= \omega\left(\begin{matrix} \begin{bmatrix} A & 0 & 0 \\ 0 & 0 & \alpha^2 A \\ 0 & \alpha A & 0 \end{bmatrix} + \begin{bmatrix} 0 & \alpha B & 0 \\ B & 0 & 0 \\ 0 & 0 & \alpha^2 B \end{bmatrix} \\ + \begin{bmatrix} 0 & 0 & \alpha^2 C \\ 0 & \alpha C & 0 \\ C & 0 & 0 \end{bmatrix} \end{matrix}\right) \\ &\leq \left(\begin{matrix} \omega\left(\begin{bmatrix} A & 0 & 0 \\ 0 & 0 & \alpha^2 A \\ 0 & \alpha A & 0 \end{bmatrix}\right) + \omega\left(\begin{bmatrix} 0 & \alpha B & 0 \\ B & 0 & 0 \\ 0 & 0 & \alpha^2 B \end{bmatrix}\right) \\ +\omega\left(\begin{bmatrix} 0 & 0 & \alpha^2 C \\ 0 & \alpha C & 0 \\ C & 0 & 0 \end{bmatrix}\right) \end{matrix}\right) \\ &= \omega(A) + \omega(B) + \omega(C). \quad \text{(by Lemma 2 (a) and (b))} \end{aligned}$$

The first inequality follows from Lemma 1 (a), Theorem 4, and Lemma 8.

$$\omega\left(\begin{bmatrix} A & \alpha B & \alpha^2 C \\ B & \alpha C & \alpha^2 A \\ C & \alpha A & \alpha^2 B \end{bmatrix}\right)$$

$$\geq \max\begin{pmatrix} \omega\left(\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & \alpha^2 A \\ 0 & \alpha A & 0 \end{bmatrix}\right), \omega\left(\begin{bmatrix} 0 & \alpha B & 0 \\ B & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\right), \\ \omega\left(\begin{bmatrix} 0 & 0 & \alpha^2 C \\ 0 & 0 & 0 \\ C & 0 & 0 \end{bmatrix}\right), \omega\left(\begin{bmatrix} A & 0 & 0 \\ 0 & \alpha C & 0 \\ 0 & 0 & \alpha^2 B \end{bmatrix}\right) \end{pmatrix}$$

$$= \max\begin{pmatrix} \omega\left(\begin{bmatrix} 0 & 0 & \alpha^2 A \\ 0 & 0 & 0 \\ \alpha A & 0 & 0 \end{bmatrix}\right), \omega\left(\begin{bmatrix} 0 & 0 & \alpha B \\ 0 & 0 & 0 \\ B & 0 & 0 \end{bmatrix}\right), \\ \omega\left(\begin{bmatrix} 0 & 0 & \alpha^2 C \\ 0 & 0 & 0 \\ C & 0 & 0 \end{bmatrix}\right), \omega(A), \omega(B), \omega(C) \end{pmatrix}$$

$$\geq \max\left(\tfrac{2}{3}\omega(A), \tfrac{2}{3}\omega(B), \tfrac{2}{3}\omega(C), (\omega(A), \omega(B), \omega(C))\right)$$

$$= \max(\omega(A), \omega(B), \omega(C)) \quad \square$$

At the end of this section, we present a general numerical radius inequalities for $3\times 3$ operator matrices. These new inequalities are based on the pinching inequalities given in Lemma 8, the triangle inequality for $\omega(.)$, Lemma 1 (a) and Lemma 2 (a), concerning the numerical radii of the diagonal parts of $3\times 3$ operator matrices, and our estimates of the numerical radii of the off-diagonal parts of these operator matrices given in Theorem 4.

**Theorem 3.** *Let $A_{ij} \in B(H)$, for all $i, j = 1, 2, 3$. Then*

$$\omega([A_{ij}])$$

$$\leq \frac{1}{3}\begin{bmatrix} \omega(A_{11}+A_{23}+A_{32}) + \omega(\alpha^2 A_{23}+A_{11}+\alpha A_{32}) + \omega(\alpha A_{23}+A_{11}+\alpha^2 A_{32}) \\ +\omega(A_{12}+A_{33}+A_{21}) + \omega(\alpha^2 A_{12}+A_{33}+\alpha A_{21}) + \omega(\alpha A_{12}+A_{33}+\alpha^2 A_{21}) \\ +\omega(A_{13}+A_{22}+A_{31}) + \omega(\alpha^2 A_{13}+A_{22}+\alpha A_{31}) + \omega(\alpha A_{13}+A_{22}+\alpha^2 A_{31}) \end{bmatrix}$$

(3.1)

*and*

$$\omega\left([A_{ij}]\right)$$
$$\geq \frac{1}{3} max \left[ \begin{array}{c} \omega\left(A_{23}+A_{32}\right), \omega\left(\alpha^2 A_{23}+\alpha A_{32}\right), \omega\left(\alpha A_{23}+\alpha^2 A_{32}\right), 3\omega\left(A_{11}\right), \\ \omega\left(A_{12}+A_{21}\right), \omega\left(\alpha^2 A_{12}+\alpha A_{21}\right), \omega\left(\alpha A_{12}+\alpha^2 A_{21}\right), 3\omega\left(A_{22}\right), \\ \omega\left(A_{13}+A_{31}\right), \omega\left(\alpha^2 A_{13}+\alpha A_{31}\right), \omega\left(\alpha A_{13}+\alpha^2 A_{31}\right), 3\omega\left(A_{33}\right) \end{array} \right].$$
(3.2)

**Proof.** To prove the inequality (3.3), note that Lemma 2 (a) and Theorem 4 imply that

$$\omega\left([A_{ij}]\right)$$

$$\leq \left( \begin{array}{c} \omega\left(\left[\begin{array}{ccc} A_{11} & 0 & 0 \\ 0 & 0 & A_{23} \\ 0 & A_{32} & 0 \end{array}\right]\right) + \omega\left(\left[\begin{array}{ccc} 0 & A_{12} & 0 \\ A_{21} & 0 & 0 \\ 0 & 0 & A_{33} \end{array}\right]\right) \\ +\omega\left(\left[\begin{array}{ccc} 0 & 0 & A_{13} \\ 0 & A_{22} & 0 \\ A_{31} & 0 & 0 \end{array}\right]\right) \end{array} \right)$$
$$= \left( \begin{array}{c} \omega\left(\left[\begin{array}{ccc} 0 & 0 & A_{23} \\ 0 & A_{11} & 0 \\ A_{32} & 0 & 0 \end{array}\right]\right) + \omega\left(\left[\begin{array}{ccc} 0 & 0 & A_{12} \\ 0 & A_{33} & 0 \\ A_{21} & 0 & 0 \end{array}\right]\right) \\ +\omega\left(\left[\begin{array}{ccc} 0 & 0 & A_{13} \\ 0 & A_{22} & 0 \\ A_{31} & 0 & 0 \end{array}\right]\right) \end{array} \right)$$
$$\leq \frac{1}{3} \left[ \begin{array}{c} \omega\left(A_{11}+A_{23}+A_{32}\right)+\omega\left(\alpha^2 A_{23}+A_{11}+\alpha A_{32}\right)+\omega\left(\alpha A_{23}+A_{11}+\alpha^2 A_{32}\right) \\ +\omega\left(A_{12}+A_{33}+A_{21}\right)+\omega\left(\alpha^2 A_{12}+A_{33}+\alpha A_{21}\right)+\omega\left(\alpha A_{12}+A_{33}+\alpha^2 A_{21}\right) \\ +\omega\left(A_{13}+A_{22}+A_{31}\right)+\omega\left(\alpha^2 A_{13}+A_{22}+\alpha A_{31}\right)+\omega\left(\alpha A_{13}+A_{22}+\alpha^2 A_{31}\right) \end{array} \right]$$

Now, it follows from Lemma 8 that

$$\omega\left([A_{ij}]\right) \geq \max \left( \begin{array}{c} \omega\left(\left[\begin{array}{ccc} 0 & 0 & 0 \\ 0 & 0 & A_{23} \\ 0 & A_{32} & 0 \end{array}\right]\right), \omega\left(\left[\begin{array}{ccc} 0 & A_{12} & 0 \\ A_{21} & 0 & 0 \\ 0 & 0 & 0 \end{array}\right]\right), \\ \omega\left(\left[\begin{array}{ccc} 0 & 0 & A_{13} \\ 0 & 0 & 0 \\ A_{31} & 0 & 0 \end{array}\right]\right), \omega\left(\left[\begin{array}{ccc} A_{11} & 0 & 0 \\ 0 & A_{22} & 0 \\ 0 & 0 & A_{33} \end{array}\right]\right) \end{array} \right)$$

$$= \max \begin{pmatrix} \omega\left(\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & A_{23} \\ 0 & A_{32} & 0 \end{bmatrix}\right), \omega\left(\begin{bmatrix} 0 & A_{12} & 0 \\ A_{21} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\right), \\ \omega\left(\begin{bmatrix} 0 & 0 & A_{13} \\ 0 & 0 & 0 \\ A_{31} & 0 & 0 \end{bmatrix}\right), \omega(A_{11}), (A_{22}), \omega(A_{33}) \end{pmatrix}$$

(by Lemma 1 (a))

$$\geq \frac{1}{3}\max \begin{bmatrix} \omega(A_{23}+A_{32}), \omega(\alpha^2 A_{23}+\alpha A_{32}), \omega(\alpha A_{23}+\alpha^2 A_{32}), 3\omega(A_{11}), \\ \omega(A_{12}+A_{21}), \omega(\alpha^2 A_{12}+\alpha A_{21}), \omega(\alpha A_{12}+\alpha^2 A_{21}), 3\omega(A_{22}), \\ \omega(A_{13}+A_{31}), \omega(\alpha^2 A_{13}+\alpha A_{31}), \omega(\alpha A_{13}+\alpha^2 A_{31}), 3\omega(A_{33}) \end{bmatrix}.$$

(by Theorem 4)

□

This proves the inequality (3.4)

**Theorem 4.** *Let $A_{ij} \in B(H)$, for all $i, j = 1, 2, 3$. Then*

$$\omega([A_{ij}])$$

$$\leq \begin{pmatrix} \max(\omega(A_{11}), (A_{22}), \omega(A_{33})) + \\ \frac{1}{3}\begin{bmatrix} \omega(A_{23}+A_{32}) + \omega(\alpha^2 A_{23}+\alpha A_{32}) + \omega(\alpha A_{23}+\alpha^2 A_{32}) \\ +\omega(A_{12}+A_{21}) + \omega(\alpha^2 A_{12}+\alpha A_{21}) + \omega(\alpha A_{12}+\alpha^2 A_{21}) \\ +\omega(A_{13}+A_{31}) + \omega(\alpha^2 A_{13}+\alpha A_{31}) + \omega(\alpha A_{13}+\alpha^2 A_{31}) \end{bmatrix} \end{pmatrix}$$

(3.3)

**Proof.** To prove the inequality (3.5), note that

$$\omega([A_{ij}])$$

$$\leq \begin{pmatrix} \omega\left(\begin{bmatrix} A_{11} & 0 & 0 \\ 0 & A_{22} & 0 \\ 0 & 0 & A_{33} \end{bmatrix}\right) + \omega\left(\begin{bmatrix} 0 & A_{12} & 0 \\ A_{21} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\right) \\ +\omega\left(\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & A_{23} \\ 0 & A_{32} & 0 \end{bmatrix}\right) + \omega\left(\begin{bmatrix} 0 & 0 & A_{13} \\ 0 & 0 & 0 \\ A_{31} & 0 & 0 \end{bmatrix}\right) \end{pmatrix}$$

$$= \left( \begin{array}{c} \max\left(\omega\left(A_{11}\right),\left(A_{22}\right),\omega\left(A_{33}\right)\right)+\omega\left(\left[\begin{array}{ccc} 0 & 0 & A_{12} \\ 0 & 0 & 0 \\ A_{21} & 0 & 0 \end{array}\right]\right) \\ +\omega\left(\left[\begin{array}{ccc} 0 & 0 & A_{23} \\ 0 & 0 & 0 \\ A_{32} & 0 & 0 \end{array}\right]\right)+\omega\left(\left[\begin{array}{ccc} 0 & 0 & A_{13} \\ 0 & 0 & 0 \\ A_{31} & 0 & 0 \end{array}\right]\right) \end{array} \right)$$

$$\leq \left( \begin{array}{c} \max\left(\omega\left(A_{11}\right),\left(A_{22}\right),\omega\left(A_{33}\right)\right)+ \\ \frac{1}{3}\left[\begin{array}{c} \omega\left(A_{23}+A_{32}\right)+\omega\left(\alpha^2A_{23}+\alpha A_{32}\right)+\omega\left(\alpha A_{23}+\alpha^2A_{32}\right) \\ +\omega\left(A_{12}+A_{21}\right)+\omega\left(\alpha^2A_{12}+\alpha A_{21}\right)+\omega\left(\alpha A_{12}+\alpha^2A_{21}\right) \\ +\omega\left(A_{13}+A_{31}\right)+\omega\left(\alpha^2A_{13}+\alpha A_{31}\right)+\omega\left(\alpha A_{13}+\alpha^2A_{31}\right) \end{array}\right] \end{array} \right). \square$$

**Remark 5.** *If $A_{ij} = A$ for all $i, j = 1, 2, 3$, then the inequality in (3.3) becomes equality, but the inequality (3.5) does not.*

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