Proyecciones Journal of Mathematics
Vol. 31, N^o 3, pp. 197-207, September 2012.
Universidad Católica del Norte
Antofagasta - Chile
DOI: 10.4067/S0716-09172012000300001

A note on rescalings of the skew-normal distribution

OSVALDO VENEGAS
Universidad Católica de Temuco, Chile
DAVID ELAL-OLIVERO
Universidad de Atacama, Chile
and
HÉCTOR W. GÓMEZ
Universidad de Antofagasta, Chile
Received : December 2011. Accepted : March 2012

Abstract

In this article, we show that certain skew-normal parametric statistical models are a result of rescalings of the skew normal model studied by Azzalini (1985). Using this procedure we define a class of skew-normal distributions and we study its moment, skewness and kurtosis coefficients. At the end of this article we will use this class of distribution to make some extensions of the skew-normal model.

Key Words : *Asymmetry, Rescalings, Skew-Normal model.*

1. Introduction

Azzalini (1985) studied the $\{SN(\lambda), \lambda \in \mathbf{R}\}$ family of Skew-normal distributions, with asymmetry parameter λ, where $SN(0)$ is the standard normal distribution. This is, $X \sim SN(\lambda)$ and its density function is:

$$\phi_\lambda(x) = 2\phi(x)\Phi(\lambda x), \qquad x, \lambda \in \mathbf{R}, \tag{1.1}$$

where ϕ and Φ are the $N(0,1)$ probability density function and cumulative distribution function, respectively. Some properties for this distribution are:

$$E(X) = \sqrt{\frac{2}{\pi}} \frac{\lambda}{\sqrt{1+\lambda^2}} \quad ; \quad Var(X) = 1 - \frac{2\lambda^2}{\pi(1+\lambda^2)} \tag{1.2}$$

$$M_X(t) = 2exp(t^2/2)\Phi\left(\frac{\lambda t}{\sqrt{1+\lambda^2}}\right) \tag{1.3}$$

$$\sqrt{\beta_1} = \frac{1}{2}(4-\pi)\left(\frac{E^2(X)}{Var(X)}\right)^{3/2} \quad ; \quad \beta_2 = 2(\pi-3)\left(\frac{E^2(X)}{Var(X)}\right)^2, \tag{1.4}$$

where $\sqrt{\beta_1}$ and β_2 are the asymmetric and kurtosis coefficients, respectively.

From (4) we get

$$-0.9953 \leq \sqrt{\beta_1} \leq 0.9953 \quad ; \quad 3.0000 \leq \beta_2 \leq 3.8692. \tag{1.5}$$

The following stochastic representation of the skew-normal model was given by Henze (1986) and Azzalini (1986), i.e., $X \sim SN(\lambda)$ if

$$X = a|X_1| + bX_2, \tag{1.6}$$

where X_1, X_2 are iid $N(0,1)$ random variables, with $a = \frac{\lambda}{\sqrt{1+\lambda^2}}$ and $b = \frac{1}{\sqrt{1+\lambda^2}}$. Henze (1986) used this representation to compute the odds moments of the skew-normal density and gave an extension of the truncated normal distribution. If $X \sim SN(\lambda)$ then the skew-normal distribution with location and scale is the distribution of $Z = \mu + \sigma X$, $\mu \in \mathbf{R}$ and $\sigma > 0$, and it will be denoted by $Z \sim SN(\mu, \sigma, \lambda)$.

Other skew-normal models has been studied, for example by Mukhopadhyay and Vidakovic (1995), Sahu et al. (2003) and Nadarajah and Kotz (2003). These models are product of rescalings of the skew-normal model of Azzalini (1985).

The purpose in these notes, is to show how the rescalings affects the representation of the skew-normal model, as well as provide some extensions of this rescaled model.

This article is organized as follows. In section 2 we give the rescalings of the skew-normal model, some properties, examples, moments and asymmetric and kurtosis coefficients. In section 3 some extensions of this rescaled model are presented. In section 4 presents the appendix.

2. Rescaling

Definition 1. *We will say that a random variable X has a Rescaled-Skew-Normal distribution with parameters λ, r and s, with $\frac{r}{s} > 0$, that will be denoted by $X \sim RSN(\lambda, r, s)$, if its density function is given by*

$$f_X(x) = \frac{2}{r}\phi\left(\frac{x}{r}\right)\Phi\left(\frac{\lambda}{s}x\right), \tag{2.1}$$

where ϕ and Φ are the standard normal density and distribution functions, respectively.

Proposition 1. *Let $U \sim N(0, \frac{r}{s})$ and $V \sim N(0,1)$ where U and V are independent and let $X = \frac{\lambda\sqrt{rsr}}{\sqrt{s^2+r^2\lambda^2}}|U| + \frac{rs}{\sqrt{s^2+r^2\lambda^2}}V$ where $\frac{r}{s} > 0$ and $\lambda \in \mathbf{R}$, then the density function f_X of the random variable X is given in* (7).

Proof This proof is in the appendix. □

Remark 1.

a) *A distribution suitable for fitting positive data is the half-normal distribution. We say that a random variable X follows a half-normal distribution with scale parameter σ if its density function is given by:*

$$f(x;\sigma) = \frac{2}{\sigma}\phi\left(\frac{x}{\sigma}\right)I\{x > 0\}$$

with $\sigma > 0$. We denote this by writing $X \sim HN(\sigma)$.

b) *We observe that in the representation we have that $\frac{r}{s} > 0$, since it is the variance of the normal model. Thus, we may construct skew-normal models that are product of rescalings and it must only satisfy that $\frac{r}{s} > 0$. On the other hand, we can see that the distribution of $|U|$ is the family of rescaled half normal distribution.*

c) If $r = s = 1$, *then* $X = \frac{\lambda}{\sqrt{1+\lambda^2}}|U| + \frac{1}{\sqrt{1+\lambda^2}}V \iff X \sim SN(\lambda)$.

d) If $Z \sim SN\left(0, \frac{r}{s}, \frac{r\lambda}{s}\right)$, *then* $X = sZ \sim RSN(\lambda, r, s)$. *Therefore, the RSN model is product of a rescaling of the SN model.*

Properties 1. *The following properties follow immediately from Definition 1.*

a) The pdf of the $RSN(0, r, s)$ *is identical to the pdf of the* $N(0, r^2)$.

b) Consider $X \sim RSN(\lambda, r, s)$. *Then*

i) As $\lambda \to \infty$, $f_X(x; \lambda, r, s)$ *tends to* $HN(0, r^2)$.

ii) As $s \to \infty$, $f_X(x; \lambda, r, s)$ *tends to* $N(0, r^2)$.

c) If $X \sim RSN(\lambda, r, s)$, *then* $-X \sim RSN(-\lambda, r, s)$.

d) If $X \sim RSN(\lambda, r, s)$ *and* $Y \sim N(0, r^2)$ *then* $|X|$ *and* $|Y|$ *have the same pdf.*

e) If $X \sim RSN(\lambda, r, s)$, *then* $X^2 \sim Gamma(1/2, 2r^2)$.

2.1. Examples of the RSN models

Example 1. *If* $f_X(x) = \frac{2}{\sqrt{1+\alpha^2}}\phi\left(\frac{x}{\sqrt{1+\alpha^2}}\right)\Phi\left(\frac{\lambda}{\sqrt{1+\beta^2}}x\right)$ *with* $\alpha \in \mathbf{R}$ *and* $\beta \in \mathbf{R}$ *then* $X \sim RSN\left(\lambda, \sqrt{1+\alpha^2}, \sqrt{1+\beta^2}\right)$. *We note that* $\frac{r}{s} = \frac{\sqrt{1+\alpha^2}}{\sqrt{1+\beta^2}} > 0$ *and* $\left(\sqrt{1+\alpha^2}, \sqrt{1+\beta^2}\right) \in]1, \infty[^2 \subseteq]0, \infty[^2$.

Such model is studied by Elal et al. (2004). If consider $\alpha = \beta = \lambda$ then the model corresponds to the one presented by Sahu et al. (2003) in the univariate case and, on the other hand, if $\alpha = \beta = 0$ it corresponds to the $SN(\lambda)$ model by Azzalini (1985).

Example 2. *If* $f_X(x) = \frac{2}{\sqrt{2}}\phi\left(\frac{x}{\sqrt{2}}\right)\Phi\left(\frac{\lambda}{\sqrt{4+2\lambda^2}}x\right)$ *then* $X \sim RSN\left(\lambda, \sqrt{2}, \sqrt{4+2\lambda^2}\right)$. *Observe that in this case* $\frac{r}{s} = \frac{\sqrt{2}}{\sqrt{4+2\lambda^2}} > 0$ *and* $\left(\sqrt{2}, \sqrt{4+2\lambda^2}\right) \in \left\{\sqrt{2}\right\} \times]2, \infty[\subseteq]0, \infty[^2$.

This model is exhibited by Mukhopadhyay and Vidakovic (1995) in an Bayesian analysis.

Example 3. *If* $f_X(x) = \frac{2}{\sigma}\phi\left(\frac{x}{\sigma}\right)\Phi\left(\frac{\lambda}{\delta}x\right)$ *where* σ *and* δ *corresponds to the standard deviations of the normal models, then* $X \sim RSN(\lambda, \sigma, \delta)$ *since* $\frac{r}{s} = \frac{\sigma}{\delta} > 0$ *and* $(\sigma,\ \delta) \in \Theta =]0, \infty[^2$*. Nadarajah and Kotz(2003) introduced such model and they called Skew-Normal-Normal distribution.*

2.2. Moments and Moment Generating Function of the RSN Model

Proposition 2. *Let* $X \sim RSN(\lambda, r, s)$ *then the moments of order* n *are:*

$$\mu_n = E[X^n] = \frac{\left(r^2\lambda\sqrt{2}\right)^n}{\sqrt{\pi}\left(s^2 + r^2\lambda^2\right)^{\frac{n}{2}}} \sum_{k=0}^{n} \binom{n}{k} \left(\frac{s}{r\lambda\sqrt{2}}\right)^k \Gamma\left(\frac{n-k+1}{2}\right) b_k, \tag{2.2}$$

where $b_k = E(V^k) = \begin{cases} 0, & k \text{ odd} \\ 2^{k/2}\Gamma(\frac{k+1}{2})/\sqrt{\pi}, & k \text{ even.} \end{cases}$

Proof This proof is in the appendix. □

Proposition 3. *Let* $X \sim RSN(\lambda, r, s)$ *then the moment generating function is:*

$$E[\exp(tX)] = 2\exp\left(\frac{r^2t^2}{2}\right)\Phi\left(\frac{r^2\lambda t}{\sqrt{s^2 + r^2\lambda^2}}\right). \tag{2.3}$$

Proof This proof is in the appendix. □

2.3. Asymmetry and Kurtosis Coefficients of the RSN model

Proposition 4. *Let* $X \sim RSN(\lambda, r, s)$ *then the asymmetry and kurtosis coefficients are:*

$$\sqrt{\beta_1} = \frac{(4-\pi)\, r^3\lambda^3\sqrt{2}}{\left(\pi s^2 + \pi r^2\lambda^2 - 2r^2\lambda^2\right)^{\frac{3}{2}}},$$

$$\beta_2 = \frac{8(\pi - 3)\, r^4\lambda^4}{\left(\pi s^2 + \pi r^2\lambda^2 - 2\lambda^2 r^2\right)^2} + 3.$$

Proof Considering that

$$\sqrt{\beta_1} = \frac{\mu_3 - 3\mu_1\mu_2 + 2\mu_1^3}{\left(\mu_2 - \mu_1^2\right)^{\frac{3}{2}}} \quad \text{and} \quad \beta_2 = \frac{\mu_4 - 4\mu_1\mu_3 + 6\mu_1^2\mu_2 - 3\mu_1^4}{\left(\mu_2 - \mu_1^2\right)^2}$$

and from Proposition 3 we have

$$\mu_1 = \frac{r^2\lambda\sqrt{2}}{\sqrt{\pi}\sqrt{s^2+r^2\lambda^2}}, \quad \mu_2 = r^2, \quad \mu_3 = \frac{r^4\lambda\sqrt{2}\left(3s^2+2r^2\lambda^2\right)}{\sqrt{\pi}\left(s^2+r^2\lambda^2\right)^{\frac{3}{2}}}, \quad \mu_4 = 3r^4.$$

This completes the result. □

Remark 2. *Since the RSN model is a result of a rescaling of the SN model, the asymmetry and kurtosis ranges of the RSN model are equal to the SN model, in other words, the intervals coincide with the given in* (5)*. On the other hand, any rescaling of the random variable that has a skew-normal distribution, do no affect the result of the asymmetry and kurtosis, as well as, the expressions given in* (4)*. If in* (9) *we consider* $r = s = 1$*, we obtain* (3)*. The* b_k *in the Proposition 2 are the moments of the standard normal distribution, see Johnson et al. (1995).*

3. Extensions of the RSN model

In this section we show extensions of the skew-normal model using the rescaled skew-normal model.

Proposition 5. *Let* $X \sim RSN\left(\lambda, r, s\right)$*,* $Y \sim N\left(0, \sigma^2\right)$*, with* X *and* Y *independent, and* $H = \frac{r}{\sqrt{r^2+\sigma^2}}X + \frac{r}{\sqrt{r^2+\sigma^2}}Y$*. Then*

$$H \sim RSN\left(\lambda, r, \sqrt{s^2 + r^{-2}s^2\sigma^2 + \lambda^2\sigma^2}\right).$$

Proof This proof is in the appendix. □

Definition 2. *Let* $X \sim N(0, \sigma^2)$*, we say that* T *has a Truncated-Normal distribution with parameters* c*,* σ^2*, will be denoted by* $T \sim \Upsilon_{\sigma^2}(c)$*, if its density function,* $f_T(t)$*, is given by:* $f_T(t) = \frac{1}{1-F_X(c)}f_X(t)$*,* $c \in \mathbf{R}$*,* $c \leq x < \infty$*, with* F_X *and* f_X *distribution and density functions of the variable* X*, respectively.*

Proposition 6. *Let* $T \sim \Upsilon_{\frac{r}{s}}\left(-\frac{\sqrt{rs}\varepsilon}{\sqrt{s^2+r^2\lambda^2}}\right)$*,* $\varepsilon \in \mathbf{R}$*,* $\lambda \in \mathbf{R}$*,* $\frac{r}{s} > 0$*,* $(r, s) \in \Theta \subseteq]0, \infty[^2$*, and let* $V \sim N\left(0, 1\right)$ *with* T *and* V *independents. If*

$$X = \frac{\lambda\sqrt{rs}r}{\sqrt{s^2+r^2\lambda^2}}T + \frac{rs}{\sqrt{s^2+r^2\lambda^2}}V, \tag{3.1}$$

then X *have as density function to* $g(x)$*, where*

$$g(x) = \frac{1}{\Phi\left(\frac{s\varepsilon}{\sqrt{s^2+r^2\lambda^2}}\right)r}\phi\left(\frac{x}{r}\right)\Phi\left(\frac{\lambda x}{s}+\varepsilon\right). \tag{3.2}$$

Proof This proof is in the appendix. □
If X is considered as in Proposition 6, then we will say that X has a extended rescaled-skew-normal distribution with parameters $\lambda, r, s, \varepsilon$ and, will be denoted by $X \sim ERSN\left(\lambda, r, s, \varepsilon\right)$.

Definition 3. *Let $X \sim ERSN\left(\lambda, r, s, \varepsilon\right)$. The family of distributions with location and scale is defined as the distribution of $Z = \mu + \sigma X$, $\mu \in \mathbf{R}$ and $\sigma > 0$, where $\boldsymbol{\theta} = (\mu, \sigma, \lambda, r, s, \varepsilon)$ and will be denoted by $ZERSN\left(\boldsymbol{\theta}\right)$.*

Proposition 7. *Let $X \mid W = w \sim N\left(w, 1\right)$ and $WERSN\left(0, 1, \lambda, r, s, 0\right)$. Then $X \sim ERSN\left(0, 1, \lambda, \sqrt{1+r^2}, \frac{\sqrt{s^2(1+r^2)^2 + r^2\lambda^2(1+r^2)}}{r^2}, 0\right)$.*

Proof By a direct computation in $f_X(x) = \int_{-\infty}^{\infty} f_{X|W}(x \mid w) f_W(w) dw$ is obtained the result. □

Corollary 1. *Let $X \mid W = w \sim N\left(w, 1\right)$ and $WERSN\left(0, 1, \lambda, r, s, 0\right)$. Then $W \mid X = x \sim ERSN\left(\frac{r^2 x}{1+r^2}, \frac{1}{\sqrt{1+r^2}}, \frac{\lambda}{\sqrt{1+r^2}}, r, s, \frac{r^2\lambda x}{s(1+r^2)}\right)$.*

Proposition 8. *Let $X \mid V = vERSN\left(0, 1, w, r, s, 0\right)$ and $V \sim N\left(\theta_1, \theta_2\right)$. Then $f_X(x) = \frac{2}{r}\phi\left(\frac{x}{r}\right)\Phi\left(\frac{\theta_1 x}{\sqrt{s^2 + \theta_2 x^2}}\right)$.*

Proof By a direct computation in $f_X(x) = \displaystyle\int_{-\infty}^{\infty} f_{X|V}(x \mid v) f_V(v) dv$ the result is obtained. □

Remark 3. *Proposition 5 shows that the addition of a random variable rescaled skew normal and a normal to scale is a rescaled skew normal. The Proposition 6 extends the result by Henze (1986) of the extended skew normal model. When $r = s = 1$ in Proposition 7, we obtain the model of the Example 2. In Bayesian context, the Corollary 1 is a result of conjugate priori. When $r = s$ in Proposition 8, we obtain the Skew-Generalized-Normal model to scale introduced by Arellano-Valle et al. (2004).*

4. Appendix

Proof of the Proposition 1:

Let $a = \frac{\lambda\sqrt{rs}r}{\sqrt{s^2 + r^2\lambda^2}}$ and $b = \frac{rs}{\sqrt{s^2 + r^2\lambda^2}}$ then

$$
\begin{aligned}
&P[X \le x] \\
&= P[a|U| + bV \le x] \\
&= \int_0^\infty P\left[V \le \frac{x-au}{b} \mid |U| = u\right] f_{|U|}(u)du = \int_0^\infty \Phi\left(\frac{x-au}{b}\right) f_{|U|}(u)du
\end{aligned}
$$

By using $ra^2 + sb^2 = r^2 s$ and differentiating with respect to x we have

$$
\begin{aligned}
f_X(x) &= \int_0^\infty \frac{1}{b}\phi\left(\frac{x-au}{b}\right) f_{|U|}(u)du \\
&= \frac{2\sqrt{s}}{b\sqrt{2\pi}\sqrt{2\pi}\sqrt{r}} \exp\left[-\frac{1}{2}\left(\frac{x}{r}\right)^2\right] \int_0^\infty \exp\left[-\frac{1}{2}\left(\frac{rsu-ax}{b\sqrt{rs}}\right)^2\right] du
\end{aligned}
$$

Taking $y = \frac{rsu-ax}{b\sqrt{rs}}$, and using the relation $\frac{a}{b} = \frac{\lambda\sqrt{rs}}{s}$, the result is obtained.

Proof of the Proposition 2:

Let $a = \frac{\lambda\sqrt{rs}r}{\sqrt{s^2+r^2\lambda^2}}$ and $b = \frac{rs}{\sqrt{s^2+r^2\lambda^2}}$

$$
\begin{aligned}
E[X^n] &= E[(a|U| + bV)^n] \\
&= \sum_{k=0}^{n} \binom{n}{k} a^{n-k} E\left[|U|^{n-k}\right] b^k E\left[V^k\right].
\end{aligned}
$$

Considering that $E\left[|U|^{n-k}\right] = \frac{1}{\sqrt{\pi}}\left(\frac{2r}{s}\right)^{\frac{n-k}{2}} \Gamma\left(\frac{n-k+1}{2}\right)$ and $E(V^k)$ are the k-th moment of the standard normal random variable then the result is obtained.

Proof of the Proposition 3:

$$
\begin{aligned}
E[\exp(tX)] &= \int_{-\infty}^\infty \exp[tx] \frac{2}{r}\phi\left(\frac{x}{r}\right)\Phi\left(\frac{\lambda x}{s}\right) dx \\
&= \frac{2}{\sqrt{2\pi}r} \int_{-\infty}^\infty \exp\left[tx - \frac{x^2}{2r^2}\right]\Phi\left(\frac{\lambda x}{s}\right) dx \\
&= \frac{2}{\sqrt{2\pi}r} \exp\left[\frac{r^2t^2}{2}\right] \int_{-\infty}^\infty \exp\left[-\frac{1}{2}\left(\frac{x-r^2t}{r}\right)^2\right]\Phi\left(\frac{\lambda x}{s}\right) dx
\end{aligned}
$$

Taking $y = \frac{x-r^2t}{r}$ we have

$$
\begin{aligned}
E[\exp(tX)] &= 2\exp\left[\frac{r^2t^2}{2}\right] \int_{-\infty}^\infty \Phi\left(\frac{r\lambda}{s}y + \frac{r^2\lambda t}{s}\right)\phi(y)dy \\
&= 2\exp\left[\frac{r^2t^2}{2}\right] E\left[\Phi\left(\frac{r\lambda}{s}T + \frac{r^2\lambda t}{s}\right)\right] \quad \text{with } TN(0,1)
\end{aligned}
$$

Thus, $E\left[\exp\left(tX\right)\right]=2\exp\left[\frac{r^2t^2}{2}\right]\Phi\left(\frac{\lambda r^2t}{\sqrt{s^2+r^2\lambda^2}}\right).$

Proof of the Proposition 5:

$$H=\frac{r}{\sqrt{r^2+\sigma^2}}X+\frac{r}{\sqrt{r^2+\sigma^2}}Y=\frac{r}{\sqrt{r^2+\sigma^2}}\left[a\left|U\right|+bV\right]+\frac{r}{\sqrt{r^2+\sigma^2}}Y,$$

where $a=\frac{\lambda\sqrt{rs}r}{\sqrt{s^2+r^2\lambda^2}}$ and $b=\frac{rs}{\sqrt{s^2+r^2\lambda^2}}$. Thus,

$$\begin{aligned}H &= \frac{ra}{\sqrt{r^2+\sigma^2}}\left|U\right|+\frac{r}{\sqrt{r^2+\sigma^2}}\left(bV+Y\right)\\ &= \frac{ra}{\sqrt{r^2+\sigma^2}}\left|U\right|+\frac{r\sqrt{b^2+\sigma^2}}{\sqrt{r^2+\sigma^2}}W \qquad \text{with } W\sim N\left(0,1\right)\end{aligned}$$

Now, we have the hypotheses of Proposition 1, then letting $A=\frac{ra}{\sqrt{r^2+\sigma^2}}$ and $B=\frac{r\sqrt{b^2+\sigma^2}}{\sqrt{r^2+\sigma^2}}$, and by using $rA^2+sB^2=r^2s$ and $\frac{A}{B}=\frac{\lambda\sqrt{rs}r}{\sqrt{r^2s^2+s^2\sigma^2+r^2\lambda^2\sigma^2}}$ follows that $H\sim SHN\left(\lambda,r,\sqrt{s^2+r^{-2}s^2\sigma^2+\lambda^2\sigma^2}\right).$

Proof of the Proposition 6:

Let $a=\frac{\lambda\sqrt{rs}r}{\sqrt{s^2+r^2\lambda^2}}$, $b=\frac{rs}{\sqrt{s^2+r^2\lambda^2}}$ and $c=-\frac{\sqrt{rs}\varepsilon}{\sqrt{s^2+r^2\lambda^2}}$

$$\begin{aligned}P\left[X\le x\right] &= P\left[aT+bV\le x\right]\\ &= \int_c^\infty P\left[V\le\frac{x-at}{b}\mid T=t\right]f_T(t)dt=\int_c^\infty\Phi\left(\frac{x-at}{b}\right)f_T(t)dt\end{aligned}$$

$$\begin{aligned}\frac{dP\left[X\le x\right]}{dx} &= \int_c^\infty\frac{\sqrt{s/r}}{b}\phi\left(\frac{x-at}{b}\right)\frac{\phi(\sqrt{s/r})t}{\Phi(\frac{s\varepsilon}{\sqrt{s^2+r^2\lambda^2}})}dt\\ &= \frac{1}{r}\phi\left(\frac{x}{r}\right)\frac{\sqrt{rs}}{\sqrt{2\pi}b\Phi(\frac{s\varepsilon}{\sqrt{s^2+r^2\lambda^2}})}\int_c^\infty\exp\left[-\frac{1}{2}\left(\frac{rst-ax}{\sqrt{rs}b}\right)^2\right]dt,\end{aligned}$$

taking $y=\frac{rst-ax}{\sqrt{rs}b}$, the result is obtained.

Acknowledgments

The authors acknowledge helpful comments and suggestions by the referee which substantially improved the presentation. The research of O. Venegas was supported by Grant DGIP 2009 | 2 | 1. The research of D. Elal-Olivero was supported by Grant DIUDA-221189. The research of H. W. Gómez was supported by FONDECYT (Chile) 1090411.

References

[1] Arellano-Valle, R. B., Gómez, H. W. and Quintana, F. A. A New Class of Skew-Normal Distributions. *Communications in Statististics: Theory and Methods.* **33(7)**, pp. 1465-1480, (2004).

[2] Azzalini, A. A class of distributions which includes the normal ones. *Scand. J. Statist.* **12**, pp. 171-178, (1985).

[3] Azzalini, A. Further results on a class of distributions which includes the normal ones. *Statistica.* **46**, pp. 199-208, (1986).

[4] Elal, D., Elal, S. and Gómez, H. W. La Distribución $SN(\alpha, \beta)$. *Revista de la Facultad de Ingeniería.* **18**, pp. 29-31, (2004).

[5] Henze, N. A probabilistic representation of the skew-normal distribution. *Scand. J. Statist.* **13**, pp. 271-275, (1986).

[6] Johnson, N., Kotz, S. L. and Balakrishann, N. Continuous univariate distributions, Second Edition. New York: Willey and Sons, (1995).

[7] Mukhopadhyay, S. and Vidakovic, B. Efficiency of Linear Bayes Rules for a Normal Mean : Skewed Priors Class. *The Statistician.* **44 (3)**, pp. 389-397, (1995).

[8] Nadarajah, N., Kotz, S. Skewed distributions generated by the normal kernel. *Statistic & Probability Letters.* **65**, pp. 269-277, (2003).

[9] Sahu, S. K., Dey, D. K. and Branco, M. A new class of multivariate skew distributions with application to Bayesian regression models. *Canad. J. Statist.* **31**, pp. 129-150, (2003).

Osvaldo Venegas
Departamento de Ciencias Matemáticas y Físicas,
Facultad de Ingeniería,
Universidad Católica de Temuco,
Chile,
e-mail: ovenegas@uct.cl

David Elal-Olivero
Departamento de Matemática,
Facultad de Ingeniería,
Universidad de Atacama,
Avenida Copayapu 485,
Copiapó,
Chile,
e-mail: delal@mat.uda.cl

and

Héctor W. Gómez
Departamento de Matemáticas,
Facultad de Ciencias Básicas,
Universidad de Antofagasta,
Antofagasta,
Chile,
e-mail: hgomez@uantof.cl