Proyecciones Journal of Mathematics
Vol. 31, N^o 4, pp. 391-402, December 2012.
Universidad Católica del Norte
Antofagasta - Chile
DOI: 10.4067/S0716-09172012000400007

The Chetaev Theorem for Ordinary Difference Equations

Cristian Cárcamo and Claudio Vidal
Universidad del Bío Bío, Chile
Received : October 2012. Accepted : November 2012

Abstract

In this work we obtain necessary conditions for the instability in the Lyapunov sense of equilibrium points of autonomous and non-autonomous difference equations through the adaptation of differential methods and techniques due to Chetaev [3].

Key words and phrases : *Lyapunov's stability, Chetaev's Theorem, equilibrium solution, instability.*

AMS (MOS) 1991 Subject Classifications : *39 A30; 39 A05.*

1. Introduction

We are interested in the qualitative behavior of solutions without actually computing them. Realizing that most of the problems that arise in practice are nonlinear and mostly unsolvable, this investigation is of vital importance to scientists, engineers, and applied mathematicians. Stability theory plays a central role in systems theory and engineering. There are different kinds of stability problems that arise in the study of dynamical systems. This work is concerned with stability of equilibrium points and our main purpose is to adapt the differential methods and techniques of Chetaev [3] (see also [10]) to difference equations in both situations: autonomous and non-autonomous cases. Similar methods can be found, for example, in [1], [7], where the works of [9], [11] and many others were adapted.

In this paper we consider the ordinary difference equations

$$x(n+1) = \mathbf{f}(x(n), n), \quad n \geq 0 \tag{1.1}$$

where $\mathbf{f} : \mathbf{R}^m \times \mathbf{N}^* \to \mathbf{R}^m$ ($\mathbf{N}^* = \mathbf{N} \cup \{0\}$) is a continuous function and we assume that $\mathbf{f}(x^*, n) = x^*$, for all $n \geq 0$, i.e., x^* is an equilibrium solution of (1.1). During this work we will use the notation $x(n, x)$ to denote the solution of (1.1) such that $x(0, x) = x$. The sequence $x_n(x)$ means $x(n, x)$.

Now, it is important to recall the notion of stability in the Lyapunov sense.

Definition 1. *The equilibrium point x^* of (1.1) is said to be stable in the Lyapunov sense if given $\epsilon > 0$ there exists $\delta = \delta(\epsilon)$ such that $\|x_0 - x^*\| < \delta$ implies that $\|x(n) - x^*\| < \epsilon$ for all $n \geq 0$, where $x_0 = x(0)$. The equilibrium point is unstable if it is not stable.*

Often it is important to know whether the solution $x(n) = x^*$ is stable, i.e., if it persists essentially unchanged on the infinite interval $\mathbf{Z}^+$ under small changes in the initial data. This is particularly important in applications where the initial data are often imperfectly known. We restrict our consideration to Lyapunov stability, wherein only perturbations of the initial data are contemplated, and thereby exclude considerations of structural stability, in which one considers perturbations of the function $\mathbf{f}$.

The original proof for ordinary differential equations can be found in [2], [3] or [10]. In his famous memoir (his Phd thesis), published in 1892, the Russian mathematician A.M. Lyapunov (see [9]) introduced a new method to investigate the stability of nonlinear differential equations. This method, known as Lyapunov's direct method, allows one to investigate the qualitative nature of solutions without actually determining the solutions themselves. It consists of the use of an auxiliary function, which generalizes the role of the energy in mechanical systems. In the Lyapunov results such as stability and instability it is necessary to know the behavior of the Lyapunov function around a neighborhood of the origin. While under the use of Chetaev's Theorem the instability theorem graphically shows that if the condition is satisfied, then the origin of (1.1) is a gap within any neighborhood of the origin, so that a solution could escape for neighborhoods of the origin along a solution in this gap. So, in this case we do not need to construct a Lyapunov function on a neighborhood of the equilibrium point, in fact, it is only necessary to detect a sector that contains the gap.

In order to point out the main difference of the autonomous and non-autonomous ordinary difference equations, we decide to include separately the proof of the Chetaev Theorem for each situation. In Section 2 we consider the autonomous case and in Section 3 we see the non-autonomous case.

It is important to emphasize that one of the importances of the Chetaev Theorem is its application to several problems whose linear part associated to each equilibrium point does not give information about the stability of the full system. i.e., linearized stability could imply instability in the Lyapunov sense for the full system, or more precisely, arbitrary small disturbing forces can make stable motions to become unstable. The investigation of stability and instability of solutions in difference equations has been considered for example in [1], [4], [5], [6], [7], [8], [12], [13], [14] and references therein.

2. The autonomous case

In this case $\mathbf{f}(x,n) = \mathbf{f}(x)$. Initially, we point out to some important definitions. Let $V : \mathbf{R}^m \to \mathbf{R}$ be defined as a real-valued function. The variation of V is defined as

$$\Delta V(x) = V(\mathbf{f}(x)) - V(x),\ x \in \mathbf{R}^m$$

and through the solutions of (1.1), the variation of V relative to (1.1) is given by

$$\Delta V(x(n)) = V(x(n+1)) - V(x(n)). \tag{2.1}$$

Notice that if the function $\Delta V(x) \geq 0$, then V is nonincreasing along solutions of (1.1).

We denote by $B_a(x^*)$ the open ball in $\mathbf{R}^m$ with center at x^* and radius a.

The set of points $x = (x_1, x_2, \dots, x_m) \in \mathbf{R}^m$ with $\|x\| \leq \eta$ (a small neighborhood of the origin) satisfying the condition $V(x) > 0$ will be called the region $V > 0$, and the surface $V = 0$ will be called the boundary of this region (see Figure 2).

We denote by $B_a(x^*)$ the open ball in $\mathbf{R}^m$ with center at x^* and radius a. Our main result in this work is the following.

Theorem 1. *(Chetaev's theorem for Ordinary Difference Equations) Let $\Omega \subset \overline{B}_a(x^*) \subset \mathbf{R}^m$ be an open and connected region and let us assume that there is a continuous scalar function V on $B_a(x^*)$ which possesses the following properties:*

1. *$V(x)$ and $\Delta V(x)$ are positive definite on Ω.*

2. *$V(x) = 0$, $\forall\, x \in \partial\Omega$, where $\partial\Omega$ is the boundary of Ω in $\overline{B}_a(x^*)$.*

Then x^* is unstable in the Lyapunov sense. In particular, there is a neighborhood $\mathcal{V}$ of the equilibrium point x^* such that all solutions which start in $\mathcal{V} \cap \Omega$ leave $\mathcal{V}$ in positive time.

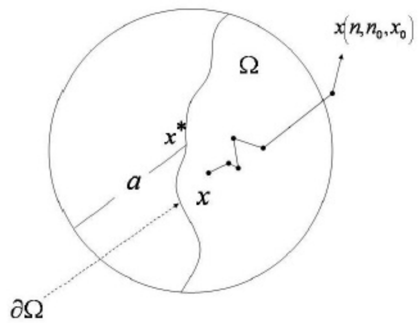

Figure 1: Unstable solution of the system (1) under Chetaev's Theorem 2.

Proof. We can take $x^* = 0$. Let $\epsilon > 0$ be so small that the closed ball of radius ϵ about 0 is contained in the domain $\overline{B}_a(x^*)$ and let $\mathcal{V} = \Omega \cap \{\|y\| < \epsilon\}$. We affirm that there are points arbitrarily close to the equilibrium point which move a distance at least ϵ from the equilibrium.

Note that $\mathcal{V}$ has points arbitrarily close to the origin, so for any $\delta > 0$ there is a point $x \in \mathcal{V}$ with $\|x\| < \delta$ and $V(x) > 0$.

Let $y(n) = V(x(n,x))$ with x as before. We affirm that $x(n,x)$ remains in $\mathcal{V}$ for all $n \geq 0$ or $x(n,x)$ crosses the boundary of $\mathcal{V}$ for the first time at a time n^*.

If $x(n,x)$ remains in $\mathcal{V}$, for all $n \geq 0$ then $y(n)$ is increasing because $\Delta V(x(n,x)) > 0$ and so $y(n) \geq y(0) = V(x) > 0$ for $n \geq 0$. The closure of $\widetilde{\Omega} = \{x(n,x) \ / \ n \geq 0\}$ is compact and then defining

$$\gamma = \min_{y \in \widetilde{\Omega}} \Delta V(y).$$

Since $\Delta V(y)$ is positive definite in $\widetilde{\Omega}$, it is clear that $\gamma > 0$. From the fact that $x(n,x) \in \widetilde{\Omega}$, for every n it follows that

$$V(x(n,x)) - V(x) = \sum_{s=0}^{n-1} \Delta V(x(s,x)) \geq \gamma n.$$

Taking $n \to \infty$ in the previous inequality, we obtain that $V(x(n,x)) \to \infty$, as $n \to \infty$. This is a contradiction because $x(n,x)$ remains in an ϵ neighborhood of the origin and V is continuous.

Therefore there exists $n^* > 0$ such that $x(n,x)$ crosses the boundary of $\mathcal{V}$ for the first time at $n = n^*$, and $\Delta V(x(n,x)) > 0$ for $0 \leq n < n^*$ and so $y(n^*) \geq V(x) > 0$. Because the complement of $\mathcal{V}$ consist of points q where $V(q) \leq 0$ or where $\|q\| \geq \epsilon$, it follows that $\|x(n^*,x)\| \geq \epsilon$. Therefore, $x^* = 0$ is unstable. $\square$

Remark 2. *The region $V > 0$ may consist of several connected components, in order to apply Chetaev's theorem 1 only one connected component Ω of the region $V > 0$ must be considered. To determine Ω by a single inequality $W > 0$ it is sufficient to consider the continuous function W, equal to V in Ω and equal to $-|V|$ outside of Ω.*

Corollary 3. *(Second version of Chetaev's theorem for autonomous Ordinary Difference Equations) Assume that there exists a continuous function V, such that the region $V > 0$ is (non empty) with $V(0) = 0$, and $\Delta V(x)$ is positive definite in the region $V > 0$. Then the equilibrium point 0 is unstable to the system (1.1).*

Proof. It is enough to define the set $\Omega = \{x \in B_a(0) \;/\; V(x) > 0\}$ and then apply Theorem 1. In Figure 2 we exhibit the local behavior shown in Corollary 3. $\square$

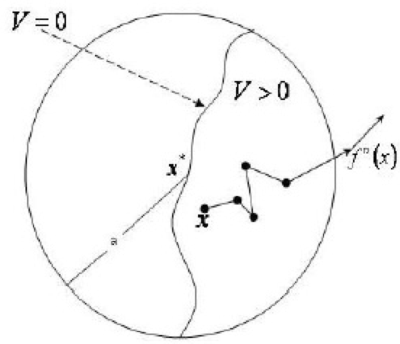

Figure 2: Local behavior of system (1.1) in Theorem 2.

Corollary 4. *(Instability Lyapunov's theorem for autonomous Ordinary Difference Equations) Assume that there exists a continuous function V, such that $V > 0$ and $\Delta V(x)$ are positive definite in the neighborhood of the origin Then the equilibrium point* 0 *is unstable to the system (1.1).*

Proof. It is enough to define the set $\Omega = \{x \in B_a(0) \;/\; V(x) > 0\}$ and then apply Theorem 1. $\square$

2.1. Examples

Example 1. We consider the ordinary difference system in $\mathbf{R}^2$

$$
\begin{aligned}
x_1(n+1) &= x_2(n) + x_1(n)x_2(n) \\
x_2(n+1) &= x_1(n) + x_1(n)x_2(n).
\end{aligned}
\tag{2.2}
$$

We affirm that the equilibrium point $(0,0)$ is unstable in the Lyapunov sense.

Let $V(x_1, x_1) = x_1\ x_2$ be a function defined on $D = \{(x_1, x_2) : x_1 > 0, x_2 > 0\}$. It is clear that $V > 0$ on D y and $V = 0$ on the boundary of D. Also, we show that

$$\Delta V(x_1, x_2) = x_1 x_2(x_1 + x_2 + x_1 x_2).$$

As $x_1 > 0$ and $x_2 > 0$ in D it follows that $\Delta V(x_1, x_2) > 0$ for all $(x_1, x_2) \in D \cap B$, where B is a small ball with center at the origin. Finally, by Theorem 1, we arrive to the fact that $(0, 0)$ is unstable. It is important to observe that the eigenvalues of the linear part around the equilibrium point $(0, 0)$ of the system (2.2) are ± 1. Thus, it follows that the linear part does not decide the kind of stability of the equilibrium, therefore the stability is decided taking into account the higher order terms.

Example 2. Let

$$\begin{aligned} x_1(n+1) &= x_1(n) + bx_1^2(n) + cx_1(n)x_2^2(n) \\ x_2(n+1) &= -x_2(n) + dx_1^2(n)x_2(n) + ex_1(n)x_2^2(n), \end{aligned} \tag{2.3}$$

where b, c, d, e are real parameters with $b > 0$. The eigenvalues of the linearized part around the equilibrium point $(0, 0)$ are 1 and -1, so the linear part does not decide the stability or instability of the solution $(0, 0)$ of the system (2.3).

We consider the function $V(x_1, x_2) = x_1^2 - x_2^2$ and the set $\Omega = \{(x_1, x_2) : 0 < |x_2| < x_1\}$. Thus, $V > 0$ in Ω and $V = 0$ in the boundary of Ω. It is verified that

$$\begin{aligned} \Delta V(x_1, x_2) = {} & 2bx_1^3 + b^2x_1^4 + 2(c+d)x_1^2x_2^2 + 2ex_1x_2^3 + \\ & 2bcx_1^3x_2^2 + (c^2 - e^2)x_1^2x_2^4 - d^2x_1^4x_2^2 - 2dex_1^3x_2^3 \\ = {} & x_1^3\Big[2b + b^2x_1 + 2(c+d)\tfrac{x_2}{x_1}x_2 + 2e\tfrac{x_2^2}{x_1^2}x_2 + \\ & 2bcx_2^2 + (c^2 - e^2)\tfrac{x_2}{x_1}x_2^3 - d^2x_1x_2^2 - 2dex_2^3\Big]. \end{aligned}$$

Since $b > 0$, and in Ω we have $-1 < \frac{x_2}{x_1} < 1$, it follows that the term in brackets is positive when $(x_1, x_2) \to (0, 0)$. Therefore, by Theorem 1, we conclude that $(0, 0)$ is unstable for any $b > 0$ and any choice of the parameters c, d and e.

Example 3. A generalization of the previous example can be obtained as follows. Let

$$\begin{aligned} x_1(n+1) &= x_1(n) + g_1(x_1, x_2) \\ x_2(n+1) &= -x_2(n) + g_2(x_1, x_2), \end{aligned} \tag{2.4}$$

where $g_1(x_1,x_2) = \alpha x_1^l + h(x_1,x_2)$, such that $l \in \mathbf{N}$, $\alpha > 0$ and h and g_2 are polynomials of degree at least $l+1$ in (x_1,x_2). We consider $V(x_1,x_2) = x_1^2 - x_2^2$ and the set $\Omega = \{(x_1,x_2) : 0 < |x_2| < x_1\}$. Thus, $V > 0$ in Ω and $V = 0$ in the boundary of Ω and as $\alpha > 0$ and in Ω we have $-1 < \frac{x_2}{x_1} < 1$ it follows from Theorem 1 that in a neighborhood sufficiently small of the origin in $\mathbf{R}^2$, we conclude that $(0,0)$ is unstable.

3. The non-autonomous case

Now, we will consider the ordinary difference equation (1.1) in the non-autonomous case and again we assume that $x^* = 0$ is an equilibrium point of the system (1.1).

In order to point out the hypotheses of the Chetaev Theorem in the non-autonomous case, we need to introduce some necessary additional definitions of a domain $V > 0$.

Let $V : \mathbf{R}^m \times \mathbf{N}^* \to \mathbf{R}$ be a function depending of x and n simultaneously. A set of values x with $\|x\| < a$ (a sufficiently small) that satisfies $V(x,n) > 0$ is called a domain $V > 0$, and the surface $V(x,n) = 0$ is called bound of that domain. It is clear that if the function V depends explicitly of n, then with the variations of n the region $V > 0$ and its bound change with n.

Definition 1. *A function $U = U(x,n)$ is called positive definite in the domain $V > 0$ if for any $\rho > 0$ arbitrarily small there exists a number $\iota = \iota(\rho) > 0$ such that for all x satisfying the condition $V(x,n) \geq \rho$ and $\|x\| \leq a$ the inequality $U(x,n) \geq \iota$ is valid for all $n \geq 0$.*

The variation of V for equation (1.1), where $V : \mathbf{R}^m \times \mathbf{N}^* \to \mathbf{R}$ is defined as

$$\Delta V(x,n) = V(\mathbf{f}(x),n) - V(x,n),\ x \in \mathbf{R}^m,\ n \geq 0$$

and through the solutions of (1.1), the variation of V is given by

$$\Delta V(x(n),n) = V(x(n+1),n+1) - V(x(n),n),\ n \geq 0. \tag{3.1}$$

In particular, we have that the function $\Delta V(x,n)$ is positive-definite in the region $V > 0$ according to Definition 1 provided that for every positive number $\epsilon > 0$, there is a positive number $\delta = \delta(\epsilon)$ such that $\Delta V(x, n) > \epsilon$ for all x such that $V(x, n) > \delta$.

The next theorem corresponds to our first version of the Chetaev Theorem for the non autonomous case.

Theorem 2. *(Chetaev's theorem for non-autonomous Ordinary Difference Equations) Suppose that for the system (1.1) there is a continuous function $V(x, n)$ defined for all $n \geq 0$ and for all $\|x\| < a$, having the property that for values of the variables x that are arbitrarily small, V is bounded and the values of ΔV along any trajectory of the equation (1.1) is positive definite in the region $V > 0$. Then, the equilibrium solution $x(n) = 0$ is unstable in the Lyapunov sense.*

Proof. Since $V = V(x, n)$ is bounded in the region $V > 0$, we have that the inequality

$$V(x, n) < L, \quad n \in \mathbf{N}^*, \ \|x\| < a \tag{3.2}$$

is satisfied for some $L > 0$. Now, we assume that for the previous points x the solution $x_n(x) = x(n, x)$ of (1.1) is stable. Thus, given ε, with $0 < \varepsilon < a$, there exists $\delta > 0$, $0 < \delta < a$, such that $\|x\| \leq \delta$ implies that $\|x(n, x)\| \leq \varepsilon$ for all $n \geq 0$. Since x is in the region $V > 0$, we have that $V_0 = V(x, 0) > 0$. We affirm that for all $n \geq 0$ the sequence $x_n(x)$ satisfies $V(x_n(x), n) > V_0$. The proof of this affirmation is based on the fact that if $V(x_s(x), s) > 0$ for $0 \leq s < \theta$ then $V(x_\theta(x), \theta) > 0$, because

$$V(x_\theta(x), \theta) - V(x, 0) = \sum_{s=0}^{\theta-1} \Delta V(x_s(x), s) > 0,$$

since ΔV is positive definite in the region $V > 0$.

Next, according to Definition 1 given $V_0 > 0$ there exists $\delta > 0$ such that

$$\begin{aligned} \Delta V(x_m(x), m) > \delta \ &\Rightarrow \sum_{m=0}^{n-1} \Delta V(x_m(x), m) > \sum_{m=0}^{n-1} \delta \\ &\Leftrightarrow V(x_n(x), n) - V(x, 0) > n\delta \\ &\Leftrightarrow V(x_n(x), n) > V_0 + n\delta. \end{aligned} \tag{3.3}$$

Making $n \to \infty$ we conclude that V is not bounded through any solution of (1.1) but this contradicts (3.2). Therefore, $x = 0$ of (1.1) is unstable. □

The region $V > 0$ may consist of several connected components. For application of Chetaev's theorem only one connected component Ω of the region $V > 0$ can be of interest. To determine Ω by a single inequality $W > 0$ it is sufficient to consider the continuous function W, equal to V in Ω and equal $-|V|$ outside of Ω.

3.1. Examples

Example 4. First, we will consider the following non-autonomous difference equations

$$\begin{aligned} x_1(n+1) &= x_1(n) + x_2^2 + n^2 x_1^2(n) \\ x_2(n+1) &= x_2(n), \end{aligned} \tag{3.4}$$

where the origin is an equilibrium solution. In order to prove the instability of this equilibrium point we define the Chetaev function $V(x_1, x_2) = x_1 + x_2$ and the set $V > 0$ as $\{(x_1, x_2) \in \mathbf{R}^2 \ / \ x_2 > -x_1\}$, where the boundary is the set $V = 0$, i.e., $\{(x_1, x_2) \in \mathbf{R}^2 \ / \ x_2 = -x_1\}$.

We have that $V(0,0) = 0$, and

$$\begin{aligned} \Delta V(x_1, x_2) &= x_1(n) + x_2^2(n) + n^2 x_1^2(n) + x_2(n) - x_1(n) - x_2(n) \\ &= x_2^2(n) + n^2 x_1^2(n) \\ &> 0. \end{aligned}$$

Therefore, by Theorem 2 we conclude that the solution $(0,0)$ of (3.4) is unstable in the Lyapunov sense.

Example 5. Consider the non-autonomous system

$$\begin{aligned} x_1(n+1) &= x_1(n) + x_2^2(n) + n^2 x_1^2(n) \\ x_2(n+1) &= x_2(n) + x_1^2(n) x_2^2(n). \end{aligned} \tag{3.5}$$

$(0,0)$ is an equilibrium point and here we define the same auxiliary function V as in the previous example and it follows that this equilibrium point is unstable in the Lyapunov sense. In fact,

$$\Delta V(x_1, x_2) = x_2^2(n) + n^2 x_1^2(n) + x_1^2(n) x_2^2(n) > 0.$$

Example 6. A generalization of the previous examples is obtained by the following form. Assume that the functions $f_1(x_1, x_2)$ and $f_2(x_1, x_2)$ are polynomials in the variable x_1 and x_2 of degree at least two, such that,

$$\begin{aligned} x_1(n+1) &= x_1(n) + x_2^2(n) + n^2 x_1^2(n) + f_1(x_1(n), x_2(n)) \\ x_2(n+1) &= x_2(n) + f_2(x_1(n), x_2(n)). \end{aligned} \tag{3.6}$$

Considering $V(x_1, x_2) = x_1 + x_2$ and since

$$\Delta V(x_1(n), x_2(n)) = x_2^2(n) + n^2 x_1^2(n) + f_1(x_1(n), x_2(n)) + f_2(x_1(n), x_2(n)),$$

we arrive to the conclusion that in a small neighborhood of the origin $\Delta V(x_1, x_2) > 0$ along any trajectory of the equation (3.6) in $V > 0$.

Therefore, by Theorem 2 we conclude that the solution $(0, 0)$ of (3.6) is unstable in the Lyapunov sense.

Acknowledgements. This work is part of the master's thesis in Mathematics of the first author in the Universidad del Bío Bío.

References

[1] Agarwal, R. P., *Difference Equations and Inequalities*, Marcel Dekker, New York, (1992).

[2] Chetaev, N. G. Concerning the stability and instability of irregular systems. (Russian) *Akad. Nauk SSSR. Prikl. Mat. Meh.*, **12**, pp. 639642 (1948).

[3] Chetaev, N. G., The stability of motion. New York: Pergamom Press (1961).

[4] Diamond, P., Finite stability domains for difference equations, *Jour. Austral. Soc.*, **22A**, pp. 177-181 (1976).

[5] Diamond, P., Discrete Liapunov function with $\delta^2 v > 0$, *Jour. Austral. Soc.*, **20B**, pp. 280-284 (1978).

[6] Elaydi, S., Asymptotics for linear difference equations I: Basic theory, *J. Difference Equ. Appl.*, **5**, pp. 563-589, (1999).

[7] Elaydi, S., *An introduction to difference equations*, Third Edition, Springer (2005).

[8] Lasalle, J. P., The stability and control of discrete processes, *Appl. Math. Sci.*, **62**, (1986).

[9] Lyapunov, A., Problème general de la stabilité du movement, *Ann. of Math., Study*, **17**, (1947).

[10] Krasovskii, N., *Stability of motion*, Stanford Univ. Press-Stanford, California, (1963).

[11] Perron, O., Über einen Satz des Herrn Poincarè, *J. Reine Angew. Math.*, **136**, pp. 17 37, (1909).

[12] Pituk, M., criterion for the exponential stability of linear difference equations, *Appl. Math. Lett.*, **181**, pp. 779-783, (2004).

[13] Sasu, B., Stability of difference equations and applications to robustness problems, *Advances in Difference Equatiuons*, **ID 869608**, 24 pages, (2010).

[14] Sugiyama, S. Difference inequalities and their applications to satability problem, *Lectures Notes in Math.*, Springer, **243**, pp. 1-15, (1971).

Cristian Cárcamo
Departamento de Matemática,
Facultad de Ciencias
Universidad del Bío Bío,
Casilla 5-C,
Concepción
Chile
e-mail : ccarcamo@ubiobio.cl

and

Claudio Vidal
Departamento de Matemática,
Facultad de Ciencias
Universidad del Bío Bío,
Casilla 5-C,
Concepción
Chile
e-mail : clvidal@ubiobio.cl