Proyecciones Journal of Mathematics
Vol. 31, N^o 4, pp. 355-361, December 2012.
Universidad Católica del Norte
Antofagasta - Chile
DOI: 10.4067/S0716-09172012000400004

Prime Submodules of Graded Modules

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Received : March 2012. Accepted : October 2012

Abstract

Let G be a group, R be a G-graded ring and M be a G-graded R-module. Suppose P is a prime ideal of R_e and $g \in G$. In this article, we define

$$M_g(P) = \{m \in M_g : Am \subseteq PM_g \text{ for some ideal } A \text{ of } R_e \text{ satisfying } A \nsubseteq P\}$$

that is an R_e-submodule of M_g, and we investigate some results on this submodule. Also, we introduce a situation where if N is a gr-prime R-submodule of M, then $(N_g : M_g)$ is a maximal ideal of R_e. We close this article by introducing a situation where if N is a gr- R-submodule of M such that N_e is a weakly prime R_e-submodule of M_e, then N_g is a prime R_e-submodule of M_g.

2010 AMS Subject Classifications : *13 A 02.*

Keywords and Phrases : *Graded rings, graded modules, prime submodules.*

Introduction

Let G be a group and R be a commutative G-graded ring which is denoted by (R, G). The elements of R_g are called homogeneous of degree g where R_g are additive subgroups of R indexed by the elements $g \in G$. Consider $\text{supp}(R, G) = \{g \in G : R_g \neq 0\}$. If $x \in R$, then x can be written uniquely as $\sum_{g \in G} x_g$, where x_g is the component of x in R_g. Moreover, R_e is a subring of R and $1 \in R_e$. Further, if $r \in R_g$ and r is a unit, then $r^{-1} \in R_{g^{-1}}$. Let, $h(R) = \bigcup_{g \in G} R_g$. Assume M is a left R-module. Then M, denoted by (M, G), is a G-graded R-module (for simplicity, we write M is gr-R- module) if there exist additive subgroups M_g of M indexed by the elements $g \in G$ such that $M = \bigoplus_{g \in G} M_g$ and $R_g M_h \subseteq M_{gh}$ for all $g, h \in G$. Also, we consider $\text{supp}(M, G) = \{g \in G : M_g \neq 0\}$. It is clear that M_g is an R_e-submodule of M for all $g \in G$. For more details, one can look in [3,4,5]. Throughout this article, R is commutative ring with unity 1 and M is a left R-module.

A G-graded ring R is said to be first strongly graded if $1 \in R_g R_{g^{-1}}$ for all $g \in \text{supp}(R, G)$, this is equivalent to say that $\text{supp}(R, G)$ is a subgroup of G and $R_g R_h = R_{gh}$ for all $g, h \in \text{supp}(R, G)$. A G-graded R-module M is said to be first strongly graded if $\text{supp}(R, G)$ is a subgroup of G and $R_g M_h = M_{gh}$ for all $g \in \text{supp}(R, G)$, $h \in G$. Clearly, (R, G) is first strong if and only if every graded R-module is first strongly graded. For more details, one can look in [6]. (R, G) is said to be crossed product over the support if R_g contains a unit for all $g \in \text{supp}(R, G)$. It is not difficult to prove that if (R, G) is crossed product over the support, then (R, G) is first strong. Also, if R_e is a field, then (R, G) is crossed product over the support. For more details, it is nice to see [1]. An R-submodule N of a G-gr-R-module M is said to be graded if $N = \bigoplus_{g \in G}(N \cap M_g)$, a submodule of a graded module need not be graded. For more details, it is good to look quickly in [7].

For a gr-R-submodule N of a gr-R-module M, we define $(N : M) = \{r \in R : rM \subseteq N\}$. Clearly, $(N : M)$ is a graded ideal of R, see [2]. A proper gr-R-submodule N of a gr-R-module M will be called a graded prime R-submodule if whenever $r \in h(R)$ and $m \in h(M)$ with $rm \in N$, then either $m \in N$ or $r \in (N : M)$. Moreover, it is easy to prove that if N is a graded prime R-submodule of M, then $(N : M)$ is a graded prime ideal of R.

Results

We begin our article by introducing a situation where every ideal A of R_e has the form $(K : M_e)$ for some R_e-submodule K of M_e:

Proposition 0.1. *Let R be a first strongly G-graded ring and M be a gr-R-module. Suppose A is an ideal of R_e. Then there exists a proper gr-R-submodule N of M such that $A = (N_e : M_e)$ if and only if $AM_g \neq M_g$ and $A = (AM_g : M_g)$ for all $g \in supp(R, G)$.*

Proof. Suppose $A = (N_e : M_e)$ for some proper gr-R-submodule N of M. Then $AM_e \subseteq N_e$ and then $AM_e \neq M_e$. Let $g \in \mathrm{supp}(R, G)$. If $AM_g = M_g$, then $M_e = R_{g^{-1}}M_g = R_{g^{-1}}AM_g = AR_{g^{-1}}M_g = AM_e$ that is a contradiction. Thus, $AM_g \neq M_g$. On the other hand, let $a \in A$. Then $aM_g \subseteq AM_g$, so $a \in (AM_g : M_g)$. Thus, $A \subseteq (AM_g : M_g)$. Let $x \in (AM_g : M_g)$. $xM_g \subseteq AM_g$ and then $xM_e = xR_{g^{-1}}M_g = R_{g^{-1}}xM_g \subseteq R_{g^{-1}}AM_g = AR_{g^{-1}}M_g = AM_e \subseteq N_e$, so $x \in (N_e : M_e) = A$. Thus $(AM_g : M_g) \subseteq A$ and hence $A = (AM_g : M_g)$. The converse is obvious. □

A gr-R-module M will be called gr-weakly prime Noetherian if for every $a \in h(R)$ and every $m \in h(M)$, the gr-R-submodule $RaRm$ is finitely generated. Let P be a prime ideal of R_e. Then we define $M_g(P) = \{m \in M_g : Am \subseteq PM_g$ for some ideal A of R_e satisfying $A \not\subseteq P\}$, $g \in G$. It is clear that $M_g(P)$ is an R_e-submodule of M_g for all $g \in G$ and $PM_g \subseteq M_g(P)$. Now, we introduce the following results about $M_g(P)$.

Proposition 0.2. *Let R be a first strongly G-graded ring and M be a gr-R-module. If P is a prime ideal of R_e and K is a gr-prime R-submodule of M such that $(K_e : M_e) = P$, then $M_g(P) \subseteq K$ for all $g \in supp(R, G)$.*

Proof. Let $g \in \mathrm{supp}(R, G)$. Suppose $m \in M_g(P)$. Then there exists an ideal of R_e such that AP and $Am \subseteq PM_g$. However, $PM_g = PR_gM_e = R_gPM_e \subseteq R_gK_e \subseteq K$ and hence $Am \subseteq K$. Since K is gr-prime, either $m \in K$ or $Am \subseteq K$. If $Am \subseteq K$, then $AM_e \subseteq K_e$, so $A \subseteq (K_e : M_e) = P$ that is a contradiction. Thus $m \in K$ and hence $M_g(P) \subseteq K$. □

Proposition 0.3. *Let R be a G-graded ring and M be a gr-R-module. Suppose P is a prime ideal of R_e and $g \in G$ such that M_g/PM_g is weakly Noetherian R_e/P-module. If $N = M_g(P)$, then $N = M_g$ or N is a prime R_e-submodule of M_g such that $P = (N : M_g)$.*

Proof. Suppose $N \neq M_g$. Let $r \in R_e$ and $m \in M_g$ such that $rm \in N$. If $r \in P$, then $rM_g \subseteq PM_g \subseteq N$. Suppose $r \notin P$. Let $A = R_e r R_e$. Then A is an ideal of R_e such that AP. Since M_g/PM_g is weakly Noetherian, $Am + PM_g = Am_1 + ... + Am_k + PM_g$ for some positive integer k and $m_i \in Am_i$, $1 \leq i \leq k$. For each $1 \leq i \leq k$, $m_i \in Am \subseteq N$ and hence there exists ideal B_i of R_e such that B_iP and $B_im_i \subseteq PM_g$. Let $B = B_1 \bigcap ...... \bigcap B_k$. Then B is an ideal of R_e such that BP because P is prime. Moreover, $BAm \subseteq Bm_1 + ... + Bm_k + PM_g \subseteq PM_g$. However, P is prime implies BAP. Thus $m \in N$. It follows that N is a prime R_e-submodule of M_g. Now, let $x \in P$. Then $xM_g \subseteq PM_g \subseteq N$, so $x \in (N : M_g)$. Thus $P \subseteq (N : M_g)$. Suppose $P \neq (N : M_g)$. Then there exists $\alpha \in (N : M_g)$ such that $\alpha \notin P$. Let $t \in M_g$. Then $R_e \alpha R_e t \subseteq N$. By above technique, $t \in N$. Thus $N = M_g$ that is a contradiction. Hence $P = (N : M_g)$. $\square$

By Proposition 0.2, $M_g(P) \neq M_g$ if M contains a gr-prime R-submodule K such that $(K_e : M_e) = P$ provided that (R, G) is first strong and $g \in$ supp(R, G). Now, we introduce another situation where $M_g(P) \neq M_g$.

Proposition 0.4. *Let R be a G-graded ring and M be a gr-R-module. Suppose P is a prime ideal of R_e and $g \in G$ such that M_g/PM_g is finitely generated and weakly Noetherian R_e/P-module. If $P = (PM_g : M_g)$, then $M_g(P) \neq M_g$.*

Proof. Suppose $M_g(P) = M_g$. Then there exists a positive integer k and elements $m_i \in M_g$, $1 \leq i \leq k$ such that $M_g = R_e m_1 + ... + R_e m_k + PM_g$. For each $1 \leq i \leq k$, there exists an ideal A_i of R_e such that A_iP and $A_im_i \subseteq PM_g$. Let $A = A_1 \bigcap ...... \bigcap A_k$. Then A is an ideal of R_e such that AP and $AM_g \subseteq PM_g$, so $A \subseteq (PM_g : M_g) = P$ that is a contradiction. Hence $M_g(P) \neq M_g$. $\square$

It is easy to prove that if N is a gr-prime R-submodule of a gr-R-module M, then $(N : M)$ is a gr-prime ideal of R, see [2, Proposition 2.4]. Similarly, one can prove that if N is a gr-prime R-submodule of a gr-R-module M, then $(N_g : M_g)$ is a prime ideal of R_e for all $g \in G$. In this article, we introduce a situation where if N is a gr-prime R-submodule of a gr-R-module M, then $(N_g : M_g)$ is a maximal ideal of R_e, see Corollary 0.7.

Proposition 0.5. *Let R be a first strongly G-graded ring and M be a gr-R-module. If M_e is Artinian and prime, then $R_e/Ann(M_g)$ is a field for all $g \in supp(R, G)$.*

Proof. Let $T = \{N : N$ is a nontrivial R_e-submodule of $M_e\}$. Suppose that N_0 is a minimal element of T. Obviously N_0 is a non-zero simple module. Hence there exists a nonzero $a \in M_e$ such that $N_0 = R_e a \approx R_e/Ann(a)$ and $Ann(a)$ is a maximal ideal of R_e. Since M_e is prime, $Ann(a) = Ann(M_e)$. Let $r \in Ann(M_e)$ and $g \in \mathrm{supp}(R, G)$. Then $rM_g = rR_gM_e = R_grM_e = R_g.\{0\} = \{0\}$, so $r \in Ann(M_g)$. Let $s \in Ann(M_g)$. Then $sM_e = sR_{g^{-1}}M_g = R_{g^{-1}}sM_g = R_{g^{-1}}.\{0\} = \{0\}$, so $s \in Ann(M_e)$. Hence $Ann(M_e) = Ann(M_g)$. Consequently, $Ann(M_g)$ is a maximal ideal of R_e and then $R_e/Ann(M_g)$ is a field. □

Corollary 0.6. *Let R be a first strongly G-graded ring and M be a gr-R-module. If M_e is Artinian, faithful and prime, then R_e is a field.*

Corollary 0.7. *Let R be a first strongly G-graded ring, M be a gr-R-module and N be a gr-prime R-submodule of M. If M_e is Artinian, then $(N_g : M_g)$ is a maximal ideal of R_e for all $g \in supp(R, G)$.*

Proof. Since N is a gr-prime R-submodule of M, by [2, Proposition. 2.5], N_e is a prime R_e-submodule of M_e. Then M_e/N_e is an Artinian prime R_e-module, consequently, by Proposition 0.5, $R_e/Ann(M_g/N_g)$ is a field for all $g \in \mathrm{supp}(R, G)$, and then $(N_g : M_g) = Ann(M_g/N_g)$ is a maximal ideal of R_e for all $g \in \mathrm{supp}(R, G)$. □

A proper R-submodule N of an R-module M is said to be weakly prime if whenever $a, b \in R$ and $x \in M$ such that $abx \in N$, then either $ax \in N$ or $bx \in N$. Obviously, any prime submodule is a weakly prime submodule, but the converse is not always correct. We close our article by introducing a situation where if N is a gr- R-submodule of M such that N_e is a weakly prime R_e-submodule of M_e, then N_g is a prime R_e-submodule of M_g.

Proposition 0.8. *Let R be a G-graded ring and M be a gr-R-module such that (R, G) is crossed product over the support and M_e satisfies the DCC on cyclic R_e-submodules. Assume that M_e is R_e-torsion free. Suppose N is a gr-R-submodule of M such that N_e is weakly prime R_e-submodule of M_e. Then N_g is prime R_e-submodule of M_g for all $g \in supp(R, G)$, $g \neq e$.*

Proof. Let $g \in \mathrm{supp}(R, G)$, $g \neq e$. Suppose $r \in R_e$, $a \in M_g$ such that $ra \in N_g$. Assume $r \notin (N_g : M_g)$. Then there exists $b \in M_g$ such that $rb \notin N_g$. Since (R, G) is crossed product over the support, $R_{g^{-1}}$ contains a unit, say x. Consider the following chain of R_e-submodules of M_e: ...... $\subseteq R_exr^3(a + b) \subseteq R_exr^2(a + b) \subseteq R_exr(a + b)$. For some positive integer n, we have $R_e x r^{n+1}(a+b) = R_e x r^n(a+b)$, that is $xr^n(rt-1)(a+b) = 0 \in N_e$ for some $t \in R_e$. If $r^n(a+b) \in N_e$, then $r(a+b) \in N_e$ and then $0 \neq r(a+b) \in N_e \bigcap M_g \subseteq M_e \bigcap M_g$ that is a contradiction since $g \neq e$. So, $x(rt-1)(a+b) \in N_e$. Now, $xrta - xa + x(rt-1)b = x(rt-1)(a+b) \in N_e$, on the other hand $xrta = xt(ra) \in R_{g^{-1}} R_e N_g \subseteq N_e$, so

$$-xa + x(rt-1)b \in N_e............(*)$$

before and then $-a+(rt-1)b \in N_g$. We get that $-ra+r(rt-1)b \in N_g$, on the other hand, $-ra \in N_g$, so $r(rt-1)b \in N_g$ and then $xr(rt-1)b \in N_e$. If $rb \in N_e$, then $0 \neq rb \in N_e \bigcap M_g \subseteq M_e \bigcap M_g$ that is a contradiction, so $x(rt-1)b \in N_e$ and then by (*), $xa \in N_e$ and then $a \in N_g$. Hence N_g is a prime R_e-submodule of M_g. $\square$

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